Elliptic rook and file numbers

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Abstract. In this work, we construct elliptic analogues of the rook numbers and file numbers by attaching elliptic weights to the cells in a board. We show that our elliptic rook and file numbers satisfy elliptic extensions of corresponding factorization theorems which in the classical case were established by Goldman, Joichi and White and by Garsia and Remmel in the file number case. This factorization theorem can be used to define elliptic analogues of various kinds of Stirling numbers of the first and second kind as well as Abel numbers. We also give analogous results for matchings of graphs, elliptically extending the result of Haglund and Remmel.

Résumé. Dans cet article nous construisons des analogues elliptiques des nombres de tour (rook numbers) et des nombres de file (file numbers), en attachant des poids elliptiques aux cases d'un damier. Nous montrons que nos nombres elliptiques de tour et de file satisfont des extensions elliptiques des théorèmes de factorisation correspondants établis dans le cas classique par Goldman, Joichi et White, et par Garsia et Remmel dans le cas des nombres de file. Ce théorème de factorisation peut être utilisé pour définir des analogues elliptiques de différentes sortes des nombres de Stirling de première et de deuxième espèces, et des nombres d'Abel. Nous donnons également des résultats analogues pour les appariements de graphes, en étendant elliptiquement le résultat de Haglund et Remmel.

Keywords. rook numbers, file numbers, elliptic analogues, r-Stirling numbers, perfect matchings

1 Introduction to rook theory

The theory of rook numbers was introduced by Kaplansky and Riordan [KR46] in 1946 and since then it has been further studied and developed by many people. We start with reviewing the q-analogue of the rook theory developed by Garsia and Remmel [GR86] and the j-attacking model of Remmel and Wachs [RW04], then extend it to the elliptic case. This is an extended abstract of [SY].

Let $\mathbb{N}$ denote the set of positive integers. We consider $\mathbb{N}\times\mathbb{N}$ grid and label the columns with $1, 2, \dots$ from left to right and also the rows from bottom to top. Let (i, j) denote the cell in the column i and row j in $\mathbb{N}\times\mathbb{N}$ grid. A finite subset of $\mathbb{N}\times\mathbb{N}$ grid shall be called a *board*.

For a given sequence of nonnegative integers $(b_1, \dots, b_n)$, we let $B(b_1, \dots, b_n)$ denote the set of cells

$$B(b_1, \dots, b_n) = \{(i, j) : 1 \le i \le n, 1 \le j \le b_i\}.$$

If a board B can be represented by the set $B(b_1, \dots, b_n)$ for some b_i's, then the board B is called a *skyline board*. Furthermore, if those b_i's are nondecreasing, then the board $B = B(b_1, \dots, b_n)$ is called a *Ferrers*

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board. Given a board B, we say that we place k *nonattacking* rooks in B to mean that we choose a k element subset of B such that no two cells have a common coordinate. That is, no two rooks lie in the same row or column. Let $\mathcal{N}_k(B)$ denote the set of nonattacking rook placements of k rooks in B. Then the k-th rook number $r_k(B)$ is defined to be $r_k(B) = |\mathcal{N}_k(B)|$.

To define the q-analogue of $r_k(B)$, we consider a Ferrers board $B = B(b_1, \dots, b_n)$, with $0 \le b_1 \le \cdots \le b_n$. Given a k rook placement $P \in \mathcal{N}_k(B)$, we define a *rook cancellation*. Namely, a rook cancels all the cells to the right in the same row and all the cells below it in the same column. Given a placement P, let $u_B(P)$ denote the number of cells in $B - P$ which are uncancelled by any rooks in P. Garsia and Remmel [GR86] defined the q-analogue of the k-th rook number by

$$r_k(q; B) = \sum_{P \in \mathcal{N}_k(B)} q^{u_B(P)}$$

and proved the q-analogue of the product formula of Goldman, Joichi and White [GJW75]

$$\prod_{i=1}^{n} [z + b_i - i + 1]_q = \sum_{k=0}^{n} r_{n-k}(q; B)\, [z]_q \!\downarrow_k, \tag{1.1}$$

where $[n]_q = \frac{1-q^n}{1-q}$ and $[n]_q \downarrow_k = [n]_q [n-1]_q \cdots [n-k+1]_q$.

Remmel and Wachs [RW04] considered more generalized case by introducing the j-*attacking model*. For a fixed integer $\mathrm{j} \ge 1$, we say that a Ferrers board $B(b_1, \dots, b_n)$ is a j-*attacking board* if for all $1 \le i < n$, $b_i \ne 0$ implies $b_{i+1} \ge b_i + \mathrm{j} - 1$. Suppose that $B(b_1, \dots, b_n)$ is a j-attacking board and P is a placement of rooks in $B(b_1, \dots, b_n)$ which has at most one rook in each column of $B(b_1, \dots, b_n)$. For any individual rook $\mathbf{r} \in P$, we say that $\mathbf{r}$ j-*attacks* a cell $c \in B(b_1, \dots, b_n)$ if c lies in a column which is strictly to the right of the column of $\mathbf{r}$ and c lies in the first j rows which are weakly above the row of $\mathbf{r}$ and which are not j-attacked by any rook which lies in a column that is strictly to the left of $\mathbf{r}$. Figure 1 shows an example of j-attack when $\mathrm{j} = 2$. The cells which are attacked by the rook $\mathbf{r}_i$ are denoted by i in the cell. Let a rook $\mathbf{r}$ in $B(b_1, \dots, b_n)$ cancel the cells below it and the cells which are attacked by $\mathbf{r}$. Given a j-attacking board, we let $\mathcal{N}_k^{\mathrm{j}}(B)$ be the set of all placements P of k j-nonattacking rooks in B.

