

The topology of the external activity complex of a matroid

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Abstract. We prove that the external activity complex $\text{Act}_{<}(M)$ of a matroid is shellable. In fact, we show that every linear extension of Las Vergnas’s external/internal order $<_{ext/int}$ on M provides a shelling of $\text{Act}_{<}(M)$. We also show that every linear extension of Las Vergnas’s internal order $<_{int}$ on M provides a shelling of the independence complex $IN(M)$. As a corollary, $\text{Act}_{<}(M)$ and M have the same h -vector. We prove that, after removing its cone points, the external activity complex is contractible if M contains $U_{3,1}$ as a minor, and a sphere otherwise.

Résumé. Nous prouvons que le complexe d’activité externe $\text{Act}_{<}(M)$ d’un matroïde est épluchable. En fait, nous montrons que toute extension linéaire de l’ordre externe/interne de Las Vergnas $<_{ext/int}$ sur M fournit un épluchage de $\text{Act}_{<}(M)$. Nous montrons aussi que toute extension linéaire de l’ordre interne de Las Vergnas $<_{int}$ sur M fournit un épluchage du complexe d’indépendance $IN(M)$. En conséquence, $\text{Act}_{<}(M)$ et M ont le même h -vecteur. Nous prouvons que, après suppression des points cones, le complexe d’activité externe est contractible si M contient $U_{3,1}$ comme mineur, et est une sphère sinon.

Keywords. matroids, simplicial complexes, shellability, external/internal order

1 Introduction

Matroid theory is a combinatorial theory of independence which has its roots in linear algebra and graph theory, but which turns out to have deep connections with many fields. There are natural notions of independence in linear algebra, graph theory, matching theory, the theory of field extensions, and the theory of routings, among others. Matroids capture the combinatorial essence that those notions share.

A matroid can be described in many equivalent ways, arising from the many contexts where matroids are found: the bases, the circuits, the lattice of flats, and the matroid polytope, among others. One important approach, which is the most relevant one to this paper, has been to model a matroid in terms of a simplicial or polyhedral complex.

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1.1 Motivation for this work.

The *external activity complex* $\text{Act}_{<}(M)$ of a matroid is a simplicial complex associated to a matroid M and a linear order $<$ on its ground set. This complex arose in work of the first author with Adam Boocher [1]. They started with a linear subspace L of affine space \mathbb{A}^n with a chosen system of coordinates. There is a natural embedding $\mathbb{A}^n \hookrightarrow (\mathbb{P}^1)^n$ into a product of projective lines, and they considered the closure \tilde{L} of L in $(\mathbb{P}^1)^n$. They proved that many geometric and algebraic invariants of the variety \tilde{L} are determined by the matroid of L .

As is common in combinatorial commutative algebra, a key ingredient of [1] was to consider the initial ideals in $_{<}\tilde{L}$ under various term orders. These initial ideals are the Stanley-Reisner ideals of the external activity complexes $\text{Act}_{<}(M)$ under the different linear orders $<$ of the ground set. This led them to consider and describe the complexes $\text{Act}_{<}(M)$.

The ideals in $_{<}\tilde{L}$ are shown to be Cohen-Macaulay in [1], and the authors asked the stronger question: Are the external activity complexes $\text{Act}_{<}(M)$ shellable? We prove they are, but furthermore, along the way we prove other results that we now describe.

1.2 Our results.

The facets of $\text{Act}_{<}(M)$ are indexed by the bases \mathcal{B} of M , and [1] suggested a possible connection between $\text{Act}_{<}(M)$ and LasVergnas's *internal order* $<_{int}$ on \mathcal{B} . Surprisingly, we find that it is the *external/internal order* $<_{ext/int}$ on \mathcal{B} , also defined in [6], which plays a key role. Our main result is the following:

Theorem 1.1 *Let $M = (E, \mathcal{B})$ be a matroid, and let $<$ be a linear order on the ground set E . Any linear extension of LasVergnas's external/internal order $<_{ext/int}$ of \mathcal{B} induces a shelling of the external activity complex $\text{Act}_{<}(M)$.*

As a corollary we obtain that these orders also shell the independence complex $IN(M)$, and in fact we show a stronger statement.

Theorem 1.2 *Any linear extension of the internal order $<_{int}$ gives a shelling order of the independence complex $IN(M)$.*

These theorems are as strong as possible in the context of LasVergnas's active orders. We also obtain the following enumerative corollary.

