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Noncrossing partitions, toggles, & homomesy†

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Abstract. We introduce $n(n-1)/2$ natural involutions ("toggles") on the set S of noncrossing partitions π of size n, along with certain composite operations obtained by composing these involutions. We show that for many operations T of this kind, a surprisingly large family of functions f on S (including the function that sends π to the number of blocks of π) exhibits the homomesy phenomenon: the average of f over the elements of a T-orbit is the same for all T-orbits. Our methods apply more broadly to toggle operations on independent sets of certain graphs.

Résumé. On introduit $n(n-1)/2$ involutions naturelles ("toggles") de l'ensemble S des partitions noncroisés π de cardinalité n, et certaines operations composites obtenu en composant ces involutions. On prouvera que pour la plupart des operations T, une étonnant grande famille de fonctionnes f sur S (contenant la fonction qui envoie π sur les nombres de bloques de π) presente le phénomène de homomesie: la moyenne de f sur les éléments d'une orbite de T, c'est la même pour toutes orbites de T. Notre épreuve s'applique plus largement à les operations de toggle sur la collection des ensembles indépendants de certain graphes.

Keywords. Coxeter element, homomesy, independent set, involution, noncrossing partition, toggle group

1 Introduction

A *partition* of $[n] := \{1, 2, \ldots, n\}$ is a collection π of disjoint sets $B_1, \ldots, B_K$ with union $[n]$. We call the B_i's "blocks" and write $|\pi| = K$. A partition π is *noncrossing* if whenever $1 \leq i < j < k < \ell \leq n$, we do not have i and k belonging to one block of π with j and ℓ belonging to a different block. Simion and Ullman (1991) define an involution λ on the set $\mathrm{NC}(n)$ of noncrossing partitions of $[n]$ (they call it α) with the property that $|\pi| + |\lambda(\pi)| = n + 1$. This map is related to a different operation on $\mathrm{NC}(n)$, the *Kreweras complementation* Kreweras (1972), denoted κ. The bijection κ is not an involution but it too

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satisfies $|\pi| + |\kappa(\pi)| = n + 1$. The actions κ and λ have very different orbit-structures, but they share the curious property that the average of $|\pi|$ over each orbit is $(n+1)/2$. That is, in the terminology of Propp and Roby (2015), the statistic $\pi \mapsto |\pi|$ is *homomesic* under the actions of κ and λ, or more specifically, c-mesic with $c = (n+1)/2$.

In this paper, we exhibit a large class of actions sharing this homomesy property with κ and λ. These actions are obtained from the *toggle-action* philosophy first studied by Cameron and Fon-Der-Flaass (1995) and more recently by Striker and Williams (2012) and Striker (2016). This philosophy invites one to act on combinatorial objects via operations generated by involutions called *toggles*. Any given toggle τ will generally fix many elements of S, but by composing toggles we obtain permutations that mix S up more than any individual toggle does. Propp and Roby (2015) add to this picture the observation that in many cases of interest, T does such a good job of mixing up S that, for some interesting numerical statistics f on S, the average of f on each T-orbit is some constant that only depends on f, not what orbit we are in.

In this article, S is the set $\mathrm{NC}(n)$ and $f(\pi)$ is $|\pi|$ or some closely related quantity. To define the toggles we use, we make use of the *linear representation* of noncrossing partitions, as shown on the left in Figure 1. This depicts the numbers $1, \ldots, n$ as equally-spaced points on a horizontal line and consists of arcs above the line joining points i and j whenever i and j are successive elements of the same block of π. Formally, the linear representation P of π consists of those pairs (i, j) with $1 \le i < j \le n$ with the property that i and j are in the same block of π but none of $i+1, i+2, \ldots, j-1$ (the "interior" of the arc (i, j)) are also in that block. The noncrossing property of the partition guarantees that if two arcs belong to P, then their interiors are disjoint, their left endpoints are distinct, and their right endpoints are distinct. That is, we never see two arcs exhibiting the three "forbidden configurations": *crossing*, *left-nesting*, and *right-nesting*, as shown in the top-left of Figure 2. Note however that nested arcs are allowed. Conversely, any collection of arcs satisfying these conditions determines a unique noncrossing partition π.

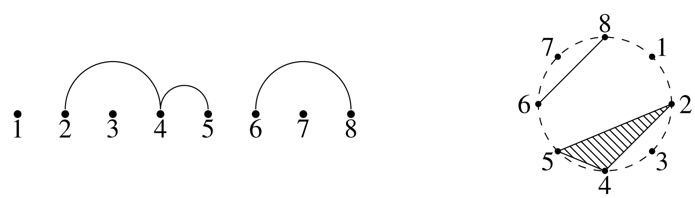

Fig. 1: The linear representation $P = \{(2,4),(4,5),(6,8)\}$ of the noncrossing partition $\pi = \{\{1\},\{2,4,5\},\{3\},\{6,8\},\{7\}\}$ at left, and the circular representation on the right.