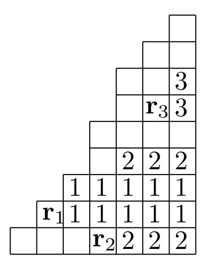

Fig. 1: $\mathrm{j} = 2$, $B = B(1, 2, 3, 5, 7, 8, 9)$.

Let $B = B(b_1, \dots, b_n)$ be a j-attacking board. Then for any placement $P \in \mathcal{N}_k^{\mathrm{j}}(B)$, denote the number of uncancelled cells in $B - P$ by $u_B^{\mathrm{j}}(P)$. Then define the q-rook number of B by

$$r_k^{\mathrm{j}}(q; B) = \sum_{P \in \mathcal{N}_k^{\mathrm{j}}(B)} q^{u_B^{\mathrm{j}}(P)}.$$

Remmel and Wachs [RW04] proved the following product formula

$$\prod_{i=1}^{n}[z+b_i-\mathrm{j}(i-1)]_q=\sum_{k=0}^{n} r_{n-k}^{\mathrm{j}}(q;B)[z]_q\downarrow_{k,\mathrm{j}}, \tag{1.2}$$

where $[z]_q\downarrow_{0,\mathrm{j}}=1$ and for $k>0$, $[z]_q\downarrow_{k,\mathrm{j}}=[z]_q[z-\mathrm{j}]_q\cdots[z-(k-1)\mathrm{j}]_q$. Note that we recover the product formula of Garsia and Remmel in the case $\mathrm{j}=1$. In this work, we establish an elliptic analogue of the product formula (1.2).

2 Elliptic Analogues

A function is called *elliptic* if it is meromorphic and doubly periodic. Define a modified Jacobi theta function with argument x and nome p by

$$\theta(x;p):=\prod_{j\geq 0}((1-p^jx)(1-p^{j+1}/x)),\qquad \theta(x_1,\dots,x_m;p)=\prod_{k=1}^{m}\theta(x_k;p),$$

where $x,x_1,\dots,x_m\neq 0$, $|p|<1$. Further, we define the *theta shifted factorial* by

$$(a;q,p)_n=\begin{cases}\prod_{k=0}^{n-1}\theta(aq^k;p), & n=1,2,\dots,\\ 1, & n=0,\\ 1/\prod_{k=0}^{-n-1}\theta(aq^{n+k};p), & n=-1,-2,\dots,\end{cases}$$

and $(a_1,a_2,\dots,a_m;q,p)_n=\prod_{k=1}^{m}(a_k;q,p)_n$, for compact notation. For $p=0$ we have $\theta(x;0)=1-x$ and hence, $(a;q,0)_n=(a;q)_n$ is a *q-shifted factorial* in base q. Refer to [GR04] for further details. The parameters q and p in $(a;q,p)_n$ are called the *base* and *nome*, respectively. One of the properties that modified Jacobi theta functions satisfy is called the *addition formula*

$$\theta(xy,x/y,uv,u/v;p)-\theta(xv,x/v,uy,u/y;p)=\frac{u}{y}\,\theta(yv,y/v,xu,x/u;p),$$

which is essential in the theory of elliptic hypergeometric series.

Inspired by earlier work of the first author regarding weighted lattice paths and elliptic binomial coefficients [Sch07, Sch], we define the *elliptic weights* $w_{a,b;q,p}(k)$ and $W_{a,b;q,p}(k)$, called *small* and *big* weights, depending on two independent parameters a and b, base q, nome p, and integer k by

$$w_{a,b;q,p}(k)=\frac{\theta(aq^{2k+1},bq^k,aq^{k-2}/b;p)}{\theta(aq^{2k-1},bq^{k+2},aq^k/b;p)}q, \tag{2.1a}$$

$$W_{a,b;q,p}(k)=\frac{\theta(aq^{1+2k},bq,bq^2,aq^{-1}/b,a/b;p)}{\theta(aq,bq^{k+1},bq^{k+2},aq^{k-1}/b,aq^k/b;p)}q^k, \tag{2.1b}$$

respectively. Observe that if k is a positive integer, Equations (2.1a) and (2.1b) imply that $W_{a,b;q,p}(k)=\prod_{j=1}^{k}w_{a,b;q,p}(j)$. Also, if we let $p\to 0$, $a\to 0$ and $b\to 0$ in this order (or $p\to 0$, $b\to 0$ and $a\to\infty$), then we recover the original q-weights $w_{0,0;q,0}(k)=q$ and $W_{0,0;q,0}(k)=q^k$.

Remark 2.1 *The small weight $w_{a,b;q,p}(k)$ (and so the big one) is indeed elliptic in its parameters, i.e., totally elliptic. If we write $q = e^{2\pi i\sigma}$, $p = e^{2\pi i\tau}$, $a = q^\alpha$ and $b = q^\beta$ with complex values σ, τ, α, β and k, then the small weight $w_{a,b;q,p}(k)$ is periodic in α with period σ^{-1}. A simple computation further shows that $w_{a,b;q,p}(k)$ is also periodic in α with period $\tau\sigma^{-1}$. The same applies to $w_{a,b;q,p}(k)$ as a function in β (or k) with the same two periods σ^{-1} and $\tau\sigma^{-1}$.*

Now we define an *elliptic number* $[z]_{a,b;q,p}$ by

$$[z]_{a,b;q,p} = \frac{\theta(q^z, aq^z, bq^2, a/b; p)}{\theta(q, aq, bq^{z+1}, aq^{z-1}/b; p)}. \tag{2.2}$$