Theorem 1.3 *The h -vector of $\text{Act}_{<}(M)$ equals the h -vector of M .*

It is easy to see that $\text{Act}_{<}(M)$ is a cone, and hence trivially contractible. It is more interesting to study the *reduced external activity complex* $\text{Act}_{<}^{\bullet}(M)$, obtained by removing all the cone points of $\text{Act}_{<}(M)$. Our main topological result is the following.

Theorem 1.4 *Let M be a matroid and $<$ be a linear order on its ground set. The reduced external activity complex $\text{Act}_{<}^{\bullet}(M)$ is contractible if M contains $U_{3,1}$ as a minor, and a sphere otherwise.*

In the present abstract we explain these statements. In the next section we introduce all necessary terminology and in the last section we illustrate all the above theorems in an extended example.

2 Background

In this section we collect the background information on matroids and shellability.

2.1 Matroids

Basic definitions. A *simplicial complex* $\Delta = (E, \mathcal{I})$ is a pair where E is a finite set and \mathcal{I} is a non empty family of subsets of E , such that if $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$. Elements of \mathcal{I} are called *faces* of the complex. The maximal elements of \mathcal{I} are called *facets*. A complex is said to be *pure* if all facets have the same number of elements.

The following is one of many equivalent ways of defining a matroid:

Definition 2.1 A matroid $M = (E, \mathcal{I})$ is a simplicial complex such that the restriction of M to any subset of E is pure.

Since there are several simplicial complexes associated to M , we will denote this one $IN(M) = (E, \mathcal{I})$. It is often called the *independence complex* of M .

The two most important motivating examples of matroids are the following.

- (Linear Algebra) Let E be a set of vectors in a vector space, and let \mathcal{I} consist of the subsets of E which are linearly independent. Then (E, \mathcal{I}) is a *linear matroid*.
- (Graph Theory) Let E be the set of edges of an undirected graph G , and let \mathcal{I} consist of the sets of edges which contain no cycle. Then (E, \mathcal{I}) is a *graphic matroid*.

For any matroid $M = (E, \mathcal{I})$, it is customary to call the sets in \mathcal{I} *independent*. The facets of a matroid are called *bases*. The set of all bases is denoted \mathcal{B} .

Example 2.2 The simplest example of a matroid is the uniform matroid $U_{k,n}$, whose ground set is $[n]$ and whose independent sets are all the subsets of $[n]$ of cardinality at most k . The uniform matroid $U_{1,3}$ is going to play an important role later.

The minimal non-faces of M , that is, the minimal dependent sets, are called *circuits*. The circuits of a matroid have a special structure [7]:

Lemma 2.3 (Circuit Elimination Property) If γ_1 and γ_2 are circuits of a matroid and $c \in \gamma_1 \cap \gamma_2$, then there is a circuit γ_3 that is contained in $\gamma_1 \cup \gamma_2 - c$.

Matroids have a notion of duality which generalizes orthogonal complements in linear algebra and dual graphs in graph theory.

Let M be a matroid with bases \mathcal{B} . Then the set

$$\mathcal{B}^* = \{E - B : B \text{ is a basis of } M\}$$

is the collection of bases of a matroid $M^* = (E, \mathcal{B}^*)$, called the *dual matroid* M^* . The circuits of the dual matroid M^* are called the *cocircuits* of M .

Definition 2.4 We say that an element $e \in E$ is a *loop* of a matroid M if it is contained in no basis; that is, if $\{e\}$ is a dependent set. Dually, e is a *coloop* if it is contained in every basis of M .

Definition 2.5 The deletion $M \setminus e$ of a non-coloop $e \in E$ is the matroid on $E - e$ whose bases are the bases of M that do not contain e . We also call this the restriction of M to $E - e$. Dually, the contraction M / e of a non-loop $e \in E$ is the matroid on $E - e$ whose bases are the subsets B of $E - e$ such that $B \cup e$ is a basis of M .

It is easy to see that any sequence of deletions and contractions of different elements commutes. We say that a matroid M' is a *minor* of a matroid M if M' is isomorphic to a matroid obtained from M by performing a sequence of deletions and contractions.

