McKay Centralizer Algebras

Georgia Benkart[1†] and Tom Halverson[2‡]

[1] *University of Wisconsin – Madison, Madison, WI 53706 USA*
[2] *Macalester College, Saint Paul, MN 55105 USA*

Abstract. For a finite subgroup G of the special unitary group $\mathbf{SU}_2$, we study the centralizer algebra $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k})$ of G acting on the k-fold tensor product of its defining representation $\mathsf{V} = \mathbb{C}^2$. The McKay correspondence relates the representation theory of these groups to an associated affine Dynkin diagram, and we use this connection to study the structure and representation theory of $\mathsf{Z}_k(\mathsf{G})$ via the combinatorics of the Dynkin diagram. When G equals the binary tetrahedral, octahedral, or icosahedral group, we exhibit remarkable connections between $\mathsf{Z}_k(\mathsf{G})$ and the Martin-Jones set partition algebras.

Résumé. Pour un sous-groupe fini G du groupe unitaire spéciale SU_2, nous étudions la centralisateur algébre $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k})$ de G agissant sur le produit k-fold de tenseur de sa représentation définissant $\mathsf{V} = \mathbb{C}^2$. La correspondance de McKay concerne la théorie des représentations de ces groupes á une associé Dynkin diagramme, et nous utiliser cette connexion pour étudier la structure et la théorie des représentations de $\mathsf{Z}_k(\mathsf{G})$ par l'intermédiaire de la combinatoire du diagramme de Dynkin. Quand G est égale à la groupe tétraédrique binaire, octaédre binaire, ou icosaédrique binaire, nous exhibons connexions remarquables entre $\mathsf{Z}_k(\mathsf{G})$ et les algébres de partitions de Martin-Jones.

Keywords. Schur-Weyl duality, McKay correspondence, partition algebra, Temperley-Lieb algebra, Dyck paths

1 Introduction

In 1980, John McKay [13] discovered that there is a natural one-to-one correspondence between the finite subgroups of the special unitary group SU_2 and the simply-laced affine Dynkin diagrams. Let $\mathsf{V} = \mathbb{C}^2$ be the defining representation of SU_2, and let G be a finite subgroup of SU_2 with irreducible modules $\mathsf{G}^\lambda, \lambda \in \Lambda(\mathsf{G})$. The representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ (also known as the McKay graph or McKay quiver) has vertices indexed by the $\lambda \in \Lambda(\mathsf{G})$ and $a_{\lambda,\mu}$ edges from λ to μ if G^μ occurs in $\mathsf{G}^\lambda \otimes \mathsf{V}$ with multiplicity $a_{\lambda,\mu}$. Almost a century earlier, Felix Klein had determined that a finite subgroup of SU_2 must be one of the following: (a) a cyclic group $\mathbf{C}_n$ of order n, (b) a binary dihedral group $\mathbf{D}_n$ of order $4n$, or (c) one of the 3 exceptional groups: the binary tetrahedral group $\mathbf{T}$ of order 24, the binary octahedral group $\mathbf{O}$ of order 48, or the binary icosahedral group $\mathbf{I}$ of order 120. McKay's observation was that the representation graph of $\mathbf{C}_n, \mathbf{D}_n, \mathbf{T}, \mathbf{O}, \mathbf{I}$ corresponds exactly to the affine Dynkin diagram $\hat{\mathsf{A}}_{n-1}$, $\hat{\mathsf{D}}_{n+2}$, $\hat{\mathsf{E}}_6$, $\hat{\mathsf{E}}_7$, $\hat{\mathsf{E}}_8$.

†Email: benkart@math.wisc.edu. G. Benkart is grateful to the Simons Foundation for the support she received as a Simons Visiting Professor at MSRI.

‡Email: halverson@macalester.edu. T. Halverson was partially supported by National Science Foundation grant DMS-0800085 and partially supported by Simons Foundation grant 283311.

In the papers [1], [2], [7], we examine the McKay correspondence from the point of view of Schur-Weyl duality. Since the McKay graph provides a way to encode the rules for tensoring by V, it is natural to consider the k-fold tensor product module $\mathsf{V}^{\otimes k}$ and to study the centralizer algebra $\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_\mathsf{G}(\mathsf{V}^{\otimes k})$ of endomorphisms that commute with the action of G on $\mathsf{V}^{\otimes k}$. The algebra $\mathsf{Z}_k(\mathsf{G})$ provides essential information about the structure of $\mathsf{V}^{\otimes k}$ as a G-module, as the projection maps from $\mathsf{V}^{\otimes k}$ onto its irreducible G-summands are idempotents in $\mathsf{Z}_k(\mathsf{G})$, and the multiplicity of G^λ in $\mathsf{V}^{\otimes k}$ is the dimension of the $\mathsf{Z}_k(\mathsf{G})$-irreducible module corresponding to λ.

The main points of this article are:

- The irreducible $\mathsf{Z}_k(\mathsf{G})$-modules are labeled by vertices of the representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$.
- The dimension of the $\mathsf{Z}_k(\mathsf{G})$-module labeled by λ is the number of k-walks from 0 to λ on $\mathcal{R}_\mathsf{V}(\mathsf{G})$.
- The dimension of $\mathsf{Z}_k(\mathsf{G})$ equals the number of $2k$-walks on $\mathcal{R}_\mathsf{V}(\mathsf{G})$ starting and ending at 0.
- $\mathsf{Z}_k(\mathsf{G})$ has generators corresponding to the nodes in $\mathcal{R}_\mathsf{V}(\mathsf{G})$, as well as generators corresponding to each embedding $\mathsf{Z}_i(\mathsf{G}) \subseteq \mathsf{Z}_{i+1}(\mathsf{G})$, and the relations are determined by the $\mathcal{R}_\mathsf{V}(\mathsf{G})$ edge structure.
- $\mathsf{Z}_k(\mathsf{G})$ has a basis of words in these generators that correspond to $2k$-walks on $\mathcal{R}_\mathsf{V}(\mathsf{G})$.
- When G is one of the exceptional groups $\mathbf{T}, \mathbf{O}, \mathbf{I}$, the centralizer $\mathsf{Z}_k(\mathsf{G})$ can be described using the Martin-Jones partition algebras and their analogs.
- New formulas for the dimensions of the irreducible representations of partition algebras are given.

When G is a subgroup of SU_2, the centralizer algebras satisfy the reverse inclusion $\mathsf{Z}_k(\mathsf{SU}_2) \subseteq \mathsf{Z}_k(\mathsf{G})$. It is well known that $\mathsf{Z}_k(\mathsf{SU}_2)$ is isomorphic to the Temperley-Lieb algebra $\mathsf{TL}_k(2)$. Thus, the centralizer algebras constructed here all contain a Temperley-Lieb subalgebra. The dimension of $\mathsf{TL}_k(2)$ is the Catalan number $\mathcal{C}_k = \frac{1}{k+1}\binom{2k}{k}$, which counts walks of $2k$ steps that begin and end at 0 on the representation graph of SU_2, i.e. the Dynkin diagram $\mathsf{A}_{+\infty}$. In this case, the walks correspond to Dyck paths.

2 McKay Centralizer Algebras

The special unitary group SU_2 is the group of 2×2 complex matrices $\left(\begin{smallmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{smallmatrix}\right)$ satisfying $\alpha\bar\alpha + \beta\bar\beta = 1$. For each $r \geq 0$, SU_2 has an irreducible module $\mathsf{V}(r)$ of dimension $r+1$. The module $\mathsf{V} = \mathsf{V}(1) = \mathbb{C}^2$ with basis $\mathsf{v}_{-1} = (1,0)^\mathsf{t}, \mathsf{v}_1 = (0,1)^\mathsf{t}$ corresponds to the natural two-dimensional representation on which SU_2 acts by matrix multiplication. These modules satisfy the Clebsch-Gordan formula,

$$\mathsf{V}(r) \otimes \mathsf{V} = \mathsf{V}(r-1) \oplus \mathsf{V}(r+1), \tag{1}$$

where $\mathsf{V}(-1) = 0$. The representation graph $\mathcal{R}_\mathsf{V}(\mathsf{SU}_2)$ is the infinite graph shown in Figure 1.