Using the addition formula for theta functions, it is not difficult to verify that the elliptic numbers satisfy

$$[z]_{a,b;q,p} = [z-1]_{a,b;q,p} + W_{a,b;q,p}(z-1).$$

We remark that in [Sch], the first author defined the *elliptic binomial coefficients*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} := \frac{(q^{1+k}, aq^{1+k}, bq^{1+k}, aq^{1-k}/b; q, p)_{n-k}}{(q, aq, bq^{1+2k}, aq/b; q, p)_{n-k}} \tag{2.3}$$

and provided a combinatorial interpretation in terms of weighted lattice paths in $\mathbb{Z}^2$. More precisely, (2.3) is the area generating function for paths starting from $(0,0)$ and ending in $(k, n-k)$ using north and east steps only, when the weight of each cell, with (s,t) being the coordinate of north-east corner, below the path is defined to be $w_{aq^{s-1},bq^{2s-2};q,p}(t)$. The elliptic number $[n]_{a,b;q,p}$ is nothing but a short-hand notation for $\begin{bmatrix} n \\ 1 \end{bmatrix}_{a,b;q,p}$, the weighted enumeration of all paths starting from $(0,0)$ and ending in $(1, n-1)$.

Now we construct an elliptic analogue of the q-rook theory. Given a $\mathfrak{j}$-attacking board $B = B(b_1, \ldots, b_n)$ and a placement $P \in \mathcal{N}_k^{\mathfrak{j}}(B)$, let $U_B^{\mathfrak{j}}(P)$ be the set of uncancelled cells in $B - P$. We define the elliptic analogue of the k-th rook number of B by

$$r_k^{\mathfrak{j}}(a, b; q, p; B) = \sum_{P \in \mathcal{N}_k^{\mathfrak{j}}(B)} wt^{\mathfrak{j}}(P),$$

where

$$wt^{\mathfrak{j}}(P) = \prod_{(s,t) \in U_B^{\mathfrak{j}}(P)} w_{a,b;q,p}(\mathfrak{j}(s-1) + 1 - t - \mathfrak{j} r_{(s,t)}(P)),$$

and $r_{(s,t)}(P)$ is the number of rooks in P which are in the north-west region of (s,t).

Theorem 2.2 *Let $B = B(b_1, \ldots, b_n)$ be a $\mathfrak{j}$-attacking board. Then we have*

$$\prod_{s=1}^n [z + b_s - \mathfrak{j}(s-1)]_{aq^{2(\mathfrak{j}(s-1)-b_s)}, bq^{\mathfrak{j}(s-1)-b_s}; q,p}$$
$$= \sum_{k=0}^n r_{n-k}^{\mathfrak{j}}(a, b; q, p; B) \prod_{t=1}^k [z - \mathfrak{j}(t-1)]_{aq^{2\mathfrak{j}(t-1)}, bq^{\mathfrak{j}(t-1)}; q,p}. \tag{2.4}$$

Let $r_k(a,b;q,p;B)$ denote $r_k^1(a,b;q,p;B)$ in the case when $\mathfrak{j}=1$. Then Theorem 2.2 gives an elliptic analogue of the product formula of Garsia and Remmel (1.1)

$$\prod_{s=1}^{n}[z+b_s-s+1]_{aq^{2(s-1-b_s)},bq^{s-1-b_s};q,p}=\sum_{k=0}^{n}r_{n-k}(a,b;q,p;B)\prod_{t=1}^{k}[z-t+1]_{aq^{2(t-1)},bq^{t-1};q,p}. \quad (2.5)$$

Proof: It suffices to prove the theorem for nonnegative integer values of $z\geq \mathfrak{j}n$. We consider the extended board, denoted by B_z, by attaching z rows of width n below the board B. We consider nonattacking placements of n rooks in B_z and compute the sum

$$\sum_{P\in\mathcal{N}_n^{\mathfrak{j}}(B_z)} wt^{\mathfrak{j}}(P),$$

where

$$wt^{\mathfrak{j}}(P)=\prod_{(s,t)\in U^{\mathfrak{j}}_{B_z}(P)} w_{a,b;q,p}(\mathfrak{j}(s-1)+1-t-\mathfrak{j}r_{(s,t)}(P)),$$

in two different ways to derive (2.4). We omit the details. □

2.1 Elliptic analogue of generalized Stirling numbers of the second kind

The *generalized* $(\mathfrak{p},q)$*-Stirling numbers* of the second kind $\tilde{S}^{\mathfrak{i},\mathfrak{j}}_{n,k}(\mathfrak{p},q)$ were introduced by Remmel and Wachs in [RW04], (here we use "$\mathfrak{p}$" to differentiate the notation from the nome p in the elliptic functions), which are defined by

$$\tilde{S}^{\mathfrak{i},\mathfrak{j}}_{n+1,k}(\mathfrak{p},q)=q^{\mathfrak{i}+(k-1)\mathfrak{j}}\tilde{S}^{\mathfrak{i},\mathfrak{j}}_{n,k-1}(\mathfrak{p},q)+\mathfrak{p}^{-(n+1)\mathfrak{j}}[k\mathfrak{j}+\mathfrak{i}]_{\mathfrak{p},q}\tilde{S}^{\mathfrak{i},\mathfrak{j}}_{n,k}(\mathfrak{p},q), \quad (2.6)$$

with $\tilde{S}^{\mathfrak{i},\mathfrak{j}}_{0,0}(\mathfrak{p},q)=1$ and $\tilde{S}^{\mathfrak{i},\mathfrak{j}}_{n,k}(\mathfrak{p},q)=0$ if $k<0$ or $k>n$. Moreover, they satisfy

$$[z+\mathfrak{i}]^n_{\mathfrak{p},q}=\sum_{k=0}^{n}\tilde{S}^{\mathfrak{i},\mathfrak{j}}_{n,k}(\mathfrak{p},q)\mathfrak{p}^{z(n-k)+\binom{n-k+1}{2}}[z]_{\mathfrak{p},q}\!\downarrow_{k,j}.$$