Definition 2.6 Given a basis B and an element $e \in E - B$ there is a unique circuit contained in $B \cup e$, called the *fundamental circuit* of e with respect to B . It is given by

$$\text{Circ}(B, e) = \{x \in E : B \cup e - x \in \mathcal{B}\}.$$

Given a basis B and an element $i \in B$ there is a unique cocircuit disjoint with $B - i$, called the *fundamental cocircuit* of i with respect to B . It is given by

$$\text{Cocirc}(B, i) = \{x \in E : B \cup x - i \in \mathcal{B}\}.$$

Note that the cocircuit $\text{Cocirc}(B, i)$ in M equals the circuit $\text{Circ}(E - B, i)$ in the dual M^* .

2.2 Basis activities.

Let $<$ be a linear order on the ground set E . For a basis B , define the sets:

$$\begin{aligned} EA(B) &= \{e \in E - B : \min(\text{Circ}(B, e)) = e\} \\ EP(B) &= \{e \in E - B : \min(\text{Circ}(B, e)) \neq e\} \end{aligned}$$

The elements of $EA(B)$ and $EP(B)$ are called *externally active* and *externally passive* with respect to B , respectively. Note that $EA(B) \uplus EP(B) = E - B$, where \uplus denotes a disjoint union.

Dually, let

$$\begin{aligned} IA(B) &= \{i \in B : \min(\text{Cocirc}(B, i)) = i\} \\ IP(B) &= \{i \in B : \min(\text{Cocirc}(B, i)) \neq i\} \end{aligned}$$

The elements of $IA(B)$ and $IP(B)$ are called *internally active* and *internally passive* with respect to B , respectively. Note that $IA(B) \uplus IP(B) = B$. Also note that the internally active/passive elements with respect to basis B in M are the externally active/passive elements with respect to basis $E - B$ in M^* .

The following elegant result of Tutte [8] (for graphs) and Crapo [4] (for arbitrary matroids) underlies many of the results of [1] and this paper.

Theorem 2.7 [4, Proposition 5.12] Let M be a matroid on the ground set E and let $<$ be a linear order on E .

1. Every subset A of E can be uniquely written in the form $A = B \cup X - Y$ for some basis B , some subset $X \subseteq EA(B)$, and some subset $Y \subseteq IA(B)$. Equivalently, the intervals $[B - IA(B), B \cup EA(B)]$ form a partition of the poset 2^E of subsets of E ordered by inclusion.
2. Every independent set I of E can be uniquely written in the form $I = B - Y$ for some basis B and some subset $Y \subseteq IA(B)$. Equivalently, the intervals $[B - IA(B), B]$ form a partition of the independence complex $IN(M)$.

The Tutte polynomial of M is

$$T_M(x, y) = \sum_{B \text{ basis}} x^{|IA(B)|} y^{|EA(B)|}.$$

It follows from the work of Crapo and Tutte [4, 8] that this polynomial does not depend on the chosen order $<$. The Tutte polynomial is the most important matroid invariant, because it answers an innumerable amount of questions about the combinatorics, algebra, geometry, and topology of matroids and related objects. For more information, see [3].

2.3 The external activity complex.

Let M be a matroid on E . Let $\bar{E} = \{\bar{e} : e \in E\}$ be a second copy of E , and let $[[E]] = E \uplus \bar{E}$. This set of size $2|E|$ will be the ground set of the external activity complex of M . For each subset $S \subseteq E$ we write $\bar{S} := \{\bar{s} \mid s \in S\} \subseteq \bar{E}$. Therefore, each subset of $[[E]]$ can be written uniquely in the form $S_1 \cup \bar{S}_2$ for $S_1, S_2 \subseteq E$.

Our main object of study is the following.

Theorem 2.1 [1] *Let $M = (E, \mathcal{B})$ be a matroid and let $<$ be a linear order on E . M . There is a simplicial complex called the external activity complex $\text{Act}_{<}(M)$ on ground set $[[E]]$ such that*

1. *The facets are $F(B) := B \cup EP(B) \cup \overline{B \cup EA(B)}$ for every basis $B \in \mathcal{B}$.*
2. *The minimal non-faces are $S(\gamma) = c \cup \overline{\gamma - c}$ for every circuit γ , where c is the $<$ -smallest element of γ .*

The complement of the facet $F(B)$ in $[[E]]$ is $G(B) = EA(B) \cup \overline{EP(B)}$.

Las Vergnas's three active orders. Given a matroid $M = (E, \mathcal{B})$ and a total order $<$ on the ground set of M , LasVergnas introduced the following three *active orders*. In each case, he proved that there are several equivalent definitions.

