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DHD-puzzles

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Abstract. In this work triangular puzzles that are composed of unit triangles with labelled edges are considered. To be more precise, the labelled unit triangles that we allow are on the one hand the puzzle pieces that compute Schubert calculus and on the other hand the flipped K-theory puzzle piece. The motivation for studying such puzzles comes from the fact that they correspond to a class of oriented triangular fully packed loop configurations. The main result that is presented is an expression for the number of these puzzles with a fixed boundary in terms of Littlewood-Richardson coefficients.

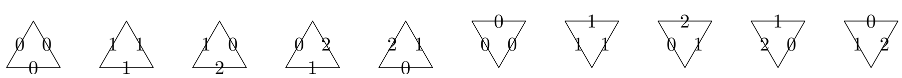
Résumé. Dans ce travail nous considérons des puzzles triangulaires composés de triangles unitaires avec côtés étiquetés. Plus précisément, les triangles que nous autorisons sont, d'une part ceux correspondant au calcul de Schubert, et d'autre part les retournements des pièces de puzzle de la K-théorie. La motivation pour étudier des tels puzzles provient du fait qu'ils correspondent à une classe de configurations de boucles compactes triangulaires (TFPL) orientées. Le résultat principal présenté est une expression du nombre de ces puzzles ayant une frontière fixée en fonction des coefficients de Littlewood-Richardson.

Keywords. Knutson-Tao puzzles, Triangular fully packed loop configurations, Littlewood-Richardson coefficients

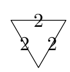
1 Introduction

In [5], triangular puzzles were introduced to compute Schubert calculus. These triangular puzzles together with additional puzzle pieces that were established in [3] will be the central objects in this article. Their study is motivated by their correspondence to a class of *oriented triangular fully packed loop configurations* as they were defined in [3]. This correspondence will be amplified at a later stage.

The puzzle pieces that were introduced in [5] are the following unit triangles with labelled edges:



In the following, they will be referred to as **ordinary puzzle pieces**. Furthermore, the additional puzzle piece that will be allowed is the following:



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This puzzle piece was established in [3] as the **DHD-puzzle piece**. A **DHD-puzzle** of size N is defined as a decomposition of an equilateral triangle with side length N, $\mathcal{T}_N$, into unit triangles each with sides labelled 0, 1 or 2 such that: (1) each unit triangle is an ordinary or DHD-puzzle piece; (2) whenever two unit triangles are adjacent their common egde has the same label; (3) no edge on the boundary of $\mathcal{T}_N$ has label 2. Examples of DHD-puzzles are given in Figure 1. The DHD-puzzles defined above generalise those defined in [3] that contained only one DHD-puzzle piece.

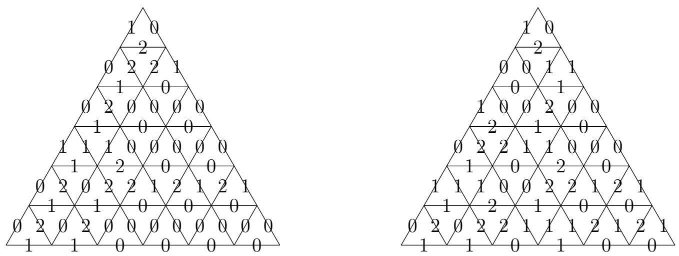

Fig. 1: Two DHD-puzzles of size 6.

The labels on the left, the right and the bottom boundary of $\mathcal{T}_N$ will be denoted by u, v and w, when read from left to right. Then u, v and w are all 01-words of length N. The boundaries of the two puzzles depicted in Figure 1 are $(001001, 010010; 110000)$ and $(010101, 010011; 110100)$. It was shown in [3] that the boundary $(u, v; w)$ of a DHD-puzzle must necessarily satisfy $|u|_1 = |v|_1 = |w|_1$ when $|\cdot|_1$ denotes the number of occurrences of 1 in the word. In the following, the number of DHD-puzzles with boundary $(u, v; w)$ will be denoted by $\mathrm{dhd}_{u,v}^w$.

From now on, to every 01-word $\omega = \omega_1 \cdots \omega_N$ a Young diagram $\lambda(\omega)$ will be assigned as follows: a path on the square lattice is constructed by drawing a $(0, 1)$-step if $\omega_i = 0$ and a $(1, 0)$-step if $\omega_i = 1$, for i from 1 to N. Additionally, a vertical line through the paths start point and a horizontal line through its ending point are drawn. Then the region enclosed by the lattice path and the two lines is the Young diagram $\lambda(\omega)$. Thus, the number of cells of $\lambda(\omega)$ coincides with the number of inversions of ω, that is, pairs $i < j$ such that $\omega_i > \omega_j$, denoted $d(\omega)$.