For each pair i, j with $1 \le i < j \le n$, the toggle operation $\tau_{i,j}$ can be summarized as follows: "*If arc (i, j) is present, then remove it. If it is missing, add it if possible.*" For example, when π is the noncrossing partition whose linear representation appears on the left of Figure 1, applying $\tau_{i,j}$ removes the edge (i, j) when (i, j) is $(2, 4)$, $(4, 5)$, or $(6, 8)$, and adds the edge (i, j) when (i, j) is $(1, 2)$, $(1, 6)$, or $(5, 6)$; for all other pairs (i, j), $\tau_{i,j}$ has no effect on π.

Our main result (Theorem 4.5) is that $\pi \mapsto |\pi|$ is $(n+1)/2$-mesic under a large class of operations T obtained as compositions of toggles (even though for most T it is not the case that $|\pi| + |T(\pi)| = n + 1$ for all π). We also show how toggling noncrossing partitions in this manner is a special case of a more general action involving independent sets of certain graphs.

2 Toggling noncrossing partitions

Though we will usually refer to elements of $\mathrm{NC}(n)$ as sets of arcs, at times it will be convenient to refer to a noncrossing partition's *block representation*, $\{B_1, \ldots, B_K\}$. We shall typically use uppercase Roman letters, especially P, to refer to the linear representation and lowercase Greek letters, especially π, to refer to the block representation. A common way to visualize the blocks is with the classical *circular representation* of noncrossing partitions. (This is useful in defining the Kreweras complement and clarifying its relation to the Simion-Ullman involution.) This depicts the numbers $1, \ldots, n$ as equally-spaced points on a circle (by convention arranged clockwise) and the blocks as convex hulls. Figure 1 shows the linear and circular representations of the noncrossing partition $\pi = \{\{1\}, \{2,4,5\}, \{3\}, \{6,8\}, \{7\}\}$. The noncrossing property ensures that the convex hulls are pairwise disjoint, i.e., the blocks are "noncrossing."

For a fixed n, there is a natural partial order on $\mathrm{NC}(n)$ by *refinement*: $\pi \leq \pi'$ if each block of π is contained in a block of π'. This endows $\mathrm{NC}(n)$ with a lattice structure. More details on this and other properties of $\mathrm{NC}(n)$ can be found in the fun survey article by McCammond (2006). Note that removing an arc from a nonempty noncrossing partition P yields another noncrossing partition strictly finer than P, but this "subset order" is *not* the same as the refinement order. For example, the noncrossing partition in $\mathrm{NC}(3)$ consisting of the single arc $(1,3)$ has two blocks. It is a refinement of, but not a subset of, the noncrossing partition consisting of the arcs $(1,2)$ and $(2,3)$, which has only one block.

Definition 2.1 *Given a pair (i,j) with $1 \leq i < j \leq n$, the* **toggle operation** $\tau_{i,j}$ *on* $\mathrm{NC}(n)$ *is defined by*

$$\tau_{i,j}(P) = \begin{cases} P \cup \{(i,j)\} & (i,j) \notin P \text{ and } P \cup \{(i,j)\} \in \mathrm{NC}(n), \\ P \setminus \{(i,j)\} & (i,j) \in P, \\ P & \text{otherwise.} \end{cases}$$

The **toggle group** W_n *is the subgroup of the permutation group $S_{\mathrm{NC}(n)}$ generated by the $\binom{n}{2}$ toggle operations. We write toggles from right-to-left, so $\tau_{i,j}\tau_{k,\ell} := \tau_{i,j} \circ \tau_{k,\ell}$.*

It is clear that each toggle operation is an involution. The object of this paper is to understand some well-behaved statistics of the toggle group and its action on $\mathrm{NC}(n)$. We define the relevant notions in Section 4. The choice of W_n to denote the toggle group is motivated by the fact that it is always a quotient of a Coxeter group, which is classically denoted by W. We will revisit this in Section 3.

First, we establish which pairs of toggles commute. Any pair of distinct arcs (i,j) and (k,ℓ) can be classified into one of six types (possibly after swapping (i,j) and (k,ℓ)):

1. $i < j < k < \ell$ (disjoint),
2. $i < k < \ell < j$ (nesting),
3. $i < j = k < \ell$ (m-shaped),
4. $i = k < j < \ell$ (left-nesting),
5. $i < k < j = \ell$ (right-nesting),
6. $i < k < j < \ell$ (crossing).

The type is sufficient to determine whether a pair of toggles commutes. The following is elementary.