Note that we recover the original Stirling numbers of the second kind when $\mathfrak{i}=0$, $\mathfrak{j}=1$, $\mathfrak{p}=1$ and $q=1$, and the q-analogue in the case $\mathfrak{i}=0$, $\mathfrak{j}=1$ and $\mathfrak{p}=1$. Here, we set $\mathfrak{p}=1$ and use the notation $\tilde{S}^{\mathfrak{i},\mathfrak{j}}_{n,k}(q)=\tilde{S}^{\mathfrak{i},\mathfrak{j}}_{n,k}(1,q)$ and $S^{\mathfrak{i},\mathfrak{j}}_{n,k}(q)=S^{\mathfrak{i},\mathfrak{j}}_{n,k}(1,q)$ for $\tilde{S}^{\mathfrak{i},\mathfrak{j}}_{n,k}(q)=q^{k\mathfrak{i}+\binom{k}{2}\mathfrak{j}}S^{\mathfrak{i},\mathfrak{j}}_{n,k}(q)$.

Let $B_{\mathfrak{i},\mathfrak{j},n}=B(\mathfrak{i},\mathfrak{i}+\mathfrak{j},\mathfrak{i}+2\mathfrak{j},\dots,\mathfrak{i}+(n-1)\mathfrak{j})$. In [RW04], Remmel and Wachs showed that

$$\tilde{S}^{\mathfrak{i},\mathfrak{j}}_{n,k}(q)=r^{\mathfrak{j}}_{n-k}(q;B_{\mathfrak{i},\mathfrak{j},n}).$$

We use $B_{\mathfrak{i},\mathfrak{j},n}$ in (2.4) to define an elliptic analogue of $\tilde{S}^{\mathfrak{i},\mathfrak{j}}_{n,k}(q)$. For $B_{\mathfrak{i},\mathfrak{j},n}$, the product formula becomes

$$([z+\mathfrak{i}]_{aq^{-2\mathfrak{i}},bq^{-\mathfrak{i}};q,p})^n=\sum_{k=0}^{n}r^{\mathfrak{j}}_{n-k}(a,b;q,p;B_{\mathfrak{i},\mathfrak{j},n})\prod_{t=1}^{k}[z-\mathfrak{j}(t-1)]_{aq^{2\mathfrak{j}(t-1)},bq^{\mathfrak{j}(t-1)};q,p}. \quad (2.7)$$

If we define $\tilde{S}^{\mathrm{i},\mathrm{j}}_{n,k}(a,b;q,p) = r^{\mathrm{j}}_{n-k}(a,b;q,p;B_{\mathrm{i},\mathrm{j},n})$, then up to whether there is a rook or not in the last column of $B_{\mathrm{i},\mathrm{j},n}$, we get the following recursion

$$\tilde{S}^{\mathrm{i},\mathrm{j}}_{n+1,k}(a,b;q,p) = W_{aq^{-2\mathrm{i}},bq^{-\mathrm{i}};q,p}(\mathrm{i}+(k-1)\mathrm{j})\tilde{S}^{\mathrm{i},\mathrm{j}}_{n,k-1}(a,b;q,p) + [\mathrm{i}+k\mathrm{j}]_{aq^{-2\mathrm{i}},bq^{-\mathrm{i}};q,p}\tilde{S}^{\mathrm{i},\mathrm{j}}_{n,k}(a,b;q,p).$$

In [RW04], they develop combinatorial interpretation for $S^{\mathrm{i},\mathrm{j}}_{n,k}(q)$ in terms of permutation statistics, colored partitions and restricted growth functions. We can modify their q-weight function to elliptic function to give a combinatorial interpretation for $S^{\mathrm{i},\mathrm{j}}_{n,k}(a,b;q,p)$ where

$$\tilde{S}^{\mathrm{i},\mathrm{j}}_{n,k}(a,b;q,p) = \left(\prod_{j=1}^{k} W_{aq^{-2\mathrm{i}},bq^{-\mathrm{i}};q,p}(\mathrm{i}+(j-1)\mathrm{j})\right) S^{\mathrm{i},\mathrm{j}}_{n,k}(a,b;q,p).$$

2.2 Elliptic r-restricted Stirling numbers of the second kind

The *r-restricted Stirling numbers of the second kind*, which we denote by $\mathcal{S}^{(r)}(n,k)$, are defined in [Bro84], for all positive integer r, by the number of set partitions on $[n]$ into k blocks such that the first r numbers $1,2,\ldots,r$ are in different blocks. Note that the case $r=1$ (or $r=0$) gives the usual Stirling numbers of the second kind. These numbers admit a rook theoretic interpretation if we consider the board $\mathsf{St}^{(r)}_n = B(0,\ldots,0,r,r+1,\ldots,n-1)$ of n columns, with the first r columns being empty. A rook placed in the (i,j) cell implies that the elements i and j are in the same block, and the numbers which have not been put in any blocks by rooks compose single element blocks. Then it is not difficult to see that each configuration of $n-k$ nonattacking rooks on $\mathsf{St}^{(r)}_n$ can be associated to a partition of $[n]$ into k blocks such that each of the first r numbers $1,2\ldots,r$ is in a different block, and vice versa.