By a result in [3], the number of DHD-puzzle pieces in a DHD-puzzle with boundary $(u, v; w)$ is given by $d(w) - d(u) - d(v)$. For puzzles where $d(w) - d(u) - d(v) = 0$ it was shown in [4] that their number is given by the Littlewood-Richardson coefficient $c_{\lambda(u),\lambda(v)}^{\lambda(w)}$ (abbreviated $c_{u,v}^w$). Furthermore, for words u, v and w with $d(w) - d(u) - d(v) = 1$, an expression for $\mathrm{dhd}_{u,v}^w$ in terms of Littlewood-Richardson coefficients was proved in [3]. The main contribution of this article will be an expression for $\mathrm{dhd}_{u,v}^w$, where u, v and w are arbitrary 01-words of length N, in terms of Littlewood-Richardson coefficients, thus generalising previously established results. The coefficients of $\mathrm{dhd}_{u,v}^w$ are products of numbers of the following kind:

Definition 1.1 *Let λ and λ^+ be two Young diagrams such that λ^+ contains λ.*

1. *The set of row- and column-strict Young tableaux of skew shape λ^+/λ with entries in the j-th column – when counted from the left – restricted to $1, 2, \ldots, j$ for all j is denoted by G_{λ,λ^+}; its cardinality by g_{λ,λ^+}.*

2. *The set of Young tableaux of skew shape λ^+/λ where entries along rows and columns are non-increasing and in the j-th column are restricted to $1, 2, \dots, j$ for all j will be denoted by F_{λ,λ^+}; its cardinality by f_{λ,λ^+}.*

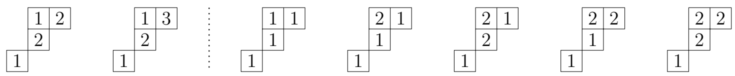

Fig. 2: Left: all tableaux of $G_{\square,\boxplus}$; right: all tableaux of $F_{\square,\boxplus}$.

Examples for both sets are given in Figure 2. To facilitate the formulation of the main result $g_{\lambda(u),\lambda(u^+)}$ is abbreviated with g_{u,u^+} and $f_{\lambda(w^-),\lambda(w)}$ with $f_{w^-,w}$.

Theorem 1.2 *Let u, v and w be 01-words of length N such that $|u|_1 = |v|_1 = |w|_1$. Then*

$$\mathrm{dhd}_{u,v}^{w} = \sum_{\substack{u^+,v^+,w^- \text{ words of length } N:\\ |u^+|_1=|v^+|_1=|w^-|_1=|u|_1}} (-1)^{d(w)-d(w^-)} g_{u,u^+} g_{v,v^+} f_{w^-,w} c_{u^+,v^+}^{w^-}. \tag{1.1}$$

Observe that Theorem 1.2 yields $\mathrm{dhd}_{u,v}^{w} = c_{u,v}^{w}$ for words u, v and w satisfying $d(u) + d(v) = d(w)$. Furthermore, for words u, v and w with $d(w) - d(u) - d(v) = 1$ the expression (1.1) and the expression proved in [3] coincide. Next, $\mathrm{dhd}_{001,001}^{100}$ will be computed using (1.1). It is easy to check that $\mathrm{dhd}_{001,001}^{100} = 0$.

Example 1.3 *For $u = 001$, $v = 001$ and $w = 100$ the sum in (1.1) runs through the triple of words $(100, 001; 100)$, $(001, 100; 100)$, $(001, 001; 001)$, $(010, 010; 100)$, $(010, 001; 010)$ and $(001, 010; 010)$. Here, the respective Littlewood-Richardson coefficients are all 1. Furthermore, $g_{001,001} = 1$, $g_{001,010} = 1$, $g_{001,100} = 0$, $f_{001,001} = 1$, $f_{001,010} = 1$ and $f_{001,100} = 1$. In summary:*

$$\begin{aligned}\mathrm{dhd}_{001,001}^{100} &= (-1)^0 \cdot 0 + (-1)^0 \cdot 0 + (-1)^2 \cdot 1 + (-1)^0 \cdot 1 + (-1)^1 \cdot 1 + (-1)^1 \cdot 1\\ &= 0\end{aligned}$$

The expression in (1.1) is inspired by an expression in terms of Littlewood-Richardson coefficients for the number of puzzles with boundary $(u, v; w)$ that are composed of ordinary puzzle pieces and flipped DHD-puzzle pieces (the so-called K-theory pieces). This expression derives from K-theoretical results in [1], [6] and [8] and is of the same form as (1.1) but with slightly different g_{λ,λ^+}'s and f_{λ,λ^+}'s. The coefficients in this expression are numbers of row- and column-strict Young tableaux of skew shape λ^+/λ with entries in the i-th row restricted to $1, 2, \dots, i-1$ for all i and numbers of semi-standard Young tableaux of skew shape λ^+/λ with entries in the i-th row restricted to $1, 2, \dots, i-1$ for all i. That said, the proof of Theorem 1.2 is independent of the mentioned K-theoretical results. In fact, it is bijective. Furthermore, the ideas presented in this work can presumably be transferred to puzzles that are composed of ordinary puzzle pieces and K-theory pieces, and facilitate a bijective proof of the expression for their number in terms of Littlewood-Richardson coefficients.

