Defining amplituhedra and Grassmann polytopes

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Abstract. The *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$ is the set of k-dimensional subspaces V of $\mathbb{R}^n$ whose nonzero Plücker coordinates all have the same sign. In their study of scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, Arkani-Hamed and Trnka (2013) considered the image (called an *amplituhedron*) of $\mathrm{Gr}_{k,n}^{\geq 0}$ under a linear map $Z : \mathbb{R}^n \to \mathbb{R}^r$, where $k \leq r$ and the $r \times r$ minors of Z are all positive. One reason they required this positivity condition is to ensure that the map $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,r}$ induced by Z is well defined, i.e. it takes every element of $\mathrm{Gr}_{k,n}^{\geq 0}$ to a k-dimensional subspace of $\mathbb{R}^r$. Lam (2015) gave a sufficient condition for the induced map $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,r}$ to be well defined, in which case he called the image a *Grassmann polytope*. (In the case $k = 1$, Grassmann polytopes are just polytopes, and amplituhedra are cyclic polytopes.) We give a necessary and sufficient condition for the induced map $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,r}$ to be well defined, in terms of sign variation. Using previous work we presented at FPSAC 2015, we obtain an equivalent condition in terms of the $r \times r$ minors of Z (assuming Z has rank r).

Résumé. La *grassmannienne totalement non négative* $\mathrm{Gr}_{k,n}^{\geq 0}$ est l'ensemble des sous-espaces V de $\mathbb{R}^n$ de dimension k dont les coordonnées plückeriennes non nulles ont toutes le même signe. Selon leur étude des amplitudes de diffusion dans la théorie supersymétrique $\mathcal{N} = 4$ de Yang-Mills, Arkani-Hamed et Trnka (2013) ont considéré l'image (appelée un *amplituèdre*) de $\mathrm{Gr}_{k,n}^{\geq 0}$ par une application linéaire $Z : \mathbb{R}^n \to \mathbb{R}^r$, où $k \leq r$ et les mineurs de l'ordre r de Z sont tous positifs. Une des raisons pour lesquelles ils ont exigé cette condition de positivité est pour s'assurer que la carte $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,r}$ induite par Z est bien définie, c'est à dire, elle prend tout élément de $\mathrm{Gr}_{k,n}^{\geq 0}$ à un sous-espace de $\mathbb{R}^r$ de dimension k. Lam (2015) a donné une condition suffisante pour que la carte induite $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,r}$ soit bien définie, et dans ce cas, il a appelé l'image un *polytope de Grassmann*. (Dans le cas $k = 1$, les polytopes de Grassmann sont polytopes, et les amplituèdres sont polytopes cycliques.) Nous donnons une condition nécessaire et suffisante pour rendre la carte induite $\mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,r}$ bien définie, en termes de variations de signe. En se basant sur un travail antérieur que nous avons présenté à SFCA 2015, nous obtenons une condition équivalente en termes des mineurs de l'ordre r de Z (en supposant que r est le rang de Z).

Keywords. total positivity, Grassmannian, sign variation, cyclic polytopes, oriented matroids, amplituhedron

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1 Introduction and main results

The *(real) Grassmannian* $\mathrm{Gr}_{k,n}$ is the set of k-dimensional subspaces of $\mathbb{R}^n$. Given $V \in \mathrm{Gr}_{k,n}$, take a $k \times n$ matrix A whose rows span V; then for k-subsets $I \subseteq [n] := \{1, \cdots, n\}$, we let $\Delta_I(V)$ be the $k \times k$ minor of A with columns I. (The $\Delta_I(V)$, called *Plücker coordinates* of V, depend on our choice of A only up to a global constant.) If all nonzero $\Delta_I(V)$ have the same sign, then V is called *totally nonnegative*, and if in addition $\Delta_I(V) \neq 0$ for all I, then V is called *totally positive*. For example, the span of $(-1, 0, 0, 1)$ and $(-1, 2, 1, 3)$ is a totally nonnegative element of $\mathrm{Gr}_{2,4}$, but it is not totally positive because the Plücker coordinate $\Delta_{\{2,3\}}$ equals 0. We define the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$ as the subset of $\mathrm{Gr}_{k,n}$ of totally nonnegative elements. $\mathrm{Gr}_{k,n}^{\geq 0}$ has become a hot topic in algebraic combinatorics in the past two decades, with applications to cluster algebras [Sco06], asymmetric exclusion processes in statistical mechanics [CW11], the KP equation [KW14], and calculating scattering amplitudes in theoretical physics [AHBC$^+$]. It is the latter connection which we explore in this paper.