Now let G be a subgroup of SU_2. Then G acts on the natural two-dimensional representation $\mathsf{V} = \mathbb{C}^2$ by restriction. Let $\{\, \mathsf{G}^\lambda \mid \lambda \in \Lambda(\mathsf{G}) \,\}$ denote a complete set of pairwise non-isomorphic irreducible finite-dimensional G-modules occurring in some $\mathsf{V}^{\otimes k}$ for $k = 0, 1, \ldots$. By convention $\mathsf{V}^{\otimes 0} = \mathsf{G}^{(0)}$ is the trivial G-module. The *representation graph* $\mathcal{R}_\mathsf{V}(\mathsf{G})$ is the graph with vertices labeled by elements of $\Lambda(\mathsf{G})$ with $a_{\lambda,\mu}$ edges between λ and μ if the decomposition of $\mathsf{G}^\lambda \otimes \mathsf{V}$ into irreducible G-modules is given by

$$\mathsf{G}^\lambda \otimes \mathsf{V} = \bigoplus_{\mu \in \Lambda(\mathsf{G})} a_{\lambda,\mu}\, \mathsf{G}^\mu. \tag{2}$$

For finite subgroups $\mathsf{G} \subseteq \mathsf{SU}_2$, the representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$ is an undirected, simple graph (see [15]). Since V is faithful (being the defining module for G) and G is finite, all irreducible G-modules

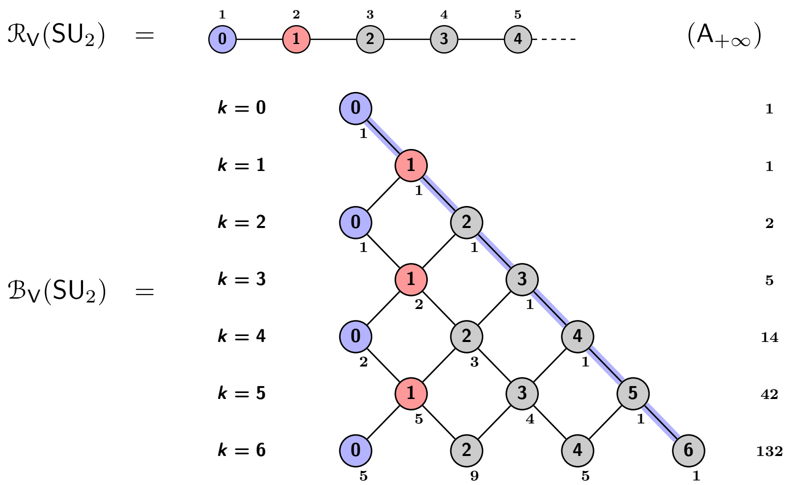

Fig. 1: The representation graph $\mathcal{R}_V(SU_2)$, which is the Dynkin diagram $A_{+\infty}$, and the first 6 levels of the corresponding Bratteli diagram $\mathcal{B}_V(SU_2)$. In $\mathcal{R}_V(SU_2)$, the label on the node is the index of the SU_2-module, and the label above the node is its dimension. The trivial module is shown in blue and the defining module V in red. In $\mathcal{B}_V(SU_2)$ the label below vertex r on level k gives the number of paths (Dyck paths in this case) from the top of the diagram to r, which is multiplicity of $V(r)$ in $V^{\otimes k}$. These numbers also count the walks of length k from 0 to r on $\mathcal{R}_V(G)$. The column to the right contains the sum of the squares of the multiplicities, which at level k is the Catalan number $\mathcal{C}_k$ and equals the dimension of the centralizer algebra $Z_k(SU_2)$.

occur in some $V^{\otimes k}$, and thus $\mathcal{R}_V(G)$ is connected. The representation graphs $\mathcal{R}_V(G)$ corresponding to $G = \mathbf{C}_n, \mathbf{D}_n, \mathbf{T}, \mathbf{O}, \mathbf{I}$, are displayed in Figure 2. McKay observed that these graphs correspond exactly to the affine Dynkin diagrams of type $\hat{A}_{n-1}$, $\hat{D}_{n+2}$, $\hat{E}_6$ $\hat{E}_7$, $\hat{E}_8$, respectively. The trivial module $G^{(0)}$ corresponds to the affine node in those cases.

For $k \geq 1$, the k-fold tensor power $V^{\otimes k}$ is 2^k-dimensional and has a basis of simple tensors $V^{\otimes k} = \text{span}_{\mathbb{C}} \{ v_{r_1} \otimes v_{r_2} \otimes \cdots \otimes v_{r_k} \mid r_j \in \{-1, 1\} \}$. Group elements $g \in G$ act on simple tensors by the diagonal action $g(v_{r_1} \otimes v_{r_2} \otimes \cdots \otimes v_{r_k}) = gv_{r_1} \otimes gv_{r_2} \otimes \cdots \otimes gv_{r_k}$. Let

$$\Lambda_k(G) = \{ \lambda \in \Lambda(G) \mid G^\lambda \text{ appears as a summand in the decomposition of } V^{\otimes k} \}. \tag{3}$$

Then, $\Lambda_k(G)$ is the set of vertices in $\mathcal{R}_V(G)$ that can be reached by paths of length k starting from 0. Furthermore, $\Lambda_k(G) \subseteq \Lambda_{k+2}(G)$, for all $k \geq 0$, since if a node can be reached in k steps, then it can also be reached in $k+2$ steps. The *Bratteli diagram* $\mathcal{B}_V(G)$ is the infinite graph with vertices labeled by $\Lambda_k(G)$ on level k and $a_{\lambda,\mu}$ edges from vertex $\lambda \in \Lambda_k(G)$ to vertex $\mu \in \Lambda_{k+1}(G)$. The Bratteli diagram for $G = SU_2$ is shown in Figure 1, and the Bratteli diagrams for $G = \mathbf{C}_n, \mathbf{D}_n, \mathbf{T}, \mathbf{O}, \mathbf{I}$ are shown in Figure 3.

A *walk* of length k on the representation graph $\mathcal{R}_V(G)$ from 0 to $\lambda \in \Lambda(G)$, is a sequence $(0, \lambda^1, \lambda^2, \ldots, \lambda^k = \lambda)$ starting at $\lambda^0 = 0$, such that $\lambda^j \in \Lambda(G)$ for each $1 \leq j \leq k$, and λ^{j-1} is connected to λ^j by an edge in $\mathcal{R}_V(G)$. Such a walk is equivalent to a unique *path* of length k on the Bratteli diagram $\mathcal{B}_V(G)$ from $0 \in \Lambda_0(G)$ to $\lambda \in \Lambda_k(G)$. Let $\mathcal{W}_k^\lambda(G)$ denote the set of walks on $\mathcal{R}_V(G)$ of length k from $0 \in \Lambda(G)$

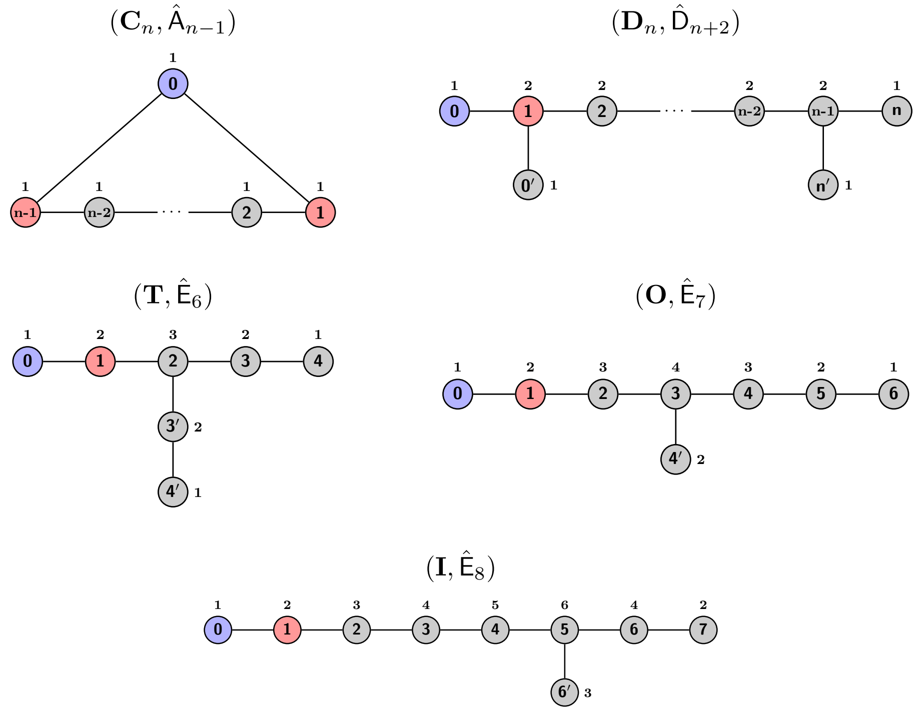

Fig. 2: The representation graphs $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ for the finite subgroups $\mathsf{G} = \mathbf{C}_n, \mathbf{D}_n, \mathbf{T}, \mathbf{O}, \mathbf{I}$ correspond to the affine Dynkin diagrams of type $\hat{\mathsf{A}}_{n-1}, \hat{\mathsf{D}}_{n+1}, \hat{\mathsf{E}}_6, \hat{\mathsf{E}}_7, \hat{\mathsf{E}}_8$. The label on the node is the index of the G-module, and the label above the node is its dimension. The trivial module (affine node) is blue and the defining module V is red.