Proposition 2.2 *Let $\tau_{i,j}$ and $\tau_{k,\ell}$ be distinct toggles. Then $\tau_{i,j}$ and $\tau_{k,\ell}$ commute if and only if the arcs (i,j) and (k,ℓ) are disjoint, nesting, or m-shaped.* □

The three types of commuting pairs are illustrated at the bottom-left of Figure 2.

Proposition 2.2, together with the following result, shows how the order of the product of two toggles is determined only by their type. This should motivate the connection to Coxeter theory.

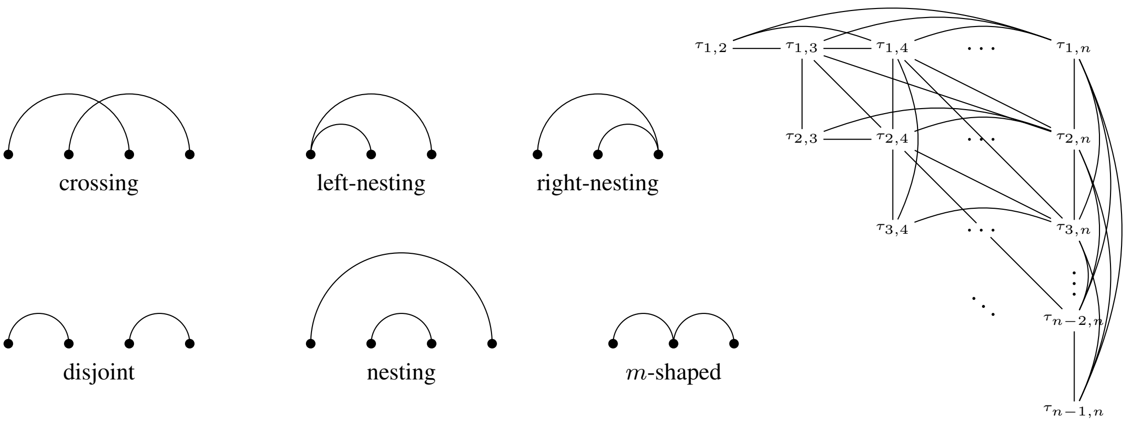

Fig. 2: *Top left*: the three disallowed pairs of arcs in a noncrossing partition, also the three noncommuting pairs of toggles. *Bottom left*: the three commuting pairs of toggles. *Right*: The base-graph Γ_n of the toggle group W_n.

Proposition 2.3 *For any pair of toggles $\tau_{i,j}$ and $\tau_{k,\ell}$, let $m(\tau_{i,j},\tau_{k,\ell})$ denote the order of $\tau_{i,j}\tau_{k,\ell}$ in W_n. Then*

$$m(\tau_{i,j},\tau_{k,\ell}) = \begin{cases} 1 & \text{if } (i,j)=(k,\ell), \\ 2 & \text{if } \tau_{i,j},\tau_{k,\ell} \text{ commute and } (i,j)\neq(k,\ell), \\ 6 & \text{if } \tau_{i,j},\tau_{k,\ell} \text{ do not commute.} \end{cases}$$

The proof of Proposition 2.3 amounts to showing that for any noncommuting pair of toggles, the action of $\tau_{i,j}\tau_{k,\ell}$ on $\mathrm{NC}(n)$ has an orbit of size 2, an orbit of size 3, and no orbit of size greater than 3.

Let C_n denote the n^{th} Catalan number, where $C_0 = C_1 = 1$. It is well-known that the cardinality $|\mathrm{NC}(n)|$ is C_n; see McCammond (2006).

The commutation relations between the toggle operations can be described by an undirected graph.

Definition 2.4 *The* **base graph** *Γ_n of W_n is the graph (V,E), where $V=\{\tau_{i,j} \mid i<j\}$, and E consists of edges of the form $\{\tau_{i,j},\tau_{k,\ell}\}$, where $\tau_{i,j}$ and $\tau_{k,\ell}$ are non-commuting toggles.*

It is easiest to arrange the vertex set in an upper-triangular grid, as shown on the right in Figure 2. Each row is a clique (complete subgraph), as is each column; these correspond to the half-nesting pairs. Finally, there are some "diagonal edges," which correspond to crossing pairs: (i,j) and (k,ℓ) where $i<k<j<\ell$. All of these are "negatively sloped" in the sense of calculus.

3 Coxeter groups

Since the toggle group is generated by involutions, it is a quotient of a Coxeter group; see Björner and Brenti (2005).