Finally, the connection between DHD-puzzles and oriented TFPLs will be explicated. This connection emanates from previous work on oriented TFPLs. In [7], it was proved that for 01-words u, v and w with $d(u) + d(v) = d(w)$, puzzles with boundary $(u, v; w)$ that are composed solely of ordinary puzzle pieces correspond to oriented TFPLs with boundary $(u, v; w)$. Later, in [3] new puzzle pieces were developed (including amongst others the DHD-puzzle piece) and it was proved that for words u, v and w with $d(u) + d(v) + 1 = d(w)$, oriented TFPLs with boundary $(u, v; w)$ correspond to puzzles with boundary $(u, v; w)$ that are composed of ordinary puzzle pieces together with one of the new puzzle pieces. Both correspondences were proved bijectively. The specific bijection for DHD-puzzles yields the following correspondence:

Proposition 1.4 ([7, 3]) *Let u, v and w be 01-words of the same length. Then DHD-puzzles with boundary $(u, v; w)$ are in one-to-one correspondence with oriented TFPLs with boundary $(u, v; w)$ that do not contain one of the following configurations:*

The correspondence in Proposition 1.4 is indicated by two examples in Figure 3. Together with Theorem 1.2, it gives rise to an enumeration result for a large class of oriented TFPLs. For this reason, Theorem 1.2 marks a significant increase in the understanding of the enumeration of oriented TFPLs, which is of interest because it facilitates enumeration results for ordinary TFPLs by a result in [3]. On the other hand, by way of [2] enumeration formulas for ordinary TFPL have potential applications in the study of *fully packed loop configurations*.

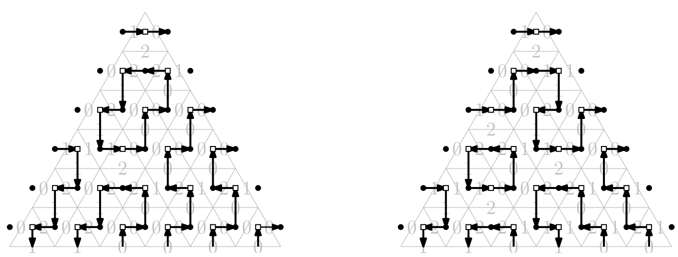

Fig. 3: The two puzzles of Figure 1 and their corresponding oriented TFPLs.

2 Preliminaries

In order to prove Theorem 1.2 we will embed DHD-puzzles into a more general class of puzzles that are composed of ordinary puzzle pieces and the following puzzle piece types: a **defect of type BD** (*resp.* **RD** *resp.* **g**) of length n, defined to be an n-tuple of unit $/$-edges (*resp.* $\backslash$-edges *resp.* horizontal edges) $e_1, \ldots, e_n$ such that: the top (*resp.* bottom *resp.* right) vertex of e_i coincides with the bottom (*resp.* top *resp.* left) vertex of e_{i+1} for all $i = 1, \ldots, n-1$; e_i is labelled with $\sigma_i, \tau_i \in \{0, 1\}$ as follows:

- ${}^{\sigma_i}\!/_{\tau_i}$ (*resp.* ${}_{\tau_i}\backslash^{\sigma_i}$ *resp.* $\frac{\sigma_i}{\tau_i}$) for all $i = 1, \ldots, n$;
- $\sigma = \sigma_1 \cdots \sigma_n < \tau_1 \cdots \tau_n = \tau$;
- $\sigma_1 = \tau_n = 0$, $\sigma_n = \tau_1 = 1$ and $|\sigma|_1 = |\tau|_1$.

Defects of length 2 have already occured in [4] and [3]. Their names were established in the latter article.

Definition 2.1 *A **puzzle** of size N is a decomposition of $\mathcal{T}_N$ into unit triangles with edges labelled* 0*,* 1 *or* 2 *satisfying the following:*

1. *there can be defects of type BD, RD or g of length at least* 2*;*
2. *the unit triangles are ordinary puzzle pieces;*
3. *whenever two unit triangles are adjacent, their common edge has the same label in both triangles;*
4. *whenever an edge of a unit triangle is part of a defect, this edge has to have the same label in the triangle and on the side of the defect that faces the triangle;*
5. *no edge on the boundary of $\mathcal{T}_N$ has label* 2*.*

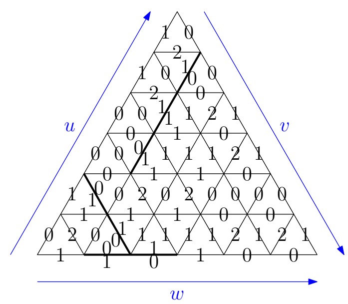

Fig. 4: A puzzle with boundary $(010011, 001101; 110100)$.

In [4] and [3], puzzles with one defect of length 2 have already appeared, while puzzles with defects of length larger than 2 are a contribution of this work. Next, it will be shown how DHD-puzzles can be converted into puzzles. To this end, note that each DHD-puzzle piece in a DHD-puzzle must necessarily be part of the configuration depicted in Figure 5. This configuration can be transformed into a configuration made up solely of ordinary puzzle pieces and a defect of type BD.

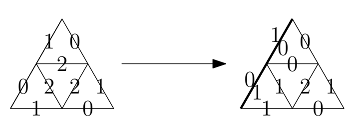

Fig. 5: The embedding of DHD-puzzles into the set of puzzles.