to $\lambda \in \Lambda(\mathsf{G})$, and let $\mathcal{P}_k^\lambda(\mathsf{G})$ denote the set of paths on $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$ of length k from $0 \in \Lambda_0(\mathsf{G})$ to $\lambda \in \Lambda_k(\mathsf{G})$. Let m_k^λ denote the multiplicity of G^λ in $\mathsf{V}^{\otimes k}$. Then, by induction on (2) we have

$$\begin{aligned} m_k^\lambda &= |\mathcal{W}_k^\lambda(\mathsf{G})| = \#(\text{walks on } \mathcal{R}_{\mathsf{V}}(\mathsf{G}) \text{ of length } k \text{ from at } 0 \text{ to } \lambda) \\ &= |\mathcal{P}_k^\lambda(\mathsf{G})| = \#(\text{paths in } \mathcal{B}_{\mathsf{V}}(\mathsf{G}) \text{ of length } k \text{ from } 0 \in \Lambda_0(\mathsf{G}) \text{ to } \lambda \in \Lambda_k(\mathsf{G})). \end{aligned} \tag{4}$$

The *centralizer of* G *on* $\mathsf{V}^{\otimes k}$ is the algebra

$$\mathsf{Z}_k(\mathsf{G}) = \mathsf{End}_{\mathsf{G}}(\mathsf{V}^{\otimes k}) = \left\{ a \in \mathsf{End}(\mathsf{V}^{\otimes k}) \;\middle|\; a(gw) = ga(w) \text{ for all } g \in \mathsf{G}, w \in \mathsf{V}^{\otimes k} \right\}. \tag{5}$$

If the group G is apparent from the context, we will simply write Z_k for $\mathsf{Z}_k(\mathsf{G})$. Since $\mathsf{V}^{\otimes 0} = \mathsf{G}^{(0)}$, we have $\mathsf{Z}_0(\mathsf{G}) = \mathbb{C}1$. There is a natural embedding $\mathsf{Z}_k(\mathsf{G}) \hookrightarrow \mathsf{Z}_{k+1}(\mathsf{G})$ given by $a \mapsto a \otimes \mathbf{1}$, where $a \otimes \mathbf{1}$ acts as a on the first k tensor factors and $\mathbf{1}$ acts as the identity in the $(k+1)$st tensor position. Iterating this embedding gives an infinite tower of algebras $\mathsf{Z}_0(\mathsf{G}) \subseteq \mathsf{Z}_1(\mathsf{G}) \subseteq \mathsf{Z}_2(\mathsf{G}) \subseteq \cdots$.

By classical double-centralizer theory (see for example [6, 3B,68]), we have:

- $\mathsf{Z}_k(\mathsf{G})$ is a semisimple $\mathbb{C}$-algebra with irreducible modules $\left\{\mathsf{Z}_k^\lambda \;\middle|\; \lambda \in \Lambda_k(\mathsf{G})\right\}$ labeled by $\Lambda_k(\mathsf{G})$.

- $\mathsf{dim}\, \mathsf{Z}_k^\lambda = m_k^\lambda = |\mathcal{W}_k^\lambda(\mathsf{G})| = |\mathcal{P}_k^\lambda(\mathsf{G})|$.
- Edges from level k to $k-1$ in $\mathcal{B}_\mathsf{V}(\mathsf{G})$ represent restriction and induction rules for $\mathsf{Z}_{k-1}(\mathsf{G}) \subseteq \mathsf{Z}_k(\mathsf{G})$.
- If $d^\lambda = \mathsf{dim}\, \mathsf{G}^\lambda$, then the tensor space $\mathsf{V}^{\otimes k}$ has the following decomposition

$$\mathsf{V}^{\otimes k} \cong \underbrace{\bigoplus_{\lambda \in \Lambda_k(\mathsf{G})} m_k^\lambda\, \mathsf{G}^\lambda}_{\text{as a } \mathsf{G}\text{-module}} \cong \underbrace{\bigoplus_{\lambda \in \Lambda_k(\mathsf{G})} d^\lambda\, \mathsf{Z}_k^\lambda}_{\text{as a } \mathsf{Z}_k(\mathsf{G})\text{-module}} \cong \underbrace{\bigoplus_{\lambda \in \Lambda_k(\mathsf{G})} \left(\mathsf{G}^\lambda \otimes \mathsf{Z}_k^\lambda\right)}_{\text{as a } (\mathsf{G}, \mathsf{Z}_k(\mathsf{G}))\text{-bimodule}} . \tag{6}$$

- By general Wedderburn theory, the dimension of $\mathsf{Z}_k(\mathsf{G})$ is

$$\mathsf{dim}\, \mathsf{Z}_k(\mathsf{G}) = \sum_{\lambda \in \Lambda_k(\mathsf{G})} (m_k^\lambda)^2 = \sum_{\lambda \in \Lambda_k(\mathsf{G})} |\mathcal{W}_k^\lambda(\mathsf{G})|^2 = \left|\mathcal{W}_{2k}^0(\mathsf{G})\right| = \mathsf{dim}\, \mathsf{Z}_{2k}^{(0)}, \tag{7}$$

  which equals the number of walks of length $2k$ that begin and end at 0 on $\mathcal{R}_\mathsf{V}(\mathsf{G})$. The third equality follows from the property that a pair of walks of length k from 0 to λ corresponds uniquely (by reversing the second walk) to a walk of length $2k$ beginning and ending at 0.

3 Paths and Dimensions

Using an inductive argument on the structure of the Bratteli diagram, we compute the dimensions of the irreducible $\mathsf{Z}_k(\mathsf{G})$-modules Z_k^λ for $\lambda \in \Lambda_k(\mathsf{G})$. They are given explicitly in [1]. This dimension also equals the multiplicity of G^λ in $\mathsf{V}^{\otimes k}$. The dimension of the centralizer algebra $\mathsf{Z}_k(\mathsf{G})$ is then the sum of the squares of these dimensions: $\mathsf{dim}\, \mathsf{Z}_k(\mathsf{G}) = \sum_\lambda \mathsf{dim}\, (\mathsf{Z}_k^\lambda)^2$.

Theorem 8 ([1] Dimension Formulas) *For $k \geq 1$, the following formulas give the dimension $\mathsf{dim}\, \mathsf{Z}_k(\mathsf{G})$ of the McKay centralizer algebra, which also equals the number of $2k$-walks on the representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$ from 0 to 0.*

(a) $\mathsf{dim}\, \mathsf{Z}_k(\mathbf{C}_n) = 2\, \mathsf{dim}\, \mathsf{Z}_k(\mathbf{D}_n) = \sum_{\substack{0 \leq a,b \leq k \\ a \equiv b \bmod \tilde{n}}} \binom{k}{a}\binom{k}{b}$ = *coefficient of z^k in $(1+z)^{2k}\big|_{z^{\tilde{n}}=1}$ which equals the the $2k$-k coefficient in Pascal's triangle on a cylinder of "diameter" $\tilde{n}$ (Fig. 3), where $\tilde{n} = n$, if n is odd, and $\tilde{n} = \frac{1}{2}n$, if n is even.*

(b) $\mathsf{dim}\, \mathsf{Z}_k(\mathbf{T}) = \dfrac{4^k + 8}{12}$ ([14] OEIS sequence A047849).

(c) $\mathsf{dim}\, \mathsf{Z}_k(\mathbf{O}) = \dfrac{4^k + 6 \cdot 2^k + 8}{24}$ ([14] OEIS sequence A007581).