Definition 2.8 *The external order $<_{ext}$ on \mathcal{B} is characterized by the following equivalent properties for two bases A and B :*

1. $A \leq_{ext} B$,
2. $A \subseteq B \cup EA(B)$,
3. $A \cup EA(A) \subseteq B \cup EA(B)$,
4. B is the lexicographically largest basis contained in $A \cup B$.

This poset is graded with $r(B) = |EA(B)|$. Adding a minimum element turns it into a lattice.

Definition 2.9 *The internal order $<_{int}$ on \mathcal{B} is characterized by the following equivalent properties for two bases A and B :*

1. $A \leq_{int} B$,
2. $A - IA(A) \subseteq B$,
3. $A - IA(A) \subseteq B - IA(B)$,
4. A is the lexicographically smallest basis containing $A \cup B$.

This poset is graded with $r(B) = r - |IA(B)|$. Adding a maximum element turns it into a lattice.

The internal and external orders are consistent in the sense that $A \leq_{int} B$ and $B \leq_{ext} A$ imply $A = B$. Therefore the following definition makes sense.

Definition 2.10 *The external/internal order $<_{ext/int}$ is the weakest order which simultaneously extends the external and the internal order. It is characterized by the following equivalent properties for two bases A and B :*

1. $A \leq_{ext/int} B$,
2. $IP(A) \cap EP(B) = \emptyset$,

This poset is a lattice. It is not necessarily graded.

We have the following proposition.

Proposition 2.11 *The lexicographic order $<_{lex}$ on \mathcal{B} is a linear extension of the three posets $<_{int}$, $<_{ext}$, and $<_{ext/int}$. In symbols, any of $A <_{int} B$, $A <_{ext} B$ or $A <_{ext/int} B$ implies $A <_{lex} B$.*

2.4 Shellability and the h -vector.

Shellability. Shellability is a combinatorial condition on a simplicial complex that allows us to describe its topology easily. A simplicial complex is shellable if it can be built up by introducing one facet at a time, so that whenever we introduce a new facet, its intersection with the previous ones is pure of codimension 1. More precisely:

Definition 2.12 *Let Δ be a pure simplicial complex. A shelling order is an order of the facets F_1, \dots, F_k such for every $i < j$ there exist $k < j$ and $f \in F_j$ such that $F_i \cap F_j \subseteq F_k \cap F_j = F_j - f$. If a shelling order exists, then we call Δ shellable.*

Given a shelling order and a facet F_j , there is a subset $\mathcal{R}(F_j)$ such that for every $A \subseteq F_j$, we have $A \not\subseteq F_i$ for all $i < j$ if and only if $\mathcal{R}(F_j) \subseteq A$. Equivalently, when we add facet F_j to the complex, the new faces that we introduce are precisely those in the interval $[\mathcal{R}(F_j), F_j]$. The set $\mathcal{R}(F_j)$ is called the *restriction set* of F_j in the shelling.

The f -vector and h -vector. The f -vector of a $(d-1)$ -dimensional simplicial complex Δ is (f_0, \dots, f_d) where f_i is the number of faces of Δ of size i . The h -vector (h_0, \dots, h_d) is an equivalent way of storing this information; it is defined by the relation

$$f_0(x-1)^d + f_1(x-1)^{d-1} + \dots + f_d(x-1)^0 = h_0x^d + h_1x^{d-1} + \dots + h_dx^0.$$

This polynomial is also known as the *shelling polynomial* $h_\Delta(x)$, due to the following description of the h -vector for shellable complexes.

Proposition 2.13 [2, Proposition 7.2.3] *If F_1, \dots, F_k is a shelling order for a $(d-1)$ -dimensional simplicial complex Δ , then*

$$h_i := |\{j : |\mathcal{R}(F_j)| = i\}|.$$

Note that it is not clear a priori that these numbers should be the same for any shelling order.

Understanding the topology of a shellable simplicial complex is easy once we know the last entry of the h -vector, thanks to the following result.

Theorem 2.14 [5, Theorem 12.2(2)] *Any geometric realization of a $(d-1)$ -dimensional shellable simplicial complex Δ is homotopy equivalent to a wedge of h_d spheres of dimension $d-1$. In particular, if $h_d = 0$, then every geometric realization of Δ is contractible.*

An important property for matroids is their shellability:

Theorem 2.15 [2, Theorem 7.3.3] *The lexicographic order $<_{lex}$ on the bases of a matroid M gives a shelling order of the independence complex $IN(M)$. Furthermore, the restriction set of a basis B in this shelling order is given by $IP(B)$.*

A straightforward consequence of the previous theorem is that the internal order poset is equal to the poset of bases of M where the order is given by inclusion of restriction sets of the lexicographic shelling order.