Proposition 2.2 *The set of DHD-puzzles with boundary $(u, v; w)$ corresponds to the set of puzzles with boundary $(u, v; w)$ that satisfy the following: there is no defect of type RD and g; each of the defects of type BD are part of the configuration that is depicted in Figure 5.*

Recall that puzzles with boundary $(u,v;w)$ that are solely composed of ordinary puzzle pieces are enumerated by the Littlewood-Richardson coefficient $c_{u,v}^{w}$. Therefore the right side in (1.1) is a signed enumeration of quadruples consisting of a $U \in G_{\lambda(u),\lambda(u^+)}$, a $V \in G_{\lambda(v),\lambda(v^+)}$, a $W \in F_{\lambda(w^-),\lambda(w)}$ and a puzzle P with boundary $(u^+,v^+;w^-)$ that is composed solely of ordinary puzzle pieces. The fundamental idea for the proof of Theorem 1.2 is to associate each of these quadruples with a puzzle $\mathrm{Q}(P;U,V,W)$ with boundary $(u,v;w)$. In doing so, we will obtain each puzzle Q with boundary $(u,v;w)$ that corresponds to a DHD-puzzle. Furthermore, the quadruples from which this Q arises will be one more in number when $d(w)-d(w^-)$ is even than when it is odd. Besides the puzzles that correspond to DHD-puzzles we will generate other puzzles too. Each such puzzle will arise equally often from quadruples for which $d(w)-d(w^-)$ is even as from quadruples for which $d(w)-d(w^-)$ is odd. Thus, the right side in (1.1) will give the number of puzzles with boundary $(u,v;w)$ that correspond to a DHD-puzzle, thereby establishing Theorem 1.2.

For the sake of convenience, the procedure by which a puzzle is assigned to a quadruple $(P;U,V,W)$ will be split into two steps. In the first step, a puzzle $\tilde{\mathrm{Q}}(P;V,W)$ is generated from the triple $(P;V,W)$. Thereafter, in the second step, $\tilde{Q}(P;U,V,W)$ is generated from U and $\tilde{\mathrm{Q}}(P;V,W)$. Each of these two steps will be treated in a separate section.

3 The map $\tilde{\mathrm{Q}}$

Throughout this section, let $v^+ \geq v$, $w^- \leq w$ and u^+ be 01-words such that $d(u^+)+d(v^+)=d(w^-)$. Furthermore, let P be a puzzle with boundary $(u^+,v^+;w^-)$ that solely consists of ordinary puzzle pieces, $V \in G_{v,v^+}$ and $W \in F_{w^-,w}$. First, we will order the cells of V and W linearly. To this end, we make the following definition:

Definition 3.1 *Let T be a Young tableau of skew shape λ^+/λ where the entries in the j-th column are restricted to $1,2,\ldots,j$ for all j, c a cell of λ^+/λ and x its entry. The* ***discrepancy*** *of c is defined as* $\operatorname{dis}(c)=j-x$.

Note that $\operatorname{dis}(c)=\operatorname{dis}(c')$ for distinct cells c and c' in V implies that c and c' are not part of the same column if they are both in V. If c and c' are both cells in W, on the other hand, from $\operatorname{dis}(c)=\operatorname{dis}(c')$ it follows that c and c' are not part of the same row.

Definition 3.2 *Let c and c' be cells in V or W, i and i' their rows, j and j' their columns and x and x' their entries. Then set $c <_{V,W} c'$ if one of the following is satisfied: (1) $\operatorname{dis}(c) < \operatorname{dis}(c')$; (2) $\operatorname{dis}(c)=\operatorname{dis}(c')$, c and c' are both in V and $j>j'$; (3) $\operatorname{dis}(c)=\operatorname{dis}(c')$, c and c' are both in W and $i<i'$; (4) $\operatorname{dis}(c)=\operatorname{dis}(c')$, c is a cell in V and c' is a cell in W.*

It is easy to check that $<_{V,W}$ is indeed a linear order. From now on, denote the cells of V and W as follows:

$$c_1 <_{V,W} c_2 <_{V,W} \cdots <_{V,W} c_{d(v^+)-d(v)+d(w)-d(w^-)}.$$

Let $s<s'$. If c_s and $c_{s'}$ are part of the same row in V, c_s has to lie to the right of $c_{s'}$, whereas c_s has to lie below $c_{s'}$, if they are both part of the same column of V. On the other hand, if c_s and $c_{s'}$ are part of the same row in W, then c_s lies to the left of $c_{s'}$, while c_s lies above $c_{s'}$, if they are part of the same column in W. By the previous observation, when the cells $c_1,\ldots,c_s$ are removed from V and W, the remaining cells give rise to a Young tableau in $G_{v,v^{(s)}}$ for a $v \leq v^{(s)} \leq v^+$ and one in $F_{w^{(s)},w}$ for a $w^- \leq w^{(s)} \leq w$ for all s. In the following, we choose $v^{(s)}$ and $w^{(s)}$ such that they are both of length

N, $|v^{(s)}|_1 = |v|_1$ and $|w^{(s)}|_1 = |w|_1$, where $|\cdot|_1$ denotes the number of occurences of 1 in a word. For the sake of convenience, we set $v^{(0)} = v^+$ and $w^{(0)} = w^-$. Now, if the cell c_s is in $\lambda(v^+)/\lambda(v)$ then there exists an $m_s \in \{1, 2, \dots, N\}$ such that $v^{(s)}_{m_s-1} = 0$, $v^{(s)}_{m_s} = 1$, $v^{(s-1)}_{m_s-1} = 1$, $v^{(s-1)}_{m_s} = 0$ and $v^{(s)}_m = v^{(s-1)}_m$ for all $m \neq m_s - 1, m_s$. On the other hand, if c_s is in $\lambda(w)/\lambda(w^-)$ then there exists an $n_s \in \{1, 2, \dots, N\}$ such that $w^{(s)}_{n_s-1} = 1$, $w^{(s)}_{n_s} = 0$, $w^{(s-1)}_{n_s-1} = 0$, $w^{(s-1)}_{n_s} = 1$ and $w^{(s)}_n = w^{(s-1)}_n$ for all $n \neq n_s - 1, n_s$.