(d) $\mathsf{dim}\, \mathsf{Z}_k(\mathbf{I}) = \dfrac{4^k + 12L_{2k} + 20}{60}$*, where L_n is the Lucas number defined by $L_0 = 2, L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$.*

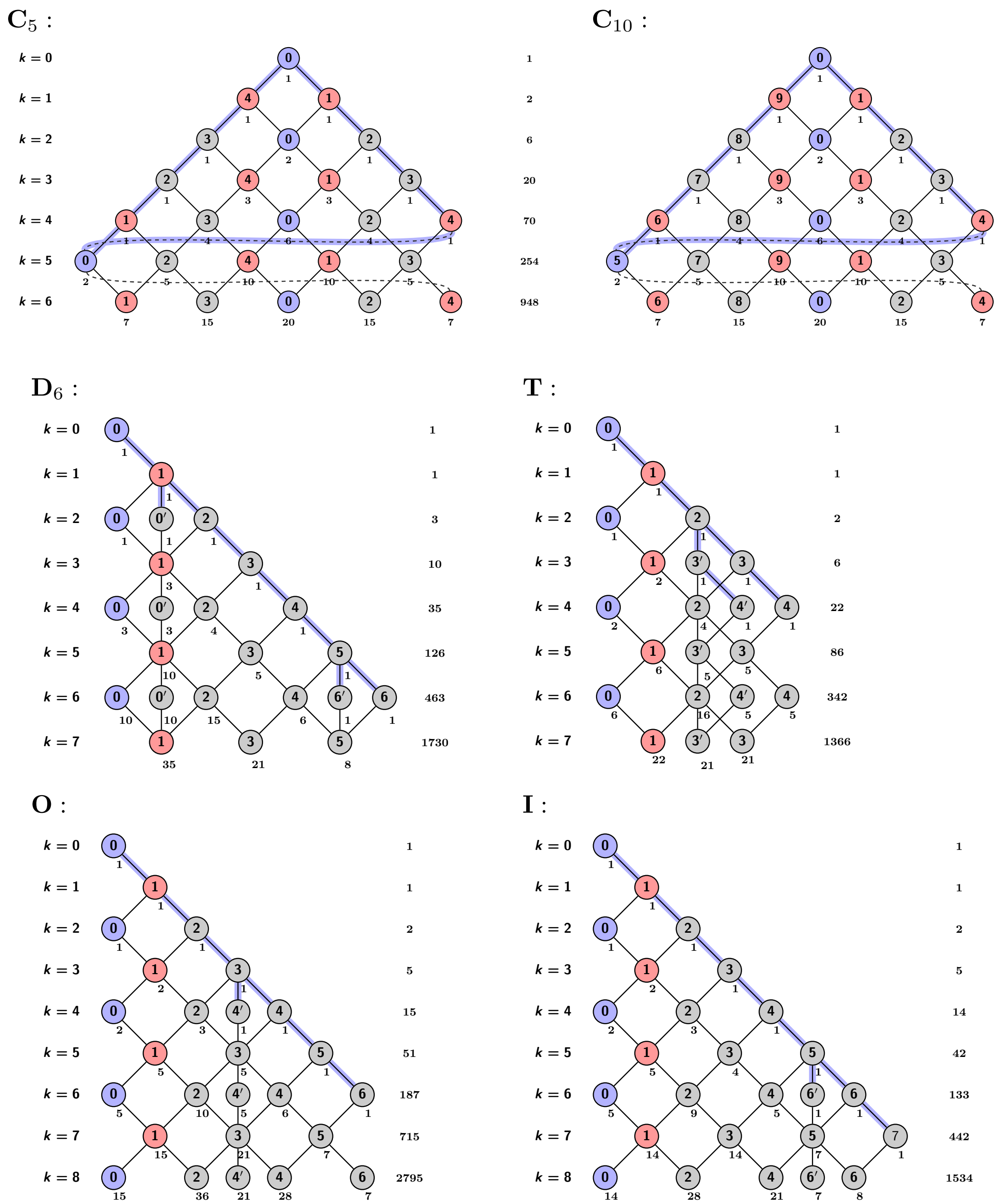

Fig. 3: The first several rows of the Bratteli diagrams $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$ for $\mathsf{G} = \mathbf{C}_5, \mathbf{C}_{10}, \mathbf{D}_6, \mathbf{T}, \mathbf{O}, \mathbf{I}$. The representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ is embedded as the shaded edges. The unshaded edges are reflections from the row above and correspond to the Jones basic construction ideal $\mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k$ (see Sec. 4). The Bratteli diagrams for $\mathbf{C}_5$ and $\mathbf{C}_{10}$ are isomorphic. The label below vertex r on level k gives the number of k-paths from the top of the diagram to r, which is also multiplicity of $\mathsf{G}^{(r)}$ in $\mathsf{V}^{\otimes k}$. These numbers also give the number of k-walks from 0 to r on the representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$. The column to the right contains the sum of the squares of the multiplicities which equals $\dim \mathsf{Z}_k(\mathsf{G})$.

4 Basic Construction

In this section we define two kinds of (essential) idempotents $\{\mathsf{e}_i \mid 1 \le i \le k-1\}$ and $\{\mathsf{f}_\nu \mid \nu \in \Lambda(\mathsf{G})\}$ which together with 1 generate $\mathsf{Z}_k(\mathsf{G})$. For $1 \le i \le k-1$, define an endomorphism e_i on $\mathsf{V}^{\otimes k}$ by

$$\mathsf{e}_i = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \mathsf{e} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \tag{9}$$

where $\mathbf{1}$ is the 2×2 identity matrix, and $\mathsf{e} : \mathsf{V} \otimes \mathsf{V} \to \mathsf{V} \otimes \mathsf{V}$ acts in tensor positions i and $i+1$ by $\mathsf{e}(\mathsf{v}_j \otimes \mathsf{v}_\ell) = \mathsf{v}_j \otimes \mathsf{v}_\ell - \mathsf{v}_\ell \otimes \mathsf{v}_j$, for $j, \ell \in \{-1, 1\}$. Thus, $\mathsf{e} : \mathsf{V}^{\otimes 2} \to \mathsf{V}^{\otimes 2}$ projects onto the antisymmetric tensors in $\mathsf{V}^{\otimes 2}$, and it is easy to confirm that $\mathsf{e}_i \in \mathsf{Z}_k(\mathsf{G})$.

For $\nu \in \Lambda(\mathsf{G})$, we let $|\nu|$ equal the distance from 0 to ν in the representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$. Thus by the way we have chosen our labels in Figure 3, $|(\ell)| = |(\ell')| = \ell$. Correspondingly, the module G^ν first appears as a constituent of $\mathsf{V}^{\otimes k}$, when $k = |\nu|$, and it appears in that tensor product with multiplicity exactly 1. Define

$$\mathsf{f}_\nu := \text{the } \mathsf{G}\text{-module homomorphism projecting onto the unique copy of } \mathsf{G}^\nu \text{ in } \mathsf{V}^{\otimes |\nu|}.$$

In [1] we show how to explicitly construct f_ν for each $\nu \in \Lambda(\mathsf{G})$. In particular, we show that

$$\mathsf{f}_\nu - \frac{d^{\nu-1}}{d^\nu} \mathsf{f}_\nu \mathsf{e}_{|\nu|} \mathsf{f}_\nu = \sum_{\mu = \nu+1} \mathsf{f}_\mu, \tag{10}$$

where $\nu - 1 \in \Lambda(\mathsf{G})$ is the unique neighbor of ν in $\mathcal{R}_\mathsf{V}(\mathsf{G})$ that is closer to 0 (i.e., $|\nu - 1| = |\nu| - 1$), where the sum is over the neighbors $\mu = \nu + 1$ of ν in $\mathcal{R}_\mathsf{V}(\mathsf{G})$ that are farther from 0 (i.e., $|\mu| = |\nu| + 1$), and where $d^\lambda = \dim \mathsf{G}^\lambda$. In most cases, ν has degree 2, and there is a unique node of the form $\mu = \nu + 1$, and thus (10) defines f_μ uniquely. If ν has degree greater than 2, then we use other methods [1, Sec. 1.8] to decompose the sum into the constituent f_μ. This recursive construction is a generalization of the construction of the Jones-Wenzl idempotent for SU_2.

A *branch node* in the representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$ is any vertex of degree greater than 2. Let $\mathsf{br}(\mathsf{G})$ denote the set of branch nodes in $\mathcal{R}_\mathsf{V}(\mathsf{G})$. In the special case of $\mathcal{R}_\mathsf{V}(\mathbf{C}_n)$ for $n \le \infty$, we consider the affine node to be the branch node. Let the *diameter* of $\mathcal{R}_\mathsf{V}(\mathsf{G})$, denoted by $\mathsf{diam}(\mathsf{G})$, be the maximum distance between any vertex $\lambda \in \Lambda(\mathsf{G})$ and $0 \in \Lambda(\mathsf{G})$. For $\mathsf{G} = \mathbf{C}_n$, we let $\mathsf{diam}(\mathsf{G}) = \tilde{n}$ as in (11).