We can use the board $\mathsf{St}^{(r)}_n$ in (2.5) to define an elliptic analogue of $\mathcal{S}^{(r)}(n,k)$. For $b_i=0$ for $i=1,\ldots,r$ and $b_i=i-1$, for $i=r+1,\ldots,n$, (2.5) becomes

$$([z]_{a,b;q,p})^{n-r}\prod_{i=1}^{r}[z-i+1]_{aq^{2(i-1)},bq^{i-1};q,p} = \sum_{k=0}^{n} r_{n-k}(a,b;q,p;\mathsf{St}^{(r)}_n)\prod_{j=1}^{k}[z-j+1]_{aq^{2(j-1)},bq^{j-1};q,p}.$$

Let $\mathcal{S}^{(r)}_{a,b;q,p}(n,k)$ denote $r_{n-k}(a,b;q,p;\mathsf{St}^{(r)}_n)$ to be the *elliptic r-restricted Stirling numbers of the second kind*. By considering whether the last column contains a rook or not, we get the recursion

$$\mathcal{S}^{(r)}_{a,b;q,p}(n+1,k) = W_{a,b;q,p}(k-1)\mathcal{S}^{(r)}_{a,b;q,p}(n,k-1) + [k]_{a,b;q,p}\mathcal{S}^{(r)}_{a,b;q,p}(n,k).$$

This recursion can characterize $\mathcal{S}^{(r)}_{a,b;q,p}(n,k)$ with the initial conditions $\mathcal{S}^{(r)}_{a,b;q,p}(n,k)=0$ for $k<r-1$ or $k>n$, and $\mathcal{S}^{(r)}_{a,b;q,p}(r-1,r-1)=1$.

3 Elliptic file numbers

In this section, we consider an elliptic analogue of file numbers introduced by Garsia and Remmel.

Given a board $B\subset[n]\times\mathbb{N}$, let $\mathcal{F}_k(B)$ be the set of placements Q of k rooks in B such that no two rooks in Q lie in the same column. We refer to such a Q as a *file placement* of k rooks in B. Thus in a file placement Q, we do allow the possibility that two rooks lie in the same row. Given a placement

$Q \in \mathcal{F}_k(B)$, we let each rook in Q cancel all the cells below it in B. Let $u_B(Q)$ be the number of cells in $B - Q$ which are not cancelled by any rook in Q. Then the q-file numbers are defined by

$$f_k(q; B) = \sum_{Q \in \mathcal{F}_k(B)} q^{u_B(Q)}. \tag{3.1}$$

Garsia and Remmel proved that for any skyline board $B = B(c_1, \ldots, c_n)$, the q-file numbers satisfy

$$\prod_{i=1}^{n} [z + c_i]_q = \sum_{k=0}^{n} f_{n-k}(q; B)([z]_q)^k. \tag{3.2}$$

We can define an elliptic analogue of the q-file numbers by assuming the same rook cancellation as in the q-case and by assigning elliptic weights to the uncancelled cells.

Given a skyline board $B = B(c_1, \ldots, c_n)$, we define the elliptic analogue of the k-th file number by

$$f_k(a, b; q, p; B) = \sum_{Q \in \mathcal{F}_k(B)} wt_f(Q), \tag{3.3}$$

where

$$wt_f(Q) = \prod_{(i,j) \in U_B(Q)} w_{a,b;q,p}(1 - j),$$

and $U_B(Q)$ denotes the set of uncancelled cells in $B - Q$ by any rooks in Q.

Theorem 3.1 *For any skyline board $B = B(c_1, \ldots, c_n)$, we have*

$$\prod_{i=1}^{n} [z + c_i]_{aq^{-2c_i}, bq^{-c_i}; q, p} = \sum_{k=0}^{n} f_{n-k}(a, b; q, p; B)([z]_{a,b;q,p})^k. \tag{3.4}$$

Proof: We consider the extended board B_z by attaching an $[n] \times [z]$ board below the board B and consider the n-file placements $\mathcal{F}_n(B_z)$ in B_z. Then (3.4) can be proved by computing the sum

$$\sum_{Q \in \mathcal{F}_n(B_z)} wt_f(Q) \tag{3.5}$$

in two ways. The left-hand side of (3.4) computes the above sum by placing rooks column by column. Since the elliptic weight used to define $wt_f(Q)$ does not depend on the column coordinate of the uncancelled cells, the weight sum in (3.5) is the product of the weight sum coming from the possible placements in each column, which is exactly the left-hand side of (3.4). The right-hand side computes (3.5) by considering the file placements in B and in the extended part separately. □

3.1 Elliptic r-restricted Stirling numbers of the first kind

The *r-restricted (signless) Stirling numbers of the first kind*, denoted by $\mathfrak{c}^{(r)}(n, k)$, are defined, for all positive r, by the number of permutations of the set $\{1, \ldots, n\}$ having k cycles, such that the numbers $1, 2, \ldots, r$ are in distinct cycles. For $r = 1$ (or $r = 0$) they reduce to the usual Stirling numbers of the first kind. In [Bro84], these Stirling numbers are treated in details and it is shown that $\mathfrak{c}^{(r)}(n,k)$ have the generating function

$$\sum_{k=0}^{n} \mathfrak{c}^{(r)}(n,k)z^k = \begin{cases} z^r(z+r)(z+r+1)\cdots(z+n-1), & n \geq r \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

We can obtain this generating function from (3.2) in the case $q \to 1$ by considering the board $\mathsf{St}_n^{(r)} = B(c_1,\dots,c_n)$ with $c_i = 0$ for $i = 1,\dots,r$ and $c_i = i-1$, for $i = r+1,\dots,n$. Thus we can identify $\mathfrak{c}^{(r)}(n,k)$ with $f_{n-k}(1;\mathsf{St}_n^{(r)})$. We can even construct a bijection between file placements of $n-k$ rooks in $\mathsf{St}_n^{(r)}$ and permutations of n numbers with k cycles, such that $1,2,\dots,r$ are in distinct cycles.