In their study of scattering theory, Arkani-Hamed and Trnka [AHT14] considered the map $Z_{\mathrm{Gr}} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,r}$ induced by some linear map $Z : \mathbb{R}^n \to \mathbb{R}^r$. In the case that $k \leq r$ and all $r \times r$ minors of Z are positive, they call the image $Z_{\mathrm{Gr}}(\mathrm{Gr}_{k,n}^{\geq 0})$ a *(tree) amplituhedron*, and use it to calculate scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (taking $r := k + 4$). (We give more background on scattering amplitudes and the amplituhedron, with examples, in Section 2.) One motivation they provide for requiring that $k \leq r$ and Z have positive $r \times r$ minors is to guarantee that Z_{Gr} is well defined, i.e. that $Z(V)$ has dimension k for all $V \in \mathrm{Gr}_{k,n}^{\geq 0}$. As a more general sufficient condition for Z_{Gr} to be well defined, Lam [Lam] requires that the row span of Z (viewed as an $r \times n$ matrix) has a k-dimensional subspace which is totally positive. (It is not obvious that Arkani-Hamed and Trnka's condition is indeed a special case of Lam's; see Section 15.1 of [Lam].) In the case that Z_{Gr} is well defined, Lam calls the image $Z_{\mathrm{Gr}}(\mathrm{Gr}_{k,n}^{\geq 0})$ a *Grassmann polytope*, since in the case $k = 1$ Grassmann polytopes are precisely projective polytopes in $\mathrm{Gr}_{1,r} = \mathbb{P}^{r-1}$ (and the amplituhedra are projective *cyclic polytopes*; see Example 2.1). Our main result is a necessary and sufficient condition for Z_{Gr} to be well defined, in terms of sign variation; we are able to translate this into a condition on the $r \times r$ minors of Z (assuming Z has rank r) using previous work we presented at FPSAC 2015 [Kar15]. As a consequence, we recover Arkani-Hamed and Trnka's and Lam's sufficient conditions.

Before stating our theorem, we introduce some notation. For $v \in \mathbb{R}^n$, let $\mathrm{var}(v)$ be the number of times v (viewed as a sequence of n numbers, ignoring any zeros) changes sign, and let

$$\overline{\mathrm{var}}(v) := \max\{\mathrm{var}(w) : w \in \mathbb{R}^n \text{ such that } w_i = v_i \text{ for all } i \in [n] \text{ with } v_i \neq 0\}.$$

(We use the convention $\mathrm{var}(0) := -1$.) For example, if $v := (1, -1, 0, -2) \in \mathbb{R}^4$, then $\mathrm{var}(v) = 1$ and $\overline{\mathrm{var}}(v) = 3$.

Theorem 1.1. *With notation as above, let d be the rank of Z and $W \in \mathrm{Gr}_{d,n}$ the row span of Z, so that $W^\perp = \ker(Z) \in \mathrm{Gr}_{n-d,n}$. The following are equivalent:*
(i) the map Z_{Gr} is well defined, i.e. $\dim(Z(V)) = k$ for all $V \in \mathrm{Gr}_{k,n}^{\geq 0}$;
(ii) $\mathrm{var}(v) \geq k$ for all nonzero $v \in \ker(Z)$; and
(iii) $\overline{\mathrm{var}}((\Delta_{I \setminus \{i\}}(W))_{i \in I}) \leq d - k$ for all $(d+1)$-subsets $I \subseteq [n]$ such that $W|_I$ has dimension d.

Note that we are not interested in knowing the map Z, only its kernel (or equivalently its row span, which is $\ker(Z)^\perp$). That is, the choice of coordinates on $\mathbb{R}^r$ (the codomain of Z) is not important.

Example 1.2. Let $Z : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear map given by the matrix $\begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & 3 \end{bmatrix}$ (so $n = 4$, $d = r = 2$), and let $W \in \mathrm{Gr}_{2,4}$ be the row span of this matrix. Let us use Theorem 1.1(iii) to determine for which k ($0 \le k \le 4$) the map $Z_{\mathrm{Gr}} : \mathrm{Gr}_{k,4}^{\ge 0} \to \mathrm{Gr}_{k,2}$ induced by Z is well defined. The 4 relevant sequences of Plücker coordinates (as I ranges over the 3-subsets $\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$ of $[4]$) are

$$\begin{aligned} (\Delta_{\{2,3\}}(W), \Delta_{\{1,3\}}(W), \Delta_{\{1,2\}}(W)) &= (-1, -3, 5), \\ (\Delta_{\{2,4\}}(W), \Delta_{\{1,4\}}(W), \Delta_{\{1,2\}}(W)) &= (-5, 5, 5), \\ (\Delta_{\{3,4\}}(W), \Delta_{\{1,4\}}(W), \Delta_{\{1,3\}}(W)) &= (4, 5, -3), \\ (\Delta_{\{3,4\}}(W), \Delta_{\{2,4\}}(W), \Delta_{\{2,3\}}(W)) &= (4, -5, -1). \end{aligned}$$

The maximum number of sign changes among these 4 sequences is 1, which is at most $2 - k$ iff $k \le 1$. Hence Z_{Gr} is well defined iff $k \le 1$.