G	SU_2	$\mathbf{C}_n$	$\mathbf{D}_n$	$\mathbf{T}$	$\mathbf{O}$	$\mathbf{I}$	$\mathbf{C}_\infty$	$\mathbf{D}_\infty$
$\mathsf{diam}(\mathsf{G})$	∞	$\tilde{n}$	n	4	6	7	∞	∞
$\mathsf{br}(\mathsf{G})$	$\emptyset$	$\{0\}$	$\{1, n\}$	$\{2\}$	$\{3\}$	$\{5\}$	$\{0\}$	$\{1\}$

$$\tilde{n} = \begin{cases} \frac{1}{2}n, & \text{if } n \text{ is even,} \\ n, & \text{if } n \text{ is odd.} \end{cases} \tag{11}$$

The *Jones basic construction* (see [8], [16], [9]) uses the ideal $\mathsf{Z}_k \mathsf{e}_k \mathsf{Z}_k \subseteq \mathsf{Z}_{k+1}$ to recursively study the structure of Z_{k+1}. We use it to prove the following theorem.

Theorem 12 ([1] Basic Construction) *Let* $\mathsf{Z}_k = \mathsf{Z}_k(\mathsf{G})$ *and let* $\mathsf{Z}_0 = \mathbb{C}1$. *Then for* $k \ge 1$, *except for the two special cases in part (d), we have*

(a) $\mathsf{Z}_{k+1} = \mathsf{Z}_k \mathsf{e}_k \mathsf{Z}_k \oplus \mathsf{N}_{k+1}$, *where* $\mathsf{Z}_k \mathsf{e}_k \mathsf{Z}_k$ *is a two-sided basic construction ideal, and* N_{k+1} *is a commutative subalgebra spanned by* $\{\, \mathsf{f}_\nu \mid |\nu| = k+1\}$. *In particular,* $\mathsf{N}_{k+1} = 0$ *if* $k \ge \mathsf{diam}(\mathsf{G})$.

(b) $\mathsf{Z}_{k+1} = \langle \mathsf{Z}_k, \mathsf{e}_k \rangle$ *if* $k \notin \mathsf{br}(\mathsf{G})$, *and* $\mathsf{Z}_{k+1} = \langle \mathsf{Z}_k, \mathsf{e}_k, \mathsf{f}_{(k+1)} \rangle$ *if* $k \in \mathsf{br}(\mathsf{G})$.

(c) Z_{k+1} *is generated by* $\{1, \mathsf{e}_1, \ldots, \mathsf{e}_k\} \cup \{\mathsf{f}_{(\ell+1)} \mid \ell \in \mathsf{br}(\mathsf{G}), \ell \leq k\}$.

(d) $\mathsf{TL}_k(2) \subseteq \mathsf{Z}_k$ *for all* $k \geq 0$*, and* $\mathsf{TL}_k(2) = \mathsf{Z}_k$ *for* $0 \leq k \leq \min(\mathsf{br}(\mathsf{G}))$.

(e) *Two special cases:* (i) *if* $\mathsf{G} = \mathbf{C}_n$*, then* $\mathsf{Z}_{\tilde{n}} = \langle \mathsf{Z}_{\tilde{n}-1}, \mathsf{e}_{\tilde{n}-1}, \mathsf{E}_{\mathsf{p},\mathsf{q}}, \text{for } \mathsf{p}, \mathsf{q} \in \{-\underline{1}, \underline{1}\} \rangle$*, where* $\mathsf{E}_{\mathsf{p},\mathsf{q}}$ *are matrix units.* (ii) *if* $\mathsf{G} = \mathbf{D}_2$*, then* $\mathsf{Z}_2 = \langle \mathsf{Z}_1, \mathsf{e}_1, \mathsf{f}_{\mu_1}, \mathsf{f}_{\mu_2} \rangle$ *where* $\mu_1, \mu_2 \in \{(0'), (2), (2')\}$*,* $\mu_1 \neq \mu_2$.

Example 13 Compare these examples with their representation graphs $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ in Figure 2.

(a) If $\mathsf{G} = \mathbf{O}$, then $\mathsf{br}(\mathbf{O}) = \{3\}$, $\mathsf{diam}(\mathbf{O}) = 6$, and

$$\begin{array}{ll}
\mathsf{Z}_1 = \mathbb{C}\mathbf{1} = \mathbb{C}\mathsf{f}_{(1)} = \mathsf{Z}_0 \cong \mathsf{TL}_1(2) & \mathsf{Z}_5 = \mathsf{Z}_4\mathsf{e}_4\mathsf{Z}_4 \oplus \mathbb{C}\mathsf{f}_{(5)} = \langle \mathsf{Z}_4, \mathsf{e}_4 \rangle \\
\mathsf{Z}_2 = \mathsf{Z}_1\mathsf{e}_1\mathsf{Z}_1 \oplus \mathbb{C}\mathsf{f}_{(2)} = \langle \mathsf{Z}_1, \mathsf{e}_1 \rangle \cong \mathsf{TL}_2(2) & \mathsf{Z}_6 = \mathsf{Z}_5\mathsf{e}_5\mathsf{Z}_5 \oplus \mathbb{C}\mathsf{f}_{(6)} = \langle \mathsf{Z}_5, \mathsf{e}_5 \rangle \\
\mathsf{Z}_3 = \mathsf{Z}_2\mathsf{e}_2\mathsf{Z}_2 \oplus \mathbb{C}\mathsf{f}_{(3)} = \langle \mathsf{Z}_2, \mathsf{e}_2 \rangle \cong \mathsf{TL}_3(2) & \mathsf{Z}_7 = \mathsf{Z}_6\mathsf{e}_6\mathsf{Z}_6 = \langle \mathsf{Z}_6, \mathsf{e}_6 \rangle \\
\mathsf{Z}_4 = \mathsf{Z}_3\mathsf{e}_3\mathsf{Z}_3 \oplus \mathbb{C}\mathsf{f}_{(4)} \oplus \mathbb{C}\mathsf{f}_{(4')} = \langle \mathsf{Z}_3, \mathsf{e}_3, \mathsf{f}_{(4)} \rangle & \mathsf{Z}_{k+1} = \mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k = \langle \mathsf{Z}_k, \mathsf{e}_k \rangle, k \geq 6.
\end{array}$$

(b) If $\mathsf{G} = \mathbf{D}_6$, then $\mathsf{br}(\mathbf{D}_6) = \{1, 5\}$, $\mathsf{diam}(\mathbf{D}_6) = 6$, and

$$\begin{array}{ll}
\mathsf{Z}_1 = \mathbb{C}\mathbf{1} = \mathbb{C}\mathsf{f}_{(1)} = \mathsf{Z}_0 \cong \mathsf{TL}_1(2) & \mathsf{Z}_5 = \mathsf{Z}_4\mathsf{e}_4\mathsf{Z}_4 \oplus \mathbb{C}\mathsf{f}_{(5)} = \langle \mathsf{Z}_4, \mathsf{e}_4 \rangle \\
\mathsf{Z}_2 = \mathsf{Z}_1\mathsf{e}_1\mathsf{Z}_1 \oplus \mathbb{C}\mathsf{f}_{(0')} \oplus \mathbb{C}\mathsf{f}_{(2)} = \langle \mathsf{Z}_1, \mathsf{e}_1, \mathsf{f}_{(2)} \rangle & \mathsf{Z}_6 = \mathsf{Z}_5\mathsf{e}_5\mathsf{Z}_5 \oplus \mathbb{C}\mathsf{f}_{(6')} \oplus \mathbb{C}\mathsf{f}_{(6)} = \langle \mathsf{Z}_5, \mathsf{e}_5, \mathsf{f}_{(6)} \rangle \\
\mathsf{Z}_3 = \mathsf{Z}_2\mathsf{e}_2\mathsf{Z}_2 \oplus \mathbb{C}\mathsf{f}_{(3)} = \langle \mathsf{Z}_2, \mathsf{e}_2 \rangle & \mathsf{Z}_7 = \mathsf{Z}_6\mathsf{e}_6\mathsf{Z}_6 = \langle \mathsf{Z}_6, \mathsf{e}_6 \rangle \\
\mathsf{Z}_4 = \mathsf{Z}_3\mathsf{e}_3\mathsf{Z}_3 \oplus \mathbb{C}\mathsf{f}_{(4)} = \langle \mathsf{Z}_3, \mathsf{e}_3 \rangle & \mathsf{Z}_{k+1} = \mathsf{Z}_k\mathsf{e}_k\mathsf{Z}_k = \langle \mathsf{Z}_k, \mathsf{e}_k \rangle, k \geq 6.
\end{array}$$

5 Linear Bases

Let $\mathcal{P}^0_{2k}(\mathsf{G})$ denote the set of paths on $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$ of length $2k$ from 0 at level 0 to 0 at level $2k$. Then $\dim \mathsf{Z}_k(\mathsf{G}) = |\mathcal{P}^0_{2k}(\mathsf{G})|$, and so it is natural to seek a basis $\{\, w_p \mid p \in \mathcal{P}^0_{2k}(\mathsf{G})\}$ of $\mathsf{Z}_k(\mathsf{G})$ where each w_p is a word in the generators $\{1, \mathsf{e}_1, \ldots, \mathsf{e}_{k-1}\} \cup \{\, \mathsf{f}_{(\ell+1)} \mid \ell \in \mathsf{br}(\mathsf{G}), \ell < k\}$. For example, when $\mathsf{G} = \mathsf{SU}_2$, the centralizer is the Temperley-Lieb algebra $\mathsf{Z}_k(\mathsf{SU}_2) \cong \mathsf{TL}_k(2)$, the dimension is the Catalan number $\mathcal{C}_k$, the paths are Dyck paths, and the bijection to a basis of words in $\mathsf{e}_1, \ldots, \mathsf{e}_{k-1}$ is given in [8, 2.8]. Here we generalize this result uniformly to the finite subgroups $\mathsf{G} \subseteq \mathsf{SU}_2$.