We can define an elliptic analogue of the r-restricted Stirling number of the first kind by using the board $\mathsf{St}_n^{(r)}$ in (3.4). Then the generating function would be

$$([z]_{a,b;q,p})^r \prod_{i=1}^{n-r} [z+r+i-1]_{aq^{2(1-i-r)},bq^{1-i-r};q,p} = \sum_{k=0}^{n} f_{n-k}(a,b;q,p;\mathsf{St}_n^{(r)})([z]_{a,b;q,p})^k. \tag{3.7}$$

Let $\mathfrak{c}_{a,b;q,p}^{(r)}(n,k)$ denote $f_{n-k}(a,b;q,p;\mathsf{St}_n^{(r)})$. By distinguishing whether there is a rook or not in the last column, we get the recurrence relation of $\mathfrak{c}_{a,b;q,p}^{(r)}(n,k)$, namely

$$\mathfrak{c}_{a,b;q,p}^{(r)}(n+1,k) = [n]_{aq^{-2n},bq^{-n};q,p}\mathfrak{c}_{a,b;q,p}^{(r)}(n,k) + W_{aq^{-2n},bq^{-n};q,p}(n)\mathfrak{c}_{a,b;q,p}^{(r)}(n,k-1).$$

This recurrence relation can be used to characterize $\mathfrak{c}_{a,b;q,p}^{(r)}(n,k)$ with the initial conditions

$$\mathfrak{c}_{a,b;q,p}^{(r)}(n,k) = 0 \text{ for } k < r-1 \text{ or } k > n, \quad \text{and} \quad \mathfrak{c}_{a,b;q,p}^{(r)}(r-1,r-1) = 1.$$

3.2 Abel boards and weighted forests

Let A_n denote the *Abel* board, the $[n-1]\times[n]$ board with column heights $(0,n,\dots,n)$. For the board A_n, the product formula involving the file numbers (3.2), when $q \to 1$, becomes

$$z(z+n)^{n-1} = \sum_{k=0}^{n} f_{n-k}(1;\mathsf{A}_n)z^k.$$

These polynomials are a special case of the general Abel polynomials $z(z+\alpha n)^{n-1}$. The coefficient $f_{n-k}(1;\mathsf{A}_n) = t_{n,k} = \binom{n-1}{k-1}n^{n-k}$ counts the number of labeled forests on n vertices composed of k rooted trees. Goldman and Haglund explained this equality bijectively in [GH00] and we have established our own bijection as well. For the Abel board A_n the product formula in Theorem 3.1 becomes

$$[z]_{a,b;q,p}([z+n]_{aq^{-2n},bq^{-n};q,p})^{n-1} = \sum_{k=0}^{n} f_{n-k}(a,b;q,p;\mathsf{A}_n)([z]_{a,b;q,p})^k. \tag{3.8}$$

We can interpret the coefficient $f_{n-k}(a,b;q,p;\mathsf{A}_n)$ as the weighted sum of labeled forests on n vertices composed of k rooted trees, by assigning elliptic weights on the branches and vertices of the rooted trees in the process of constructing the bijection. The coefficients in (3.8) have a nice closed form

$$f_{n-k}(a,b;q,p;\mathsf{A}_n) = \binom{n-1}{k-1}\left(W_{aq^{-2n},bq^{-n};q,p}(n)\right)^{k-1}\left([n]_{aq^{-2n},bq^{-n};q,p}\right)^{n-k}, \tag{3.9}$$

which can be proved by considering the file placements in A_n.

We can furthermore consider the case of r-restricted Abel boards $\mathsf{A}_n^{(r)} = B(0,\ldots,0,n,\ldots,n)$ which consists of r columns of height zero and $n-r$ columns of height n. The file number for this board $f_{n-k}(1;\mathsf{A}_n^{(r)})$ equals $\binom{n-r}{k-r}n^{n-k}$ which counts the number of labeled forests on n vertices composed of k rooted trees such that the numbers $1,2,\ldots,r$ are in distinct trees and the $r-1$ numbers $2,\ldots,r$ are the roots. For the r-restricted Abel board $\mathsf{A}_n^{(r)}$ the product formula in Theorem 3.1 becomes

$$([z]_{a,b;q,p})^r([z+n]_{aq^{-2n},bq^{-n};q,p})^{n-r} = \sum_{k=r-1}^{n} f_{n-k}(a,b;q,p;A_n^{(r)})([z]_{a,b;q,p})^k. \tag{3.10}$$

The coefficients in (3.10) have a nice closed form

$$f_{n-k}(a,b;q,p;\mathsf{A}_n^{(r)}) = \binom{n-r}{k-r}\left(W_{aq^{-2n},bq^{-n};q,p}(n)\right)^{k-r}\left([n]_{aq^{-2n},bq^{-n};q,p}\right)^{n-k}. \tag{3.11}$$

4 Rook theory for matchings

Haglund and Remmel [HR01] extended the rook theory by replacing permutations with perfect matchings. Rather than $[n]\times[n]$ which corresponds to the board for permutations, consider the following board B_{2n} described in the left of Figure 2. Note that any rook placement P in $[n]\times[n]$ is a partial permutation which can be extended to a placement P_σ corresponding to some permutation $\sigma\in S_n$. For the board B_{2n}, we replace permutations by perfect matchings of the complete graph K_{2n} on vertices $1,2,\ldots,2n$. That is, for each perfect matching M of K_{2n} consisting of n pairwise vertex disjoint edges in K_{2n}, we let

$$P_M = \{(i,j) \mid i<j \text{ and } \{i,j\}\in M\}$$

where (i,j) denotes the cell in row i and column j of B_{2n} according to the labeling of rows and columns pictured in Fig. 2. We now define a rook placement to be a subset of some P_M for a perfect matching M of K_{2n}. Given a board $B\subseteq B_{2n}$, we let $\mathcal{M}_k(B)$ denote the set of k element rook placements in B. The analogue of a skyline board in this setting is a board $B(a_1,a_2,\ldots,a_{2n-1}) = \{(i,i+j) \mid 1\le i\le 2n-1, 1\le j\le a_i\}$. It is called a *shifted Ferrers board* if $2n-1\ge a_1\ge a_2\ge\cdots\ge a_{2n-1}\ge 0$ and the nonzero entries of a_i's are strictly decreasing. A rook in (i,j) with $i<j$ in a rook placement cancels all cells (i,s) in B_{2n} with $i<s<j$ and all cells (t,j) and (t,i) with $t<i$. The picture in the right-hand side in Figure 2 shows an example of rook cancellation.