Note that for $k \ge 2$, we can obtain a certificate $V \in \mathrm{Gr}_{k,4}^{\ge 0}$ with $\dim(Z(V)) < k$ (showing that Z_{Gr} is not well defined) as follows: take a nonzero $v \in \ker(Z)$ with $\mathrm{var}(v) < k$, and extend v to $V \in \mathrm{Gr}_{k,4}^{\ge 0}$. For example, if $k = 2$ we can take $v = (1, -3, -5, 0) \in \ker(Z)$ and extend it to the row span $V \in \mathrm{Gr}_{2,4}^{\ge 0}$ of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -5 & 0 \end{bmatrix}$. Then $Z(V) = \mathrm{span}(\{(2,1)\})$, so $\dim(Z(V)) = 1 < k$. (The fact that we can always extend such a v to $V \in \mathrm{Gr}_{k,n}^{\ge 0}$ is Lemma 4.1(i), which is key to proving Theorem 1.1.) $\diamond$

We explain how to use Theorem 1.1 to deduce the sufficient conditions of Arkani-Hamed and Trnka, and of Lam, for Z_{Gr} to be well defined. If the $r \times r$ minors of Z are all positive, then $d = r$ and W is totally positive element of $\mathrm{Gr}_{r,n}$, so the condition (iii) holds for any $k \le r$. Alternatively, it follows from a result of Gantmakher and Krein [GK50] (see Corollary 3.3) that $\mathrm{var}(v) \ge r$ for all nonzero $v \in \ker(Z)$, so the condition (ii) holds for any $k \le r$. This recovers the sufficient condition of Arkani-Hamed and Trnka [AHT14]. On the other hand, if W has a totally positive subspace $V \in \mathrm{Gr}_{k,n}$, then by the aforementioned result of Gantmakher and Krein, we have $\mathrm{var}(v) \ge k$ for all nonzero $v \in V^\perp$. This implies that condition (ii) holds since $\ker(Z) = W^\perp \subseteq V^\perp$, recovering the sufficient condition of Lam [Lam]. It is not known whether or not Lam's sufficient condition for the amplituhedron map to be well defined is also necessary, i.e. whether or not for every W satisfying the conditions of Theorem 1.1 has a totally positive k-dimensional subspace.

We also obtain a result similar to Theorem 1.1 characterizing when the map induced by Z on the totally positive part of $\mathrm{Gr}_{k,n}$ (rather than the totally nonnegative part) is well defined.

Theorem 1.3. *Suppose that $Z : \mathbb{R}^n \to \mathbb{R}^r$ is a linear map. Let d be the rank of Z and $W \in \mathrm{Gr}_{d,n}$ the row span of Z, so that $W^\perp = \ker(Z) \in \mathrm{Gr}_{n-d,n}$. The following are equivalent:*
(i) the map induced by Z on the totally positive part of $\mathrm{Gr}_{k,n}$ is well defined, i.e. $\dim(Z(V)) = k$ for all totally positive $V \in \mathrm{Gr}_{k,n}$;
(ii) $\overline{\mathrm{var}}(v) \ge k$ for all nonzero $v \in \ker(Z)$; and
(iii) we can perturb W into a generic $W' \in \mathrm{Gr}_{d,n}$ without changing the sign of any nonzero Plücker coordinate of W, such that $\mathrm{var}((\Delta_{I \setminus \{i\}}(W'))_{i \in I}) \le d - k$ for all $(d+1)$-subsets $I \subseteq [n]$. (A subspace is generic if all its Plücker coordinates are nonzero.)

Outline. In Section 2 we give background on scattering amplitudes and amplituhedra, and explain how Grassmann polytopes are precisely polytopes (and amplituhedra cyclic polytopes) in the case $k = 1$. In Section 3 we review results relating sign variation and total positivity in the Grassmannian, and use them to sketch the proof of the equivalence of (ii) and (iii) in Theorem 1.1 and Theorem 1.3. In Section 4 we sketch the proof of the equivalence of (i) and (ii) in Theorem 1.1 and Theorem 1.3. Complete proofs appear in Section 4 of our preprint [Kar].

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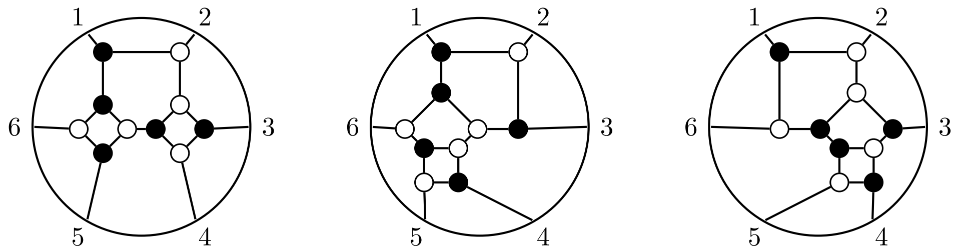
2 Scattering amplitudes and amplituhedra

In this section we give some background on scattering amplitudes and amplituhedra, and their connection to the Grassmannian. A *scattering amplitude* is a complex number associated to a *scattering process* with certain parameters, whose modulus squared gives the probability density of observing this process with such parameters. Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, and Trnka [AHBC$^+$] related scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory to the geometry and combinatorics of the totally nonnegative Grassmannian. For example, the scattering process of n gluons, k of which have negative helicity and $n - k$ of which have positive helicity, is related to $\mathrm{Gr}_{k,n}^{\geq 0}$ and $\mathrm{Gr}_{k-2,n}^{\geq 0}$. More specifically, the Britto-Cachazo-Feng-Witten (BCFW) recursion produces certain *on-shell diagrams*, and [AHBC$^+$] expresses scattering amplitudes up to zeroth order as a sum of terms, each term corresponding to such an on-shell diagram. (We do not describe here how this sum is constructed.) These on-shell diagrams are (a subset of) *plabic graphs*, which Postnikov [Pos] had used to label cells in the cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$. See Figure 1. (This cell decomposition is the stratification of $\mathrm{Gr}_{k,n}^{\geq 0}$ according to whether each Δ_I is zero or nonzero.)