For $\ell = \min(\mathsf{br}(\mathsf{G}))$, i.e., the first branch point, define

$$\mathsf{b}_i = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(i-1)\text{ factors}} \otimes \mathsf{f}_{(\ell+1)} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(k-i-\ell)\text{ factors}}, \qquad 1 \leq i \leq k - \ell, \qquad \text{for each } \mathsf{G},$$

and define

$$\mathsf{c}_i = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(i-1)\text{ factors}} \otimes \mathsf{f}_{(n)} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(k-i-n+1)\text{ factors}}, \qquad 1 \leq i \leq k - n + 1, \quad \text{for } \mathsf{G} = \mathbf{D}_n.$$

A path $p \in \mathcal{P}^0_{2k}$ is a sequence $p = (0 = p_0, p_1, \ldots, p_{2k-1}, p_{2k} = 0)$, where each $p_i \in \Lambda_i(\mathsf{G})$ is a label of an irreducible G-module that appears in $\mathsf{V}^{\otimes i}$. A *peak* in a path p is a label p_i such that $|p_{i-1}| < |p_i|$ and $|p_i| > |p_{i+1}|$. If a peak p_i is marked with the prime symbol (e.g., $0'_i$ or $4'_i$, etc), then p_i is a *nonstandard* peak. Otherwise it is said to be *standard*. To each peak p_i we associate a product of generators called *block* as follows

$$\mathcal{B}(p_i) = \mathsf{e}_\alpha \mathsf{e}_{\alpha-1} \mathsf{e}_{\alpha-2} \cdots \mathsf{e}_\beta, \text{ with } \alpha = \frac{i + |p_i| - 2}{2} \text{ and } \beta = \frac{i - |p_i| + 2}{2}, \tag{14}$$

$$\mathcal{B}(p'_i) = \mathsf{b}_\beta, \text{ with } \beta = \frac{i - |p_i| + 2}{2}, \tag{15}$$

where the b_β is replaced by a c_β in $\mathbf{D}_n$ if $p_i' = n_i'$. When $\mathsf{G} = \mathbf{T}$, we have the following special case,

$$(\mathsf{G} = \mathbf{T}) \qquad \mathcal{B}(4_i') = \begin{cases} \mathsf{b}_{\alpha-2} e_\alpha \mathsf{e}_{\alpha-1}\mathsf{e}_{\alpha-2}, \text{ with } \alpha = \frac{i+2}{2}, & \text{if } p_{i-2} = 2, \\ \mathsf{e}_\alpha \mathsf{e}_{\alpha-1}\mathsf{e}_{\alpha-2}, \text{ with } \alpha = \frac{i+2}{2}, & \text{if } p_{i-2} = 4. \end{cases} \tag{16}$$

If $\mathsf{G} = \mathbf{C}_n$ with $n \leq \infty$ then there are further special cases for $\mathcal{B}(p_i')$ and $\mathcal{B}(\tilde{n}_i^{\pm})$ described in [7].

If $p \in \mathcal{P}_{2k}^0(\mathsf{G})$ is a path in the Bratteli diagram, then we define the *word* w_p of p as the product blocks for each peak in p (see Example 19):

$$w_p = \mathcal{B}(p_{i_1})\mathcal{B}(p_{i_2})\cdots\mathcal{B}(p_{i_\ell}), \qquad \text{where } p_{i_1}, p_{i_2}, \ldots, p_{i_\ell} \text{ are the peaks in } p. \tag{17}$$

Theorem 18 ([7] Basis Theorem) *For* $\mathsf{G} = \mathsf{SU}_2, \mathbf{C}_n, \mathbf{D}_n, \mathbf{C}_\infty, \mathbf{D}_\infty, \mathbf{T}, \mathbf{O}$, *or* $\mathbf{I}$, *and* $k \geq 0$, *the set* $\{\, w_p \mid p \in \mathcal{P}_{2k}^0(\mathsf{G}) \,\}$ *is a basis for* $\mathsf{Z}_k(\mathsf{G})$.

Example 19 (Paths and their Corresponding Words) *The following are examples of paths p in the Bratteli diagram* $\mathcal{B}_{\mathsf{V}}(\mathsf{G})$ *with their peaks circled and the corresponding words* w_p.

(a) *In* $\mathsf{Z}_{10}(\mathbf{O})$

$$p = (0_0, 1_1, 2_2, 3_3, 4_4, \enclose{circle}{5_5}, 4_6, 3_7, 4_8, 5_9, \enclose{circle}{6_{10}}, 5_{11}, 4_{12}, 3_{13}, 2_{14}, 1_{15}, 2_{16}, \enclose{circle}{3_{17}}, 2_{18}, 1_{19}, 0_{20})$$

$$w_p = (\mathsf{e}_4\mathsf{e}_3\mathsf{e}_2\mathsf{e}_1)\,(\mathsf{e}_7\mathsf{e}_6\mathsf{e}_5\mathsf{e}_3\mathsf{e}_3)\,(\mathsf{e}_9\mathsf{e}_8)$$

$$p = (0_0, 1_1, 2_2, 3_3, \enclose{circle}{4_4}, 3_5, 4_6, 5_7, \enclose{circle}{6_8}, 5_9, 4_{10}, 3_{11}, \enclose{circle}{4_{12}'}, 3_{13}, 2_{14}, 1_{15}, 0_{16}, 1_{17}, \enclose{circle}{2_{18}}, 1_{19}, 0_{20})$$

$$w_p = (\mathsf{e}_3\mathsf{e}_2\mathsf{e}_1)\,(\mathsf{e}_6\mathsf{e}_5\mathsf{e}_4\mathsf{e}_3\mathsf{e}_2)\,(\mathsf{b}_3)\,(\mathsf{e}_9)$$

(b) *In* $\mathsf{Z}_{10}(\mathbf{T})$

$$p = (0_0, 1_1, 2_2, \enclose{circle}{3_3}, 2_4, \enclose{circle}{3_5}, 2_6, 1_7, 0_8, 1_9, 2_{10}, 3_{11}', \enclose{circle}{4_{12}'}, 3_{13}', 2_{14}, 3_{15}, \enclose{circle}{4_{16}}, 3_{17}, 2_{18}, 1_{19}, 0_{20})$$

$$w_p = (\mathsf{e}_2\mathsf{e}_1)\,(\mathsf{e}_3\mathsf{e}_2)\,(\mathsf{b}_5\mathsf{e}_7\mathsf{e}_6\mathsf{e}_5)\,(\mathsf{e}_9\mathsf{e}_8\mathsf{e}_7)$$

$$p = (0_0, 1_1, 2_2, 3_3, \enclose{circle}{4_4}, 3_5, \enclose{circle}{4_6}, 3_7, \enclose{circle}{4_8}, 3_9, 2_{10}, 1_{11}, 0_{12}, 1_{13}, 2_{14}, 3_{15}', \enclose{circle}{4_{16}'}, 3_{17}', 2_{18}, 1_{19}, 0_{20})$$

$$w_p = (\mathsf{e}_3\mathsf{e}_2\mathsf{e}_1)\,(\mathsf{e}_4\mathsf{e}_3\mathsf{e}_2)\,(\mathsf{e}_5\mathsf{e}_4\mathsf{e}_3)\,(\mathsf{b}_7\mathsf{e}_9\mathsf{e}_8\mathsf{e}_7)$$