Given a shifted Ferrers board $B = B(a_1,\ldots,a_{2n-1})\subseteq B_{2n}$, define

$$m_k(q;B) = \sum_{P\in\mathcal{M}_k(B)} q^{u_B(P)},$$

where $u_B(P)$ is the number of cells in $B-P$ which are not cancelled by any rook in P. In this setting, Haglund and Remmel [HR01] proved the product formula

$$\prod_{i=1}^{2n-1}[z+a_{2n-i}-2i+2]_q = \sum_{k=0}^{n} m_k(q;B)[z]\downarrow\downarrow_{2n-1-k} \tag{4.1}$$

where $[z]_q\downarrow\downarrow_k = [z]_q[z-2]_q\cdots[z-2k+2]_q$.

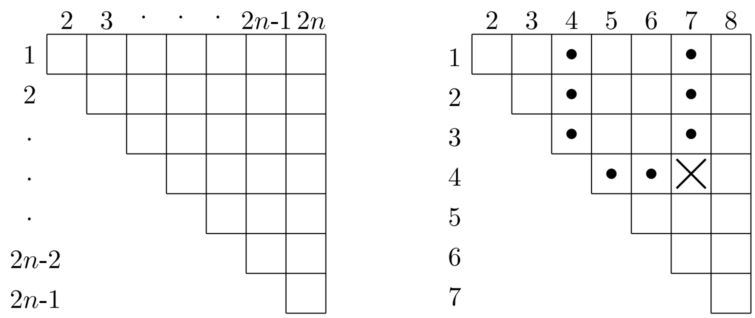

Fig. 2: B_{2n} and the rook cancellation by a rook in $(4, 7)$

We generalize the board B_{2n} and construct an elliptic analogue of the product formula which includes an elliptic extension of (4.1). Let $\mathbf{l} = (l_1, \dots, l_N)$ be a fixed N-dimensional vector of positive integers. Define $L_j = \sum_{s=1}^{j} l_s$, so that $l_j = L_j - L_{j-1}$, for $1 \le j \le N$ with $L_0 = 0$. The $\mathbf{l}$-*shifted board*, denoted by B_N^{l}, is an extension of B_{2n} with N rows and $L_N = l_1 + \cdots + l_N$ columns as described in Figure 3. A rook placed in $B_N^{\mathbf{l}}$, say $\mathbf{r} \in (i, j)$, attacks the cells in the same row, the same column,

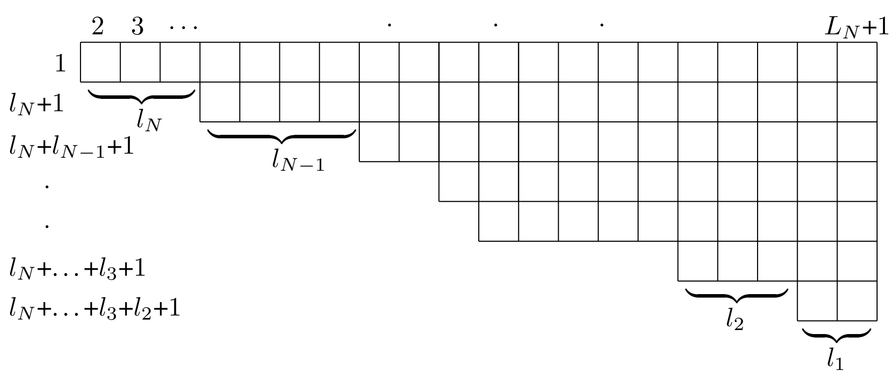

Fig. 3: $B_N^{\mathbf{l}}$.

and the cells in the column i. We can interpret a rook placement in $B_N^{\mathbf{l}}$ in the following way. We call a labeled graph of at most $L_N + 1$ vertices from the set $\{1, 2, \dots, L_N + 1\}$ *lazy* with respect to $\mathbf{l} = (l_1, \dots, l_N)$ (or, an $\mathbf{l}$-*lazy graph*, in short) if it only contains edges (i, j) for $i < j$ when i is of the form $l_N + \cdots + l_{N-s+1} + 1 = L_N - L_{N-s} + 1$ for $s \in \{0, 1, \dots, N - 1\}$. Then a k-rook placement on $B_N^{\mathbf{l}}$ is a k-matching of $K_{L_N+1}^{\mathbf{l}}$, the complete $\mathbf{l}$-lazy graph on $L_N + 1$ vertices. Given a board $B \subseteq B_N^{\mathbf{l}}$, we let $\mathcal{M}_k^{\mathbf{l}}(B)$ denote the set of k rook placements in B. An $\mathbf{l}$-*shifted skyline board* is a board of the form $B(a_1, a_2, \dots, a_N) = \{(L_N - L_{N-i+1} + 1, L_N - L_{N-i+1} + 1 + j) \mid 1 \le i \le N, 1 \le j \le a_i\}$. It is called an $\mathbf{l}$-*shifted Ferrers board* if $L_N \ge a_1 \ge a_2 \ge \cdots \ge a_N \ge 0$ and the nonzero entries of a_i satisfy $a_i - a_{i+1} \ge l_{N+1-i}$ for $1 \le i \le N - 1$. A rook in (i, j) with $i < j$ in a rook placement cancels all cells (i, s) in $B_N^{\mathbf{l}}$ with $i < s < j$ and all cells (t, j) and (t, i) in $B_N^{\mathbf{l}}$ with $t < i$.