Definition 3.3 *Define the* ***diagonal*** *of* c_s *as* $\mathrm{diag}(c_s) = m_s$*, if* $c_s \in \lambda(v^+)/\lambda(v)$*, or as* $\mathrm{diag}(c_s) = n_s$*, if* $c_s \in \lambda(w)/\lambda(w^-)$*.*

In the course of the application of the map $\tilde{\mathrm{Q}}$ the cells of V and W will be treated one after the other in increasing order. Thereby, for each cell c_s a defect d_s of length 2 will be inserted on the boundary of the puzzle P and will be moved a certain number of times by the moves in Figure 6 and Figure 7. The moves in Figure 6 were developed in [3]. A defect d' will be permitted to share an edge e with another defect d once it has been moved for the last time. If this is the case, d and d' are merged and the label on the right side of e, together with the label on the left side of the edge of d' that coincides with e, are deleted.

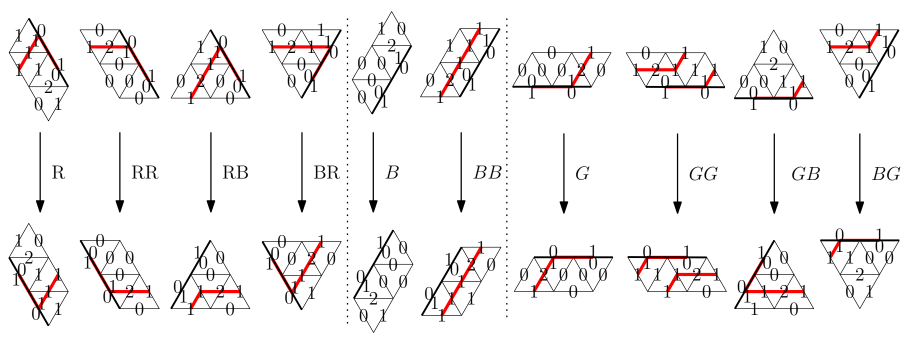

Fig. 6: The moves by which the inserted defects are moved.

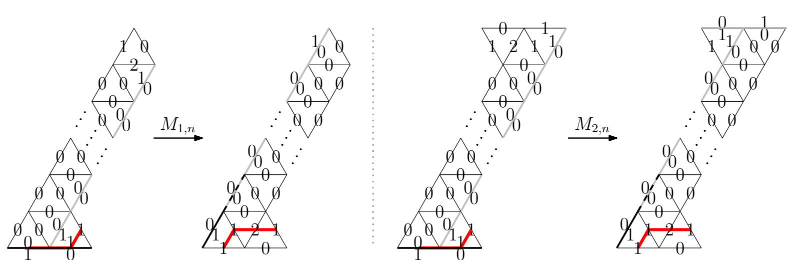

Fig. 7: The two supplementary moves $M_{1,n}$ and $M_{2,n}$ for $n \geq 2$, where n denotes the length of the defect of type BD in the preimage.

Definition 3.4 *The puzzle* $\tilde{\mathrm{Q}}(P; S, T)$ *is determined as follows: run through the cells of* V *and* W *in increasing order, that is, let* $s = 1, 2, \dots, d(v^+) - d(v) + d(w) - d(w^-)$*.*

1. *If c_s is a cell in $\lambda(v^+)/\lambda(v)$ with entry x_s insert a defect of type RD on the right boundary whose centre lies on the $\mathrm{diag}(c_s)$-th /-diagonal of $\mathcal{T}_N$. Then move it by the moves B, BB, BR, R, RR, RB (see Figure 6) until a move in $\{BB, RB\}$ is applied for the x_s-th time.*

2. *If c_s is a cell in $\lambda(w)/\lambda(w^-)$ with entry x_s and column j_s then insert a defect of type g on the bottom boundary whose centre lies on the $\mathrm{diag}(c_s)$-th /-diagonal of $\mathcal{T}_N$. Then move it by the moves B, BB, BG, G, GG, GB until one of the following two conditions is satisfied: (1) a move in $\{BB, GG, GB\}$ has been applied for the x_s-th time to d_s; (2) $x_s - 1$ many moves in $\{BB, GG, GB\}$ and either $M_{1,n}$ or $M_{2,n}$ for $n \geq 2$ (see Figure 7) have been applied to d_s.*