Arkani-Hamed and Trnka [AHT14] later furthered this connection by defining for $r \geq k$ the *amplituhedron* $\mathcal{A}_{n,k,r-k}(Z)$ associated to a linear map $Z : \mathbb{R}^n \to \mathbb{R}^r$ with positive $r \times r$ minors, as the image $Z_{\mathrm{Gr}}(\mathrm{Gr}_{k,n}^{\geq 0})$ of the map $Z_{\mathrm{Gr}} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,r}$ induced by Z. The case of particular interest in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is $r = k+4$, but $\mathcal{A}_{n,k,r-k}(Z)$ is an interesting mathematical object for any r. (There is reason, however, for requiring that r and k have the same parity. This is because $\mathrm{Gr}_{k,n}^{\geq 0}$ exhibits the following cyclic symmetry: if $\left[x^{(1)}|\cdots|x^{(n)}\right]$ is a $k \times n$ matrix of column vectors whose row span is totally nonnegative, then the row span of the matrix $\left[x^{(2)}|\cdots|x^{(n)}|(-1)^{k-1}x^{(1)}\right]$ obtained by multiplying the first column by $(-1)^{k-1}$ and moving it to the end is also totally nonnegative. The set of Z with positive $r \times r$ minors exhibits a similar cyclic symmetry, but with sign $(-1)^{r-1}$. Hence the two cyclic symmetries are compatible iff r and k have the same parity.)

Recall that [AHBC$^+$] expresses a scattering amplitude up to zeroth order as a sum over certain on-shell diagrams, and these on-shell diagrams correspond to cells of $\mathrm{Gr}_{k,n}^{\geq 0}$. Arkani-Hamed and Trnka [AHT14] interpret each term of this sum as an integral over the image of the corresponding cell of $\mathrm{Gr}_{k,n}^{\geq 0}$ under Z in the amplituhedron $\mathcal{A}_{n,k,4}(Z)$. They verified experimentally that these cells 'triangulate' the amplituhedron $\mathcal{A}_{n,k,4}(Z)$, and that this sum of terms can be collapsed into a single integral over the entire amplituhedron (for any choice of Z with positive maximal minors).

We should emphasize that behind some of these claims are purely mathematical statements which have not been rigorously proven. For example, it is only known in the case $k = 1$ (by the work of Rambau



$I = \{1, 3, 4, 5, 6\}$ $\quad$ $I = \{1, 2, 3, 5, 6\}$ $\quad$ $I = \{1, 2, 3, 4, 5\}$

Fig. 1: The (1+2,6)-scattering amplitude to zeroth order can be expressed as a sum of three terms, arising from the three on-shell diagrams, or plabic graphs, shown, which we obtain by the BCFW recursion. These graphs label cells of $\mathrm{Gr}_{3,6}^{\geq 0}$, and there is a further step (which we do not discuss here) which decreases the index k by 2, giving three cells of $\mathrm{Gr}_{1,6}^{\geq 0}$, each indexed by a subset of $[6]$. The images of these cells under Z in $\mathrm{Gr}_{1,5}$ triangulate the amplituhedron $\mathcal{A}_{6,1,4}(Z)$ (for general k, n this is only conjecturally true), using which the sum of 3 terms can be collapsed into a single integral over $\mathcal{A}_{6,1,4}(Z)$.

[Ram97] on cyclic polytopes) that these cells do in fact 'triangulate' the amplituhedron, and it has not been proven that different choices of the triangulation give the same expression for the scattering amplitude. For precise statements of such conjectures, see the mathematical introduction to amplituhedra by Lam [Lam]. A further important open problem in this area is to construct a 'dual amplituhderon,' using which, Arkani-Hamed and Trnka [AHT14] conjecture, one can give an intrinsic expression for the scattering amplitude (not depending on a choice of triangulation of the amplituhedron). However, there is not even a conjectural definition of the dual amplituhedron, except in the case $k = 1$.

As an example, let us consider the case $k = 1$ in detail.

Example 2.1. Let $k := 1$. In this case $\mathrm{Gr}_{1,n}$ is projective space $\mathbb{P}^{n-1}$, and its totally nonnegative part is the set of $(v_1 : v_2 : \cdots : v_n)$ with $\frac{v_i}{v_j} > 0$ for all $i, j \in [n]$ with $v_i, v_j \neq 0$. Hence $\mathrm{Gr}_{1,n}^{\geq 0}$ is a projective $(n-1)$-simplex; explicitly, it is isomorphic to the $(n-1)$-simplex $\{v \in \mathbb{R}^n : v_1, \cdots, v_n \geq 0, v_1 + \cdots + v_n = 1\}$ by

$$\{(v_1 : \cdots : v_n) \in \mathbb{P}^{n-1} : v_1 + \cdots + v_n \neq 0\} \hookrightarrow \mathbb{R}^n,$$
$$(v_1 : \cdots : v_n) \mapsto \frac{1}{v_1 + \cdots + v_n}(v_1, \cdots, v_n)$$

is the . The cells of $\mathrm{Gr}_{1,n}^{\geq 0}$ correspond to the faces of this simplex, and are in bijection with nonempty subsets of $[n]$, where the cell corresponding to $I \subseteq [n]$ equals

$$\{(v_1 : \cdots : v_n) \in \mathrm{Gr}_{1,n}^{\geq 0} : v_i \neq 0 \text{ for all } i \in I, v_j = 0 \text{ for all } j \in [n] \setminus I\}.$$