(c) *In* $\mathsf{Z}_{10}(\mathbf{D}_5)$

$$p = (0_1, 1_1, \enclose{circle}{2_2'}, 1_3, 2_4, 3_5, 4_6, \enclose{circle}{5_7}, 4_8, \enclose{circle}{5_9'}, 4_{10}, 3_{11}, 2_{12}, 1_{13}, 0_{14}, 1_{15}, \enclose{circle}{2_{16}}, 1_{17}, \enclose{circle}{2_{18}'}, 1_{19}, 0_{20})$$

$$w_p = (\mathsf{b}_1)\,(\mathsf{e}_5\mathsf{e}_4\mathsf{e}_3\mathsf{e}_2)\,(\mathsf{c}_3)\,(\mathsf{e}_8)\,(\mathsf{b}_9)$$

$$p = (0_0, 1_1, 2_2, 3_3, \enclose{circle}{4_4}, 3_5, 4_6, \enclose{circle}{5_7}, 4_8, 3_9, 2_{10}, 3_{11}, \enclose{circle}{4_{12}}, 3_{13}, 2_{14}, 1_{15}, 2_{16}, \enclose{circle}{3_{17}}, 2_{18}, 1_{19}, 0_{20})$$

$$w_p = (\mathsf{e}_3\mathsf{e}_2\mathsf{e}_1)\,(\mathsf{e}_5\mathsf{e}_4\mathsf{e}_3\mathsf{e}_2)\,(\mathsf{e}_7\mathsf{e}_6\mathsf{e}_5)\,(\mathsf{e}_9\mathsf{e}_8)$$

As above, if $\nu \in \Lambda(\mathsf{G})$, then $|\nu|$ equals the distance from 0 to ν in $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$. Let $\mathsf{e}_\nu := \mathsf{e}_{|\nu|}$, and let $d^\nu = \dim \mathsf{G}^\nu$. For $\nu \neq 0$, $\nu - 1 \in \Lambda(\mathsf{G})$ is the unique node that is connected to ν by an edge with $|\nu - 1| = |\nu| - 1$. We say that $\mu = \nu + 1$ if $\mu \in \Lambda(\mathsf{G})$ is connected to ν by an edge and $|\mu| = |\nu| + 1$. Finally we say that $\mu \prec \nu$ if μ is on the shortest path from 0 to ν.

Theorem 20 (Presentation on Generators and Relations) *For $k \geq 1$ and $k < \tilde{n}$ if $\mathsf{G} = \mathbf{C}_n$, the algebra $\mathsf{Z}_k(\mathsf{G})$ is generated by $\{1, \mathsf{e}_1, \dots, \mathsf{e}_{k-1}\} \cup \{\mathsf{f}_\nu \mid \nu \in \Lambda(\mathsf{G}), |\nu| \leq k\}$ subject to the following relations:*

(1) $\mathsf{e}_i^2 = 2\mathsf{e}_i$; $\quad \mathsf{e}_i\mathsf{e}_{i\pm1}\mathsf{e}_i = \mathsf{e}_i$; $\quad \mathsf{e}_i\mathsf{e}_j = \mathsf{e}_j\mathsf{e}_i,\ |i-j| > 1$;

(2) $\mathsf{f}_\nu^2 = \mathsf{f}_\nu$; $\quad \mathsf{f}_\nu\mathsf{e}_i = \mathsf{e}_i\mathsf{f}_\nu = 0$ *for* $i < |\nu|$; $\quad \mathsf{f}_\nu\mathsf{e}_i = \mathsf{e}_i\mathsf{f}_\nu$ *for* $i > |\nu|$; $\quad \mathsf{e}_\nu\mathsf{f}_\nu\mathsf{e}_\nu = \frac{d^\nu}{d^{\nu-1}}\mathsf{f}_{\nu-1}\mathsf{e}_\nu$;

(3) $\displaystyle\sum_{\mu=\nu+1} \mathsf{f}_\mu = \mathsf{f}_\nu - \frac{d^{\nu-1}}{d^\nu}\mathsf{f}_\nu\mathsf{e}_{\nu-1}\mathsf{f}_\nu$ *(and this equals 0 if no such μ exists);*

(4) *If $|\mu| \leq |\nu|$, then $f_\mu f_\nu = f_\nu f_\mu = f_\mu$, if $\mu \prec \nu$, and $f_\mu f_\nu = f_\nu f_\mu = 0$, if $\mu \not\prec \nu$.*

The case of $k \geq \tilde{n}$, for $\mathsf{G} = \mathbf{C}_n$ requires some further notation and is handled in [7].

6 Exceptional McKay Centralizers and Partition Algebras

The exceptional groups are referred to as the binary tetrahedral, binary octahedral, and binary icosahedral groups because modulo their centers $\mathcal{Z}(\mathsf{G}) = \{1, -1\}$, we have the following isomorphisms:

$\mathbf{T}/\{1,-1\} \cong \mathbf{A}_4$, the alternating group on 4 letters (rotation group of a tetrahedron),
$\mathbf{O}/\{1,-1\} \cong \mathbf{S}_4$, the symmetric group on 4 letters (rotation group of an octahedron),
$\mathbf{I}/\{1,-1\} \cong \mathbf{A}_5$, the alternating group on 5 letters (rotation group of an icosahedron).

Group elements act on $\mathsf{V}^{\otimes 2} = \mathsf{V} \otimes \mathsf{V}$ diagonally: $g \cdot (\mathsf{v}_i \otimes \mathsf{v}_j) = g\mathsf{v}_i \otimes g\mathsf{v}_j$. If $g \in \mathcal{Z}(\mathsf{G}) = \{1, -1\}$ then g acts on V as multiplication by 1 and -1, so it acts trivially on $\mathsf{V}^{\otimes 2}$. Thus, $\mathcal{Z}(\mathsf{G})$ is in the kernel of the action on tensor powers $\mathsf{V}^{\otimes k}$, with k even, and $\mathbf{T}$, $\mathbf{O}$, and $\mathbf{I}$ act the same as $\mathbf{A}_4, \mathbf{S}_4$, and $\mathbf{A}_5$, respectively

Furthermore, $\dim(\mathsf{V}^{\otimes 2}) = 4$, and as a module for $\mathbf{A}_4, \mathbf{S}_4$, and $\mathbf{A}_5$ it decomposes in the following way:

$$\begin{aligned}
&\mathsf{V}^{\otimes 2} \cong \mathbf{A}_4^{(4)} \oplus \mathbf{A}_4^{(3,1)} \cong \mathsf{M}, \text{ the permutation module for } \mathbf{A}_4,\\
&\mathsf{V}^{\otimes 2} \cong \mathbf{S}_4^{(4)} \oplus \mathbf{S}_4^{(2,1,1)} \cong \widetilde{\mathsf{M}}, \text{ a ``twisted'' permutation module for } \mathbf{S}_4,\\
&\mathsf{V}^{\otimes 2} \cong \mathbf{A}_5^{(5)} \oplus \mathbf{A}_5^{(3,1,1)^+} \cong \widetilde{\widetilde{\mathsf{M}}}, \text{ which is not a permutation module for } \mathbf{A}_5,
\end{aligned}$$

where here we are using usual integer partition notation to label the irreducible modules for $\mathbf{S}_n$ and $\mathbf{A}_n$. This means that

$$\mathsf{Z}_{2k}(\mathbf{T}) \cong \mathsf{End}_{\mathbf{A}_4}(\mathsf{M}^{\otimes k}), \qquad \mathsf{Z}_{2k}(\mathbf{O}) \cong \mathsf{End}_{\mathbf{S}_4}(\widetilde{\mathsf{M}}^{\otimes k}), \qquad \mathsf{Z}_{2k}(\mathbf{I}) \cong \mathsf{End}_{\mathbf{A}_5}(\widetilde{\widetilde{\mathsf{M}}}^{\otimes k}). \tag{21}$$

The Martin-Jones partition algebras $\mathsf{P}_k(n)$ maps surjectively onto $\mathsf{End}_{\mathbf{S}_n}(\mathsf{M}^{\otimes k})$ for all n and isomorphically for $n \geq 2k$, where M is the n-dimensional permutation representation of $\mathbf{S}_n$ (see [11], [12], [10]). When M is restricted from $\mathbf{S}_n$ to $\mathbf{A}_n$ the corresponding partition algebra $\tilde{\mathsf{P}}_k(n)$ is studied by Bloss [5]. The partition algebra $\mathsf{P}_k(n)$ has a basis indexed by set partitions of $\{1, 2, \dots, 2k\}$ and a multiplication given by set partition diagram concatenation. See [9] for a survey on partition algebras.