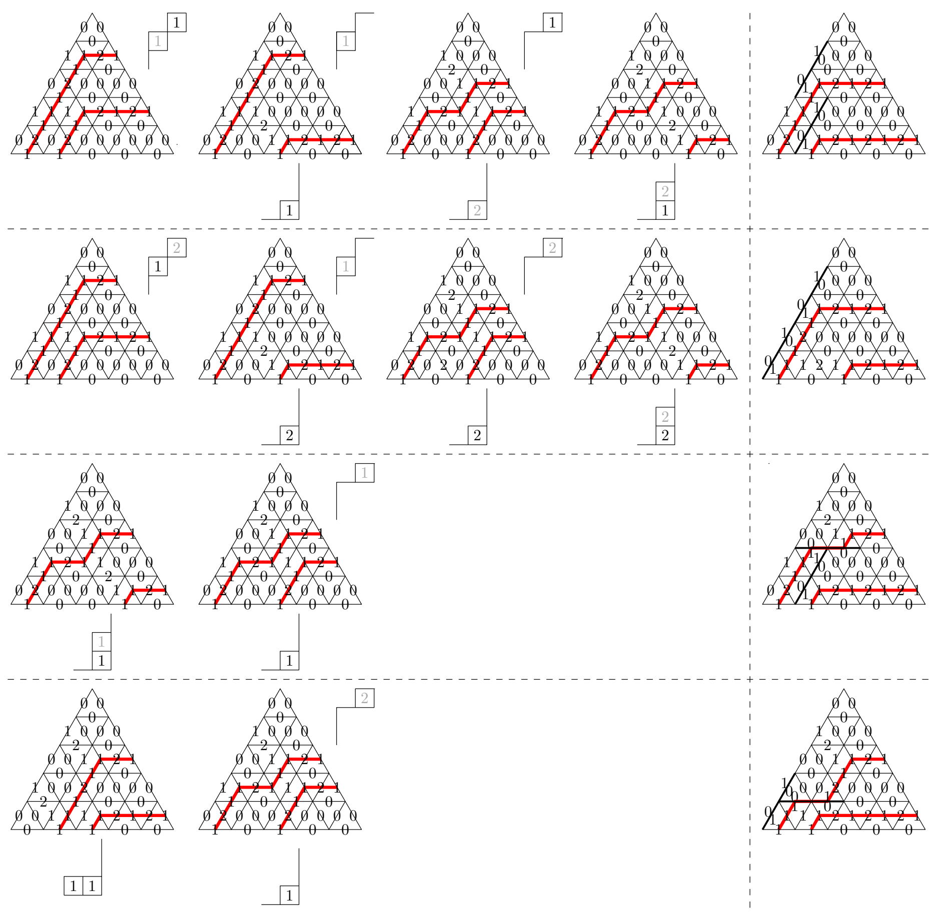

Fig. 8: For all $v^+ \geq 00101$ and $w^- \leq 11000$ with $d(01010) + d(v^+) = d(w^-)$, the triples made up of a puzzle with boundary $(u^+, v^+; w^-)$, a tableau in G_{00101,v^+} and a tableau in $F_{w^-,11000}$ (left) and their images under $\tilde{\mathrm{Q}}$ (right). The cell that is treated first by $\tilde{\mathrm{Q}}$ is indicated in grey.

Examples for $\tilde{\mathrm{Q}}$ are given in Figure 8. To prove that each inserted defect can be moved the required number of times the puzzles are equipped with tuples of pairwise non-crossing *red paths*, as is indicated in Figure 8. While being moved a defect may overcome several red paths. If a defect is inserted on the right boundary, then while being moved it can overcome the red path r it is part of when it is inserted and the red paths that intersect the right boundary above r. On the other hand, if a defect is inserted on the bottom boundary, then it may overcome the red path r it is part of when it is inserted and the red paths that intersect the bottom boundary to the left of the red path r. Thus, in both cases the number of red paths a defect can overcome at most is given by the index of the column of the cell that corresponds to the defect. From Figure 6 it can be seen that solely by the moves RB, BB, GG and GB a red path is overcome. Furthermore, a defect may never be moved below a red path that it has already overcome. Thus, the number of times a move RB, BB, GG or GB is applied to a defect coincides with the number of red paths it has overcome while being moved. On the other hand, the order $<_{V,W}$ guarantees that no defect is barred from being moved by a previously added defect. Proof of this fact is omitted here in the interest of saving space.

It can be seen from Figure 6 that puzzles $\tilde{\mathrm{Q}}(P;V,W)$, where each defect was last moved by RB or GB, are amongst the puzzles that correspond to DHD-puzzles. Using the red paths one can show that each puzzle that corresponds to a DHD-puzzle is the image of a triple $(P;V,W)$ under $\tilde{\mathrm{Q}}$.

4 The map Q

Throughout this section, let $u^+ \geq u$, $v^+ \geq v$ and $w^- \leq w$ be words such that $d(u^+) + d(v^+) = d(w^-)$ and let P be a puzzle with boundary $(u^+, v^+; w^-)$. To begin with, we will order the cells of U in $G_{\lambda(u),\lambda(u^+)}$ linearly and this linear order will be denoted by $<_U$. Let c and c' be two cells in U, j and j' their columns, i and i' their rows and x and x' their entries. Then $c <_U c'$ if one of the two following conditions is satisfied: $\mathrm{dis}(c) < \mathrm{dis}(c')$; $\mathrm{dis}(c) = \mathrm{dis}(c')$ and $j > j'$. From now on, write

$$c_1 <_U c_2 <_U \cdots <_U c_{d(u^+)-d(u)}$$

for the cells of U.