Now define $Z : \mathbb{R}^n \to \mathbb{R}^r$ by

$$Z := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{r-1} & t_2^{r-1} & \cdots & t_n^{r-1} \end{bmatrix}, \tag{1}$$

where $t_1 < t_2 < \cdots < t_n$. Then every $r \times r$ minor of Z is a Vandermonde determinant, and hence positive. (In general, we can choose any Z with positive maximal minors.) The amplituhedron of Z is

$$\mathcal{A}_{n,1,r-1}(Z) = Z_{\mathrm{Gr}}(\mathrm{Gr}_{1,n}^{\geq 0}) = \left\{ \sum_{i=1}^{n} v_i(1 : t_i : \cdots : t_i^{r-1}) : v_1, \cdots, v_n \geq 0 \text{ not all } 0 \right\} \subseteq \mathbb{P}^{r-1},$$

which is a projective *cyclic polytope*. To see this, let $U := \{(x_0 : x_1 : \cdots : x_{r-1}) \in \mathbb{P}^{r-1} : x_0 \neq 0\}$, so that $U \cong \mathbb{R}^{r-1}$ by $\pi : U \to \mathbb{R}^{r-1}, (1 : x_1 : \cdots : x_{r-1}) \mapsto (x_1, \cdots, x_{r-1})$. Note that $\mathcal{A}_{n,1,r-1}(Z) \subseteq U$, and

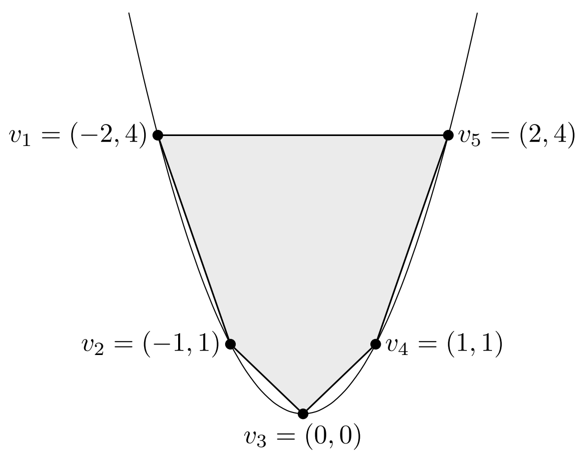
$$\pi(\mathcal{A}_{n,1,r-1}(Z)) = \left\{ \sum_{i=1}^{n} v_i(t_i, t_i^2, \cdots, t_i^{r-1}) : v_0, \cdots, v_n \geq 0, v_1 + \cdots + v_n = 1 \right\},$$

which is (by definition) a cyclic polytope in $\mathbb{R}^{r-1}$. Cyclic polytopes are of interest beyond their connections to total positivity. For example, the upper bound theorem of Stanley [Sta75] states that for any triangulation of an $(r-2)$-sphere with n vertices, the number of faces of dimension d (for any $0 \leq d \leq r-2$) is at most the number of d-dimensional faces of the cyclic polytope above.

The case of particular interest for scattering amplitudes when $r = 5$, which corresponds to 4-dimensional cyclic polytopes with n vertices. Since it is difficult to visualize a 4-dimensional polytope, let us consider a lower-dimensional example. Take $k := 1$, $n := 5$, $r := 3$, and $(t_1, t_2, t_3, t_4, t_5) := (-2, -1, 0, 1, 2)$, so that $Z : \mathbb{R}^5 \to \mathbb{R}^3$ given by the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 1 & 1 & 4 \end{bmatrix}.$$

Then $\pi(\mathcal{A}_{5,1,2}(Z)) \subseteq \mathbb{R}^2$ is the pentagon



with vertices $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, 4)$.

What happens if we relax the requirement that $Z : \mathbb{R}^n \to \mathbb{R}^r$ has positive $r \times r$ minors? Then the image $Z_{\mathrm{Gr}}(\mathrm{Gr}_{1,n}^{\geq 0})$ is an arbitrary projective polytope in $\mathbb{P}^{r-1}$ with at most n vertices, but we still need