3 Example

Instead of proving the theorems, we want to illustrate them in an example. Consider the graphical matroid given by the graph of Figure 1. Its bases are all the 3-subsets of $[5]$ except $\{1, 2, 3\}$ and $\{1, 4, 5\}$. Under the standard order $1 < 2 < 3 < 4 < 5$ on the ground set, Table 1 records the basis activity of the various bases.

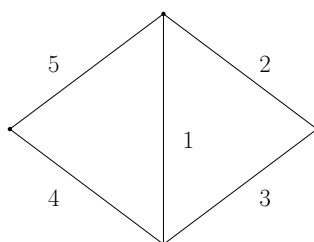


Fig. 1: A graphical matroid.

B	$EP(B)$	$EA(B)$	$IP(B)$	$IA(B)$
124	35	\emptyset	\emptyset	124
125	45	\emptyset	5	12
134	25	\emptyset	3	14
135	24	\emptyset	35	1
234	5	1	23	4
235	4	1	235	\emptyset
245	3	1	45	2
345	\emptyset	12	345	\emptyset

Tab. 1: The bases B together with their sets of externally passive, externally active, internally passive, and internally active elements.

The resulting internal, external, and external/internal orders $<_{ext}, <_{int}, <_{ext/int}$ are shown in Figure 2. By Definitions 2.8, 2.9, and 2.10, these three orders are isomorphic to the three families of sets $\{B \cup EA(B) : B \text{ basis}\}, \{B - IA(B) : B \text{ basis}\},$ and $\{B \cup EA(B) - IA(B) : B \text{ basis}\},$ partially ordered by containment.

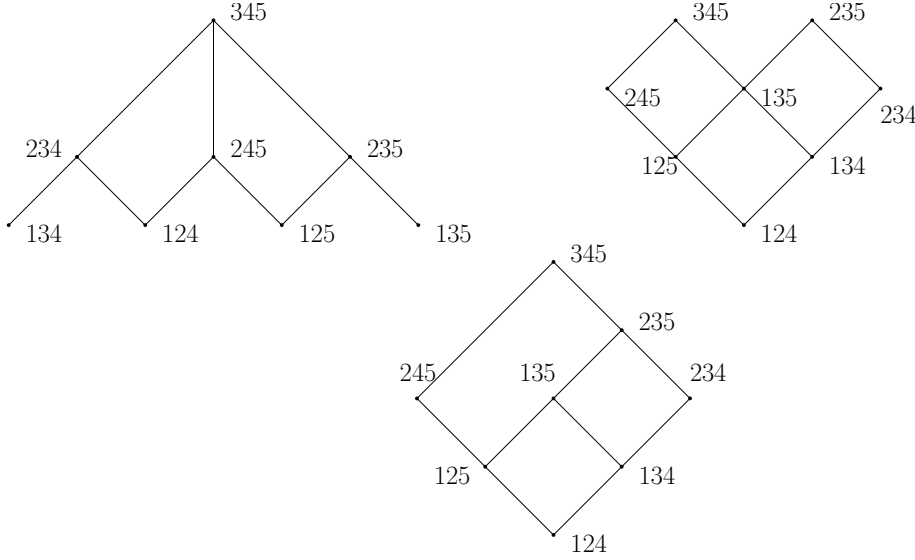


Fig. 2: The active orders \prec_{ext} , \prec_{int} , and $\prec_{ext/int}$, respectively.

Table 1 lists the bases in lexicographic order \prec_{lex} , and this is a shelling order for the independence complex $IN(M)$ by Theorem 2.15. The restriction set for each basis B is $\mathcal{R}(B) = IP(B)$. For example, when we add facet 134 in the third step of the shelling, this means that the new faces that appear are the four sets in the interval $[\mathcal{R}(134), 134] = [3, 134]$; that is, faces 3, 13, 34, and 134.