By an analogous argument to that found in the previous section, there exist unique words $u^{(s)}$ with $|u^{(s)}|_1 = |u|_1$ of length N for which $\lambda(u^{(s)})/\lambda(u)$ is the skew diagram obtained from $\lambda(u^+)/\lambda(u)$ by deleting the cells $c_1, c_2, \ldots, c_s$ for $s = 1, 2, \ldots, d(u^+) - d(u)$. For the sake of convenience we set $u^{(0)} = u^+$. Now for each s there exists an $m_s \in \{1, 2, \ldots, N\}$ such that $u^{(s)}_{m_s-1} = 0$, $u^{(s)}_{m_s} = 1$, $u^{(s-1)}_{m_s-1} = 1$, $v^{(s-1)}_{m_s} = 0$ and $u^{(s)}_m = u^{(s-1)}_m$ for all $m \neq m_s - 1, m_s$.

Definition 4.1 *Keep the notations from above. Then the* ***diagonal*** *of c_s is defined as $\mathrm{diag}(c_s) = m_s$ for each $s = 1, 2, \ldots, d(u^+) - d(u)$.*

The map Q will be implemented in two steps: in the first step, the puzzle $\tilde{\mathrm{Q}}(P;V,W)$ will be generated. Thereafter, defects will be added on the left boundary of $\tilde{\mathrm{Q}}(P;V,W)$ and will be moved by $G^{-1}, GG^{-1}, GB^{-1}, B^{-1}, BB^{-1}, BG^{-1}$ and $M^{-1}_{1,n}$ or $M^{-1}_{2,n}$ until the point at which a move in $\mathcal{M}_L := \{GG^{-1}, GB^{-1}\} \cup \{M^{-1}_{1,n}, M^{-1}_{2,n}\}_{n\geq 2}$ will be applied to it for the x-th time, where x is the entry of the corresponding cell. Other than when generating $\tilde{\mathrm{Q}}(P;V,W)$, defects here can be barred from being moved the required number of times by other defects. If this is the case, we will move another defect

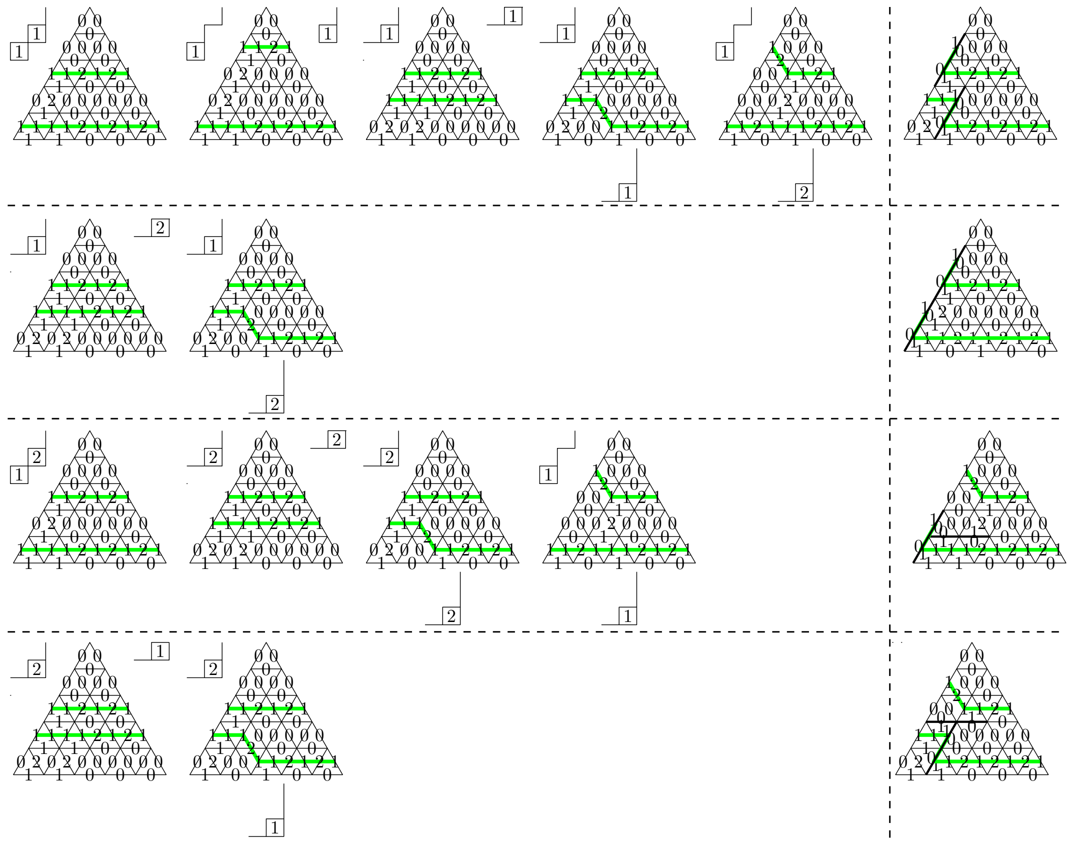

Fig. 9: For all $u^+ > 01010$, $v^+ \geq 00101$ and $w^- \leq 11000$ with $d(u^+) + d(v^+) = d(w^-)$, the triples made up of a puzzle with boundary $(u^+, v^+; w^-)$, a tableau in G_{01010,u^+}, a tableau in G_{00101,v^+} and a tableau in $F_{w^-,11000}$ (left) and their images under Q (right).

instead of the barred one. In addition, we will follow supplementary rules for each defect configurations. In the interest of clarity, only two of these supplementary rules are explicated.