$Z_{\mathrm{Gr}} : \mathrm{Gr}_{1,n}^{\geq 0} \to \mathrm{Gr}_{1,r}$ to be well defined, i.e. $\dim(Z(V)) = 1$ for all $V \in \mathrm{Gr}_{1,n}^{\geq 0}$. If there exists a nonzero $v \in \ker(Z)$ whose coordinates are all nonnegative or all nonpositive, then $\mathrm{span}(v)$ is a totally nonnegative element of $\mathrm{Gr}_{1,n}$ and $Z(\mathrm{span}(v)) = \{0\}$, so Z_{Gr} is not well defined. Conversely, if Z_{Gr} is not well defined, then there exists $V \in \mathrm{Gr}_{1,n}^{\geq 0}$ with $Z(V) = 0$. Taking any nonzero $v \in V$, we have that the coordinates of v are all nonnegative or all nonpositive, and $v \in \ker(Z)$. This proves the equivalence of (i) and (ii) in Theorem 1.1 in the case $k = 1$, since for nonzero $v \in \mathbb{R}^n$ we have $\mathrm{var}(v) = 0$ iff the coordinates of v are all nonnegative or all nonpositive. $\diamond$

3 Sign variation and the Grassmannian

In this section we review results relating sign variation and the Grassmannian which we use to prove the equivalence of (ii) and (iii) in Theorem 1.1 and Theorem 1.3. Recall that for $v \in \mathbb{R}^n$, $\mathrm{var}(v)$ is the number of times v (viewed as a sequence of n numbers, ignoring any zeros) changes sign, and

$$\overline{\mathrm{var}}(v) = \max\{\mathrm{var}(w) : w \in \mathbb{R}^n \text{ such that } w_i = v_i \text{ for all } i \in [n] \text{ with } v_i \neq 0\}.$$

We begin by making some historical remarks. The theory of total positivity originated in the 1930's with Schoenberg [Sch30], who (answering a question of Pólya) showed that if $A : \mathbb{R}^k \to \mathbb{R}^n$ is an injective linear map, then $\mathrm{var}(A(x)) \leq \mathrm{var}(x)$ for all $x \in \mathbb{R}^k$ iff for $1 \leq j \leq k$, all nonzero $j \times j$ minors of A have the same sign. Also in the 1930's, Gantmakher and Krein began studying total positivity because of its applications to the oscillation theory of mechanical systems [GK50]. For example, they showed [GK37] that if an $n \times n$ matrix X is *totally positive* (i.e. all $\binom{2n}{n}$ minors of X are positive), then the n eigenvalues of X are distinct positive reals. They also gave a characterization of (what would later be called) the totally nonnegative and totally positive Grassmannians in terms of sign variation.

Theorem 3.1 (Theorems 5.3, 5.1 of [GK50]).
(i) $V \in \mathrm{Gr}_{k,n}$ is totally nonnegative iff $\mathrm{var}(v) \leq k-1$ for all $v \in V$.
(ii) $V \in \mathrm{Gr}_{k,n}$ is totally positive iff $\overline{\mathrm{var}}(v) \leq k-1$ for all nonzero $v \in V$.

(Part (i) above was proved independently by Schoenberg and Whitney [SW51].) For example, the two vectors $(1,0,0,-1)$ and $(-1,2,1,3)$ each change sign exactly once, and we can check that any vector in their span V changes sign at most once, which is equivalent to V being totally nonnegative. On the other hand, $\overline{\mathrm{var}}((1,0,0,-1)) = 3$, so V is not totally positive. Every element of $\mathrm{Gr}_{k,n}$ has a vector which changes sign at least $k-1$ times (put a $k \times n$ matrix whose rows span V into reduced row echelon form, and take the alternating sum of the rows), so the totally nonnegative elements are those whose vectors change sign as few times as possible.

The duality of total positivity in the Grassmannian allows us to give a 'dual version' of Theorem 3.1. Namely, define $\mathrm{alt} : \mathbb{R}^n \to \mathbb{R}^n$ by $\mathrm{alt}(v) := (v_1, -v_2, v_3, -v_4, \cdots, (-1)^{n-1}v_n)$ for $v \in \mathbb{R}^n$. Then alt is related to sign variation and taking orthogonal complements of subspaces as follows.

Proposition 3.2. *(i) We have* $\mathrm{var}(v) + \overline{\mathrm{var}}(\mathrm{alt}(v)) = n-1$ *for all nonzero* $v \in \mathbb{R}^n$.
(ii) A subspace $V \in \mathrm{Gr}_{k,n}$ and its orthogonal complement $V^\perp \in \mathrm{Gr}_{n-k,n}$ have the same Plücker coordinates, up to sign:

$$\Delta_I(V) = \Delta_{[n]\setminus I}(\mathrm{alt}(V^\perp)) \qquad \text{for all } k\text{-subsets } I \subseteq [n].$$

In particular, (ii) implies that $V \in \mathrm{Gr}_{k,n}$ is totally nonnegative (or totally positive) iff $\mathrm{alt}(V^\perp)$ is totally nonnegative (or totally positive), giving the following corollary of Theorem 3.1.

Corollary 3.3. *(i) $V \in \mathrm{Gr}_{k,n}$ is totally nonnegative iff $\overline{\mathrm{var}}(v) \ge k$ for all nonzero $v \in V^{\perp}$.*
(ii) $V \in \mathrm{Gr}_{k,n}$ is totally positive iff $\mathrm{var}(v) \ge k$ for all nonzero $v \in V^{\perp}$.