In [3, 4] we study the partition algebras $\mathsf{End}_{\mathbf{S}_n}(\mathsf{M}^{\otimes k})$ and $\mathsf{End}_{\mathbf{A}_n}(\mathsf{M}^{\otimes k})$ for $n < 2k$ ("low rank") and for both the permutation module $\mathsf{M} = \mathbf{S}_n^{(n)} \oplus \mathbf{S}_n^{(n-1,1)}$ and its twisted counterpart $\widetilde{\mathsf{M}} = \mathbf{S}_n^{(n)} \oplus \mathbf{S}_n^{(2,1^{n-1})}$. This work gives us alternative precise descriptions of the centralizer algebras $\mathsf{Z}_{2k}(\mathbf{T})$ and $\mathsf{Z}_{2k}(\mathbf{O})$. In [3] we do the following:

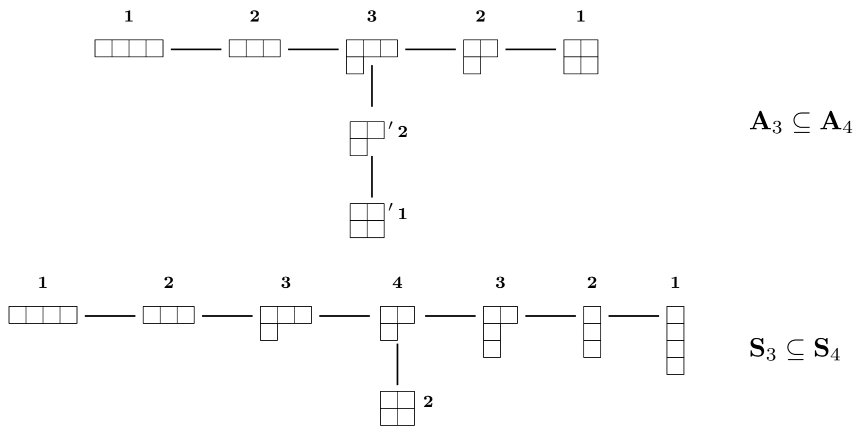

Fig. 4: Restriction-induction graphs for $\mathbf{A}_3 \subseteq \mathbf{A}_4$ and $\mathbf{S}_3 \subseteq \mathbf{S}_4$. These graphs are isomorphic to the representation graphs $\mathcal{R}_{\mathsf{V}}(\mathbf{T})$ and $\mathcal{R}_{\mathsf{V}}(\mathbf{O})$, respectively, which in turn equal the Dynkin diagrams of type $\hat{\mathsf{E}}_6$ and $\hat{\mathsf{E}}_7$. See Figure 2.

- Explicitly describe the kernel of $\mathsf{P}_k(n) \to \mathsf{End}_{\mathsf{G}}(\mathsf{M}^{\otimes k})$, when $n < 2k$, for $\mathsf{G} = \mathbf{S}_n, \mathbf{A}_n$ and for either the permutation module $\mathsf{M} = \mathbf{S}^{(n)} \oplus \mathbf{S}^{(n-1,1)}$ or its twist $\mathbf{S}^{(n)} \oplus \mathbf{S}^{(2,1^{n-1})}$.
- Give two linear bases for the image $\mathsf{End}_{\mathsf{G}}(\mathsf{M}^{\otimes k})$ in terms of a restricted collection of set partitions. Combinatorially describe multiplication in both bases and give the change of basis matrix between them in terms of the refinement ordering in the partition lattice Π_n.

Furthermore, these partition algebras can be realized in terms of restriction and induction (this perspective is emphasized in [9]) in the following way. If U is any $\mathbf{S}_n$ module, then upon restriction from $\mathbf{S}_n$ to $\mathbf{S}_{n-1}$ followed by induction back to $\mathbf{S}_n$ we get $\mathsf{Ind}_{\mathbf{S}_{n-1}}^{\mathbf{S}_n} \mathsf{Res}_{\mathbf{S}_{n-1}}^{\mathbf{S}_n}(\mathsf{U}) \cong \mathsf{U} \otimes \mathsf{M}$. This is an application of the "tensor identity" (see [9, 3.18]). It follows that the module $\mathsf{M}^{\otimes k}$ is isomorphic to k iterations of restriction and induction (starting with the trivial module). This process works both for $\mathbf{S}_{n-1} \subseteq \mathbf{S}_n$ and $\mathbf{A}_{n-1} \subseteq \mathbf{A}_n$. In Figure 6 we see this amazing correspondence: the restriction and induction graphs for $\mathbf{A}_3 \subseteq \mathbf{A}_4$ and $\mathbf{S}_3 \subseteq \mathbf{S}_4$ correspond exactly to the representation graphs $\mathcal{R}_{\mathsf{V}}(\mathbf{T})$ and $\mathcal{R}_{\mathsf{V}}(\mathbf{O})$ and thus also to the affine Dynkin diagrams $\hat{\mathsf{E}}_6$ and $\hat{\mathsf{E}}_7$. In particular, this reveals why the partition algebras also work to describe the centralizer algebras $\mathsf{Z}_k(\mathbf{T})$ and $\mathsf{Z}_k(\mathbf{O})$ for k odd as well as k even. It remains an open question to fully understand the connections between $\mathsf{Z}_k(\mathbf{I})$ and the generalized partition algebra $\mathsf{End}_{\mathbf{A}_5}(\widetilde{\widetilde{\mathsf{M}}}^{\otimes k})$, where $\widetilde{\widetilde{\mathsf{M}}}$ is the $\mathbf{A}_5$ module $\mathbf{A}_5^{(5)} \oplus \mathbf{A}_5^{(3,1,1)^+}$.

The irreducible representations of $\mathsf{P}_k(n)$ are labeled by integer partitions $\lambda \vdash n$ with $|\lambda^{\#}| \leq k$, where if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ then $\lambda^{\#} = (\lambda_2, \ldots, \lambda_\ell)$. Using an inductive argument in [4], we give formulas for these dimensions which to our knowledge are new:

$$\dim \mathsf{P}_k^\lambda(n) = \sum_{r=1}^{\ell} (-1)^{r-1} F_r^\lambda \left(\sum_{t=n-\lambda_r+r-1}^{n-2} \binom{t}{n-\lambda_r+r-1} \left\{ {k \atop t} \right\} \right) + f^\lambda \left(\left\{ {k \atop n-1} \right\} + \left\{ {k \atop n} \right\} \right), \tag{22}$$

where f^λ is the number of standard tableaux of shape λ, $\left\{ {k \atop j} \right\}$ is the Stirling number of the 2nd kind, and F_r^λ is defined by: (1) $F_1^\lambda = f^{\lambda^\#}$, (2) $F_r^\lambda = 0$, if $r > \lambda_r$, and (3) $F_r^\lambda = \sum_{\mu \subseteq \lambda,\ \mu_{r-1}=\mu_r=\lambda_r} F_{r-1}^\mu \cdot f^{\lambda/\mu}$, if $r > 1$. We have analogous formulas in [4] for the partition algebras corresponding to $\mathbf{A}_n$.

References

[1] J. M. Barnes, G. Benkart, and T. Halverson. McKay centralizer algebras. *Proc. London Math. Soc.*, 112(2):373–414, 2016.

[2] G. Benkart and T. Halverson. Exceptional McKay centralizer algebras. *In preparation*.

[3] G. Benkart and T. Halverson. Partition algebras $\mathsf{P}_k(n)$ with $n < 2k$. *In preparation*.

[4] G. Benkart and T. Halverson. Tensor power multiplicities for symmetric and alternating groups and dimensions of irreducible modules for partition algebras. *In preparation*.

[5] M. Bloss. The partition algebra as a centralizer algebra of the alternating group. *Comm. Algebra*, 33(7):2219–2229, 2005.

[6] C. W. Curtis and I. Reiner. *Methods of representation theory. Vol. I and Vol. II.* Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1987.

[7] L. Gagnon and T. Halverson. Bases for McKay centralizer algebras. *In preparation*.

[8] F. M. Goodman, P. de la Harpe, and V. F. R. Jones. *Coxeter graphs and towers of algebras*, volume 14 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1989.

[9] T. Halverson and A. Ram. Partition algebras. *European J. Combin.*, 26(6):869–921, 2005.

[10] V. F. R. Jones. The Potts model and the symmetric group. In *Subfactors (Kyuzeso, 1993)*, pages 259–267. World Sci. Publ., River Edge, NJ, 1994.

[11] P. Martin. *Potts models and related problems in statistical mechanics*, volume 5 of *Series on Advances in Statistical Mechanics*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.

[12] P. Martin. The structure of the partition algebras. *J. Algebra*, 183(2):319–358, 1996.

[13] J. McKay. Graphs, singularities, and finite groups. In *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, volume 37 of *Proc. Sympos. Pure Math.*, pages 183–186. Amer. Math. Soc., Providence, R.I., 1980.

[14] N. J. A. Sloane. The on-line encyclopedia of integer sequences (OEIS).

[15] R. Steinberg. Finite subgroups of SU_2, Dynkin diagrams and affine Coxeter elements. *Pacific J. Math.*, 118(2):587–598, 1985.

[16] H. Wenzl. On tensor categories of Lie type E_N, $N \neq 9$. *Adv. Math.*, 177(1):66–104, 2003.