The first rule deals with the defect configuration where a defect d of type g and of length 2 is barred by a defect d′ of type BD in the way that the top vertex of d′ coincides with the central vertex of d. An example of this configuration can be found in the fourth row in Figure 9. In that case, to d and d′ the move $M_{2,n}$ is applied, if by the time d is barred by d′ it is supposed to be moved until the point at which a move in $\mathcal{M}_L$ will be applied to it for the k-th time for a $k > 1$. Thereafter, the defect of type g in the image of the move $M_{2,n}$ shall be moved instead of d until the point at which a move in $\mathcal{M}_L$ will be applied to it for the $k-1$-st time. On the other hand, if $k = 1$, d is stopped being moved in the position in which it is barred by d′ and the next defect is inserted.

The second rule treats the defect configuration in which a defect d of type BD and of length 2 is barred by a defect d′ of type g and of length n that has labels $\tau_i = \sigma_i = 1$ for all $1 < i < n$. An example of this configuration can be seen in the third row in Figure 9. Suppose that by the time d is barred by d′ it shall be moved until the point at which a move in $\mathcal{M}_L$ will be applied to it for the k-th time for a $k > 1$. Then demerge a defect of length 2 on the right sight of d' and move it until the point at which a move in $\mathcal{M}_L$ will be applied to it for the k-th time. Thereafter, one after the other demerge defects of length 2 from d' and move them until the point at which a move in $\mathcal{M}_L$ will be applied to it for the second time. Eventually, d is no longer barred from being moved. At that time, d shall be moved until the point at which a move in $\mathcal{M}_L$ will be applied to it for the second time. On the other hand, if $k = 1$ the defect d is stopped being moved in the position where it is barred by d' and the next defect is inserted.

A specific aim of the previous two rules is the following: it avoids that a defect in a puzzle Q that is the image of two different quadruples (P, U, V, W) and (P', U', V', W') under the map Q has been inserted on the left boundary or has been moved last instead of a defect that has been inserted on the left boundary in the generation of both $\mathrm{Q}(P; U, V, W)$ and $\mathrm{Q}(P'; U', V', W')$.

Definition 4.2 *Keep the notation from above. The puzzle* $\mathrm{Q}(P; U, V, W)$ *is determined as follows: generate* $\tilde{\mathrm{Q}}(P; V, W)$. *Then run through the cells of* U *in increasing order, that is, for each* $s = 1, 2, \ldots, d(u^+) - d(u)$*:*

1. *insert a defect* d_s *on the left boundary of* $\tilde{\mathrm{Q}}(P; V, W)$ *with its center on the* $\mathrm{diag}(c_s)$*-th* $\backslash$*-diagonal of* $\mathcal{T}_N$*;*
2. *apply moves in* $\{G^{-1}, GG^{-1}, GB^{-1}, B^{-1}, BB^{-1}, BG^{-1}\}$ *or in* $\{M_{1,n}^{-1}, M_{2,n}^{-1}\}_{n\geq 2}$ *to* d_s *until a move in* $\mathcal{M}_L$ *would be applied to* d_s *for the* x_s*-th time; in doing so, whenever a defect is barred proceed according to the rules made for the respective defect configuration.*

Examples of quadruples and their images under Q can be seen in Figure 9. The proof that each inserted defect or a defect that is moved instead of it can be moved the required number of times uses pairwise non-crossing *green paths*, as is indicated in Figure 9. While being moved a defect can overcome the green path g it is part of when it is inserted and the green paths that intersect the left boundary above g. Thus, the number of green paths a defect can overcome at most is given by the index of the column of the cell that corresponds to the defect. It is easy to check that the sole moves by which a defect can overcome a green path are the moves in $\mathcal{M}_L$. Furthermore, a defect may never be moved above a green path that it has already overcome. However, the rules according to which it is proceeded when a defect is barred from being moved have to be checked separately. This will be omitted here.

In Figure 8 and Figure 9, for all $u^+ \geq 01010$, $v^+ \geq 00101$ and $w^- \leq 11000$ with $d(u+1) + d(v^+) = d(w^-)$ all quadruples $(P; U, V, W)$ consisting of a puzzle with boundary $(u^+, v^+; w^-)$, a tableau in G_{01010,u^+}, a tableau in G_{00101,v^+} and a tableau in $F_{w^-,11000}$ (left) and their images under Q (right) are given. From the figures it can be seen that for each puzzle Q on the right side that does not correspond to a DHD-puzzle there are equally many quadruples where $d(w) - d(w^-)$ is even as where $d(w) - d(w^-)$ is odd that are mapped to Q. On the other hand, there are five quadruples where $d(w) - d(w^-)$ is even and four where $d(w) - d(w^-)$ is odd that are mapped to the sole puzzle with boundary $(01010, 00101; 11000)$ that corresponds to a DHD-puzzle. The proof of Theorem 1.2 is omitted here. Its basic idea is to determine all possible quadruples from which a puzzle Q with boundary $(u, v; w)$ is generated by the map Q.

The author plans to upload a full version of this work on arxiv.

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