Our goal is to shell the external activity complex $\text{Act}_{\prec}(M)$ whose facets, listed in Table 2, are the sets $F(B) = B \cup EP(B) \cup \overline{B} \cup EA(B)$. Since $\overline{1}, 3, 4,$ and 5 are in all facets of $\text{Act}_{\prec}(M)$, we remove them, and shell the resulting *reduced external activity complex* $\text{Act}_{\prec}^{\bullet}(M)$. Our main result, Theorem 1.1, states that any linear extension of the external/internal order $\prec_{ext/int}$ gives a shelling order for this complex. For example, we may again consider the lexicographic order, which is indeed a linear extension of $\prec_{ext/int}$.

B	$F(B)$	$F(B)^{\bullet}$	$\mathcal{R}(F(B))$
124	12345124	1224	\emptyset
125	12345125	1225	$\overline{5}$
134	12345134	1234	$\overline{3}$
135	12345135	1235	$\overline{35}$
234	23451234	2234	$\overline{23}$
235	23451235	2235	$\overline{235}$
245	23451245	2245	$\overline{45}$
345	34512345	2345	$\overline{345}$

Tab. 2: The bases B of M , the corresponding facets $F(B)$ and $F(B)^{\bullet}$ of $\text{Act}_{\prec}(M)$ and $\text{Act}_{\prec}^{\bullet}(M)$, and their (shared) restriction set $\mathcal{R}(F(B))$ in the shelling.

For each basis B , Table 2 lists the corresponding facet $F(B)$ of $\text{Act}_{<}(M)$, the corresponding facet $F(B)^\bullet$ of $\text{Act}_{<}^\bullet(M)$, and the restriction set of the facet $F(B)$ in the shelling. This restriction set is $\mathcal{R}(F(B)) = \overline{IP(B)}$. For example, when we add facet $12\bar{3}4$ to the complex $\text{Act}_{<}^\bullet(M)$ in the third step of the shelling, the new faces that appear are the eight sets in the interval $[\mathcal{R}(12\bar{3}4), 12\bar{3}4] = [\bar{3}, 12\bar{3}4]$.

Notice that we can embed $IN(M) \rightarrow \text{Act}_{<}^\bullet(M)$ by sending $1 \rightarrow 1, 2 \rightarrow \bar{2}, 3 \rightarrow \bar{3}, 4 \rightarrow \bar{4}, 5 \rightarrow \bar{5}$. The latter complex has the same h -vector and is contractible. Therefore, it is no coincidence that the shellings of $IN(M)$ and $\text{Act}_{<}(M)$ are related. In fact, we will prove that any shelling order for $\text{Act}_{<}(M)$ is a shelling order for $IN(M)$. Theorem 1.1 then gives:

any linear extension of $<_{ext/int}$ is a shelling order for $IN(M)$ and $\text{Act}_{<}(M)$. (1)

We conclude this section with two examples showing that the linear extensions of the internal and external orders $<_{int}$ and of $<_{ext}$ are not necessarily shelling orders for $\text{Act}_{<}(M)$.

Example 3.1 Consider any linear extension of $<_{ext}$ starting with 124 and 135 in that order, such as:

124, 135, 125, 134, 234, 235, 245, 345.

This is not a shelling order for $IN(M)$ because the second facet 135 intersects the first facet 124 in codimension 2. By Corollary ?? (or directly by inspection), this is not a shelling order for $\text{Act}_{<}(M)$ either. Therefore:

a linear extension of $<_{ext}$ need not be a shelling order for $IN(M)$ or for $\text{Act}_{<}(M)$. (2)

Example 3.2 Consider the following linear extension of $<_{int}$:

124, 125, 134, 135, 245, 345, 234, 235

which gives the following order on the facets:

$12\bar{2}4, 12\bar{2}5, 12\bar{3}4, 12\bar{3}5, \bar{2}245, \bar{2}345, \bar{2}234, \bar{2}235,$

This is a shelling of $IN(M)$ by Theorem 1.2. However, it is not a shelling of $\text{Act}_{<}(M)$ and $\text{Act}_{<}^\bullet(M)$. To see this, suppose we introduce the facets of $\text{Act}_{<}^\bullet(M)$ in the order above. When we introduce the sixth facet $\bar{2}345$ we introduce two new minimal faces: $\bar{2}3$ and 345 ; so this is not a shelling order for $\text{Act}_{<}(M)$. Hence

a linear extension of $<_{int}$ is a shelling order for $IN(M)$, but not necessarily for $\text{Act}_{<}(M)$. (3)

In summary, combining (1), (2), and (3), we see that the hypotheses of Theorems 1.1 and 1.2 are as strong as possible in the context of Las Vergnas's active orders.

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