In work presented at FPSAC 2015 [Kar15], we generalized Theorem 3.1 by relating $\max_{v \in V} \mathrm{var}(v)$ and $\max_{v \in V \setminus \{0\}} \overline{\mathrm{var}}(v)$ to the Plücker coordinates of V.

Theorem 3.4 ([Kar]). *Suppose that $V \in \mathrm{Gr}_{k,n}$, and $m \ge k-1$.*
(i) If $\mathrm{var}(v) \le m$ for all $v \in V$, then

$$\mathrm{var}((\Delta_{I \cup \{i\}}(V))_{i \in [n] \setminus I}) \le m - k + 1 \text{ for all } k\text{-subsets } I \subseteq [n],$$

and the converse holds if V is generic. (A subspace is generic if all its Plücker coordinates are nonzero.) In fact, we have $\mathrm{var}(v) \le m$ for all $v \in V$ iff we can perturb V into a generic $V' \in \mathrm{Gr}_{k,n}^{\ge 0}$, such that we do not change the sign of any nonzero Plücker coordinate of V and

$$\mathrm{var}((\Delta_{I \cup \{i\}}(V'))_{i \in [n] \setminus I}) \le m - k + 1 \text{ for all } k\text{-subsets } I \subseteq [n].$$

That is, in $\{W \in \mathrm{Gr}_{k,n} : \mathrm{var}(v) \le m$ for all $v \in W\}$, the generic elements are dense.
(ii) We have $\overline{\mathrm{var}}(v) \le m$ for all nonzero $v \in V$ iff

$$\overline{\mathrm{var}}((\Delta_{I \cup \{i\}}(V))_{i \in [n] \setminus I}) \le m - k + 1$$

for all k-subsets $I \subseteq [n]$ such that $\Delta_{I \cup \{i\}}(V) \ne 0$ for some $i \in [n]$.

We proved this result more generally for oriented matroids. In the case of (i) above, we gave an explicit algorithm for constructing V' from V so that we can test whether $\mathrm{var}(v) \le m$ for all $v \in V$. We also note that if we take $m := k-1$, then we recover Theorem 3.1 of Gantmakher and Krein.

Example 3.5. Let $V \in \mathrm{Gr}_{2,4}$ be the row span of the matrix $\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 2 & 1 & 4 \end{bmatrix}$, so $k := 2$. Then by Theorem 3.4(ii), the fact that $\overline{\mathrm{var}}(v) \le 2 =: m$ for all $v \in V \setminus \{0\}$ is equivalent to the fact that the 4 sequences

$$\begin{aligned}
(\Delta_{\{1,2\}}(V), \Delta_{\{1,3\}}(V), \Delta_{\{1,4\}}(V)) &= (2, 1, 4), \\
(\Delta_{\{1,2\}}(V), \Delta_{\{2,3\}}(V), \Delta_{\{2,4\}}(V)) &= (2, 4, -6), \\
(\Delta_{\{1,3\}}(V), \Delta_{\{2,3\}}(V), \Delta_{\{3,4\}}(V)) &= (1, 4, -11), \\
(\Delta_{\{1,4\}}(V), \Delta_{\{2,4\}}(V), \Delta_{\{3,4\}}(V)) &= (4, -6, -11)
\end{aligned}$$

each change sign at most $m - k + 1 = 1$ time. $\diamond$

Example 3.6. Let $V \in \mathrm{Gr}_{2,4}$ be the row span of the matrix $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, so $k := 2$. Note that the 4 sequences of Plücker coordinates

$$\begin{aligned}
(\Delta_{\{1,2\}}(V), \Delta_{\{1,3\}}(V), \Delta_{\{1,4\}}(V)) &= (1, 0, 1), \\
(\Delta_{\{1,2\}}(V), \Delta_{\{2,3\}}(V), \Delta_{\{2,4\}}(V)) &= (1, -1, 0), \\
(\Delta_{\{1,3\}}(V), \Delta_{\{2,3\}}(V), \Delta_{\{3,4\}}(V)) &= (0, -1, 1), \\
(\Delta_{\{1,4\}}(V), \Delta_{\{2,4\}}(V), \Delta_{\{3,4\}}(V)) &= (1, 0, 1)
\end{aligned}$$

each change sign at most $m - k + 1 = 1$ time (where we take $m := 2$), but the vector $(1, -1, 1, -1) \in V$ changes sign 3 times. This shows that Theorem 3.4(i) does not hold if we remove the assumption that V is generic. Also note that (as implied by Theorem 3.4(i)) we cannot perturb V into a generic subspace without making one of the 4 sequences above change twice. For example, if we are forced to pick a sign for $\Delta_{\{1,3\}}(V)$, then either the first or third sequence above would change sign twice. $\diamond$

In the same way that we deduced Corollary 3.3, we can use Proposition 3.2 to obtain a dual version of Theorem 3.4, which relates $\min_{v \in V^\perp \setminus \{0\}} \mathrm{var}(v)$ and $\min_{v \in V^\perp \setminus \{0\}} \overline{\mathrm{var}}(v)$ to the Plücker coordinates of V. The equivalence of (ii) and (iii) in Theorem 1.1 and Theorem 1.3 follow from the dual versions of Theorem 3.4(ii) and Theorem 3.4(i), respectively.