Now we work out an elliptic analogue. We assume the same rook cancellation. However, for the purpose of conveniently computing the elliptic weights of cells, we label the columns from 1 to L_N, from right to left, and label the rows from 1 to N from the bottom. When we use this labeling, we denote the board by ${}^{w}B_N^{\mathbf{l}}$ and use $(i,j)^w$ to denote a cell with respect to this labeling.

Given an $\mathbf{l}$-shifted Ferrers board $B = B(a_1, \ldots, a_N) \subseteq B_N^{\mathbf{l}}$ and a rook placement $P \in \mathcal{M}_k^{(\mathbf{l})}(B)$, let $U_B^{\mathbf{l}}(P)$ denote the set of uncancelled cells in $B - P$ by any rooks in P. Define

$$wt_m(P) = \sum_{(i,j)^w \in U_B^{\mathbf{l}}(P)} w_{a,b;q,p}\left(i + j - 1 - l_i - 2r_{(i,j)}(P) - s_{(i,j)}(P)\right), \tag{4.2}$$

where the elliptic weight $w_{a,b;q,p}(l)$ of an integer l is defined in (2.1a), $r_{(i,j)}(P)$ is the number of rooks in P positioned south-east of $(i,j)^w$ such that the two columns cancelled by those rooks are to the right of the column j, and $s_{(i,j)}(P)$ is the number of rooks in P which are in the south-east region of $(i,j)^w$ such that only one cancelled column is to the right of column j. Then we have the following theorem.

Theorem 4.1 *For an* $\mathbf{l}$*-shifted Ferrers board* $B = B(a_1, \ldots, a_N) \subseteq B_N^{\mathbf{l}}$*, define*

$$m_k^{(\mathbf{l})}(a,b;q,p;B) = \sum_{P \in \mathcal{M}_k^{\mathbf{l}}(P)} wt_m(P). \tag{4.3}$$

Then we have

$$\prod_{i=1}^{N} [z + a_{N-i+1} - 2i + 2]_{aq^{2(L_{i-1}+i-1-a_{N-i+1})}, bq^{L_{i-1}+i-1-a_{N-i+1}}; q,p}$$
$$= \sum_{k=0}^{N} m_k^{(\mathbf{l})}(a,b;q,p;B) \prod_{j=1}^{N-k} [z - 2j + 2]_{aq^{2(L_{j-1}+j-1)}, bq^{L_{j-1}+j-1}; q,p}. \tag{4.4}$$

Remark 4.2 *If we let* $N = 2n-1$ *and* $\mathbf{l} = (1,1,\ldots,1)$*, then* (4.4) *becomes*

$$\prod_{i=1}^{2n-1} [z + a_{2n-i} - 2i + 2]_{aq^{2(2i-2-a_{2n-i})}, bq^{2i-2-a_{2n-i}}; q,p}$$
$$= \sum_{k=0}^{n} m_k^{(1,1,\ldots,1)}(a,b;q,p;B) \prod_{j=1}^{2n-1-k} [z - 2j + 2]_{aq^{4j-4}, bq^{2j-2}; q,p} \tag{4.5}$$

which becomes an elliptic analogue of the product formula of Haglund and Remmel (4.1)*. We can derive*

$$m_n^{(1,1,\ldots,1)}(a,b;q,p;B) = \frac{\prod_{i=1}^{2n-1} [a_{2n-i} + 2n - 2i]_{aq^{2(2i-2-a_{2n-i})}, bq^{2i-2-a_{2n-i}}; q,p}}{\prod_{i=1}^{n-1} [2n-2i]_{aq^{4i-4}, bq^{2i-2}; q,p}},$$

which, in the case of the full shifted Ferrers board $B = B_{2n} = B(2n-1, 2n-2, \ldots, 1)$*, gives the elliptic enumeration of perfect matchings on* K_{2n}

$$m_n^{(1,1,\ldots,1)}(a,b;q,p;B_{2n}) = \frac{\prod_{i=1}^{2n-1} [2n-i]_{aq^{2i-4}, bq^{i-2}; q,p}}{\prod_{i=1}^{n-1} [2n-2i]_{aq^{4i-4}, bq^{2i-2}; q,p}}$$
$$= [2n-1]_{aq^{-2}, bq^{-1}; q,p} [2n-3]_{aq^2, bq; q,p} \ldots [1]_{aq^{4n-6}, bq^{2n-3}}.$$

Proof of Theorem 4.1: We extend the board $B_N^{\mathbf{1}}$ by attaching z many columns of height N to the right of $B_N^{\mathbf{1}}$ and denote it by $B_{N,z}^{\mathbf{1}}$. We consider N-rook placements in $B_{N,z}^{\mathbf{1}}$ and define the rook cancellation in $B_{N,z}^{\mathbf{1}}$ so that each rook cancels two columns in the rows above the row containing that rook. Then (4.4) is the result of computing the sum

$$\sum_{P\in\mathcal{N}_N(B_{N,z}^{\mathbf{1}})} wt_m(P)$$

in two different ways, where $wt_m(P)$ is the product of weights of uncancelled cells in $B_{N,z}^{\mathbf{1}} - P$. We omit the details of the proof. □

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