4 When is Z_{Gr} well defined?

In this section we sketch the proof of the equivalence of (i) and (ii) in Theorem 1.1 and Theorem 1.3. (We explained the equivalence of (ii) and (iii) in Section 3.) Recall that $Z : \mathbb{R}^n \to \mathbb{R}^r$ is a linear map, and we want to determine when map induced by Z from either the totally nonnegative (or totally positive) part of $\mathrm{Gr}_{k,n}$ to $\mathrm{Gr}_{k,r}$ is well defined, i.e. $\dim(Z(V)) = k$ for all totally nonnegative (or totally positive) $V \in \mathrm{Gr}_{k,n}$. In particular, we want to show that this induced map is well defined iff $\mathrm{var}(v) \geq k$ (or $\overline{\mathrm{var}}(v) \geq k$ in the totally positive case) for all nonzero $v \in \ker(Z)$. The key to the proof is the following lemma.

Lemma 4.1. *Let $v \in \mathbb{R}^n$ be nonzero, and $k \leq n$.*
(i) There exists a totally nonnegative element of $\mathrm{Gr}_{k,n}$ *containing v iff* $\mathrm{var}(v) \leq k - 1$.
(ii) There exists a totally positive element of $\mathrm{Gr}_{k,n}$ *containing v iff* $\overline{\mathrm{var}}(v) \leq k - 1$.

We sketch the proof of the lemma. The forward directions of (i) and (ii) follow from the result of Gantmakher and Krein (Theorem 3.1). For the reverse direction of (i), given v with $\mathrm{var}(v) \leq k - 1$, we can explicitly construct $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ containing v. For example, if $v = (2, 5, -1, -4, -1, 3, 2) \in \mathbb{R}^7$ and $k = 3$, then we may take $V \in \mathrm{Gr}_{3,7}^{\geq 0}$ as the row span of the matrix

$$\begin{bmatrix} 2 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 2 \end{bmatrix}.$$

For the reverse direction of (ii), given v with $\overline{\mathrm{var}}(v) \leq k - 1$, we cannot so easily construct a totally positive (not just totally nonnegative) $V \in \mathrm{Gr}_{k,n}$ containing v, so we use a different argument. By the theory of oriented matroids [BLVS$^+$99], the signs of Plücker coordinates (i.e. zero, positive, or negative) of any $W \in \mathrm{Gr}_{k,n}$ are uniquely determined up to sign by the set of sign vectors $\{\mathrm{sign}(w) \in \{0, +, -\}^n : w \in W\}$, and vice versa (where, for example, $\mathrm{sign}(5, 0, -1, 2) = (+, 0, -, +)$). In particular, since all totally positive elements of $\mathrm{Gr}_{k,n}$ have the same Plücker coordinates up to sign, they all have the same set of sign vectors. It thus suffices to show that $\mathrm{sign}(v)$ appears in this common set of sign vectors, because then we can take any totally positive $W \in \mathrm{Gr}_{k,n}$, which will contain v after we rescale the coordinates of $\mathbb{R}^n$ (in other words, rescale the columns of a $k \times n$ matrix whose row span is W) by appropriate positive constants. For example, a totally positive $W \in \mathrm{Gr}_{2,4}$ is the row span of the matrix

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

If $v = (1, 1, -1, -1)$, then we observe that $(2, 1, -1, -3) \in W$ has the same sign vector as v. Rescaling the columns of the matrix above by $1/2$, 1, 1, $1/3$ gives

$$\begin{bmatrix} 1/2 & 0 & -1 & -2/3 \\ 0 & 1 & 1 & 1/3 \end{bmatrix},$$

whose row span $V \in \mathrm{Gr}_{2,4}$ is totally positive and contains v. Our argument giving the set of sign vectors of a totally positive element of $\mathrm{Gr}_{k,n}$ uses tools from the theory of oriented matroids, so we omit it here.

We now explain how to prove the equivalence of (i) and (ii) in Theorem 1.1 using Lemma 4.1(i). Suppose that (i) of Theorem 1.1 does not hold, i.e. there exists $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ such that $\dim(Z(V)) < k$. Let $\{v_1, \cdots, v_k\} \subseteq \mathbb{R}^n$ be a basis of V. Then by assumption $Z(v_1), \cdots, Z(v_k)$ are linearly dependent, so $\sum_{i=1}^k c_i Z(v_i) = 0$ for some $c_1, \cdots, c_k \in \mathbb{R}$ not all zero. Then $v := \sum_{i=1}^k c_i v_i \in V \cap \ker(Z)$ is nonzero, and $\mathrm{var}(v) \leq k - 1$ by Theorem 3.1(i), so (ii) of Theorem 1.1 does not hold. Conversely, suppose that (ii) of Theorem 1.1 does not hold, i.e. there exists a nonzero $v \in \ker(Z)$ with $\mathrm{var}(v) \leq k - 1$. Then by Lemma 4.1(i), there exists $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ containing v. Since $Z(v) = 0$ we have $\dim(Z(V)) < k$, showing that (i) of Theorem 1.1 does not hold. The proof of the equivalence of (i) and (ii) in Theorem 1.3 is similar.

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