Definition 3.1 *A* **Coxeter system** *of rank r is a pair (W,S) consisting of a group W generated by a set $S=\{s_1,\dots,s_r\}$ of involutions with presentation*

$$W = \left\langle s_1,\dots,s_r \mid s_i^2 = (s_is_j)^{m_{ij}} = 1 \right\rangle,$$

where $m(s_i, s_j) := m_{ij} \geq 2$ for $i \neq j$. A **reduced expression** *of $w \in W$ is an expression $w = s_{x_1} s_{x_2} \cdots s_{x_\ell}$ such that ℓ is minimal, and ℓ is called the* **length** *of w. The* **Coxeter graph** *Γ of (W, S) is the undirected graph with vertex set S and undirected edges $\{s_i, s_j\}$ for each $m_{ij} > 2$. Edges are labeled with m_{ij}, though labels of 3 are usually suppressed because they are the most common in Coxeter theory. It is also possible for m_{ij} to be infinity, meaning $s_i s_j$ has infinite order, but this is never the case for groups we encounter in this paper. A* **Coxeter element** *of W is a product $\prod_{i=1}^{r} s_{\sigma(i)}$ for some permutation $\sigma \in S_r$, i.e. a product of all generators, each used exactly once. The set of Coxeter elements is denoted $\mathrm{C}(W, S)$, or $\mathrm{C}(W)$ if (W, S) is understood.*

Remark 3.2 *Fix an ordering $\{\tau_1, \ldots, \tau_r\}$ of the $r = \binom{n}{2}$ generators of the toggle group W_n. There is a canonical quotient $W \to W_n$, sending $s_i \mapsto \tau_i$, where W is the Coxeter group of rank r whose Coxeter graph is the base-graph Γ_n with all edge weights 6.*

Though W_n can be realized as the quotient of many Coxeter groups, W is in some sense "minimal" in that it satisfies the following universal property: For the toggle group $W_n = \langle \tau_1, \ldots \tau_n \rangle$, the Coxeter group $W = \langle s_1, \ldots, s_n \rangle$ and quotient $f \colon s_i \mapsto \tau_i$ has the property that for any other Coxeter group $W' = \langle s'_1, \ldots, s'_n \rangle$ and quotient $g : s'_i \mapsto \tau_i$, there is a unique homomorphism $h : W' \to W$ such that $f \circ h = g$.

If ω is an acyclic orientation of a graph Γ, then the pair (Γ, ω) defines a canonical partial order $P(\Gamma, \omega)$ on the vertex set, where $i <_{P(\Gamma,\omega)} j$ if there is an ω-directed path from i to j. If Γ is understood, then we denote this partial order by $P_\omega := P(\Gamma, \omega)$, and say it is a *poset over* Γ. Each Coxeter element $c \in \mathrm{C}(W, S)$ defines an acyclic orientation $\omega(c)$ of the Coxeter graph Γ: orient the edge $\{s_i, s_j\}$ as $s_i \to s_j$ iff s_i appears before s_j in c. Shi (1997) showed that this acyclic orientation is well-defined. Thus, each Coxeter element $c \in \mathrm{C}(W)$ defines a poset $P_{\omega(c)} = P(\Gamma, \omega(c))$ over the Coxeter graph. Conversely, the Coxeter elements $c = s_1 \cdots s_r$ and $c' = s'_1 \cdots s'_r$ are equal as group elements iff they are linear extensions of the same poset P_ω.

Vertices that are sources (respectively, sinks) in $\omega(c)$ are called *initial* (respectively, *terminal*) in c, and these are precisely the generators that appear first (resp. last) in some reduced expression of c. Notice that if s is initial in c, then s is terminal in scs, which is a cyclic shift of some reduced expression for c, since

$$s_{x_1}(s_{x_1} s_{x_2} \cdots s_{x_\ell}) s_{x_1} = s_{x_2} \cdots s_{x_\ell} s_{x_1} .$$

In other words, conjugating a Coxeter element c by an initial generator s cyclically shifts some reduced expression, and the corresponding acyclic orientations $\omega(c)$ and $\omega(scs)$ differ by converting a source into a sink. This generates an equivalence relation $\equiv$ on the set $\mathrm{Acyc}(\Gamma)$ of acyclic orientations, where we declare two acyclic orientations to be *torically equivalent* if one can be obtained from the other by a sequence of these source-to-sink operations. The name comes from Develin et al.. Also, Eriksson and Eriksson (2009) showed that $c, c' \in \mathrm{C}(W)$ are conjugate iff $\omega(c) \equiv \omega(c')$. Said differently, two Coxeter elements are conjugate if and only if one can be transformed into the other via a sequence of cyclic shifts and/or transpositions of commuting generators.

In summary, for a fixed Coxeter system (W, S), we have bijections between Coxeter elements and acyclic orientations, and between conjugacy classes and toric equivalence classes. Specifically, these bijections are defined by

$$\begin{aligned} \mathrm{Acyc}(\Gamma) &\longrightarrow \mathrm{C}(W,S) \\ \omega &\longmapsto w_1 \cdots w_r \end{aligned} \qquad\qquad \begin{aligned} \mathrm{Acyc}(\Gamma)/\!\equiv &\longrightarrow \mathrm{Conj}(\mathrm{C}(W,S)) \\ [\omega] &\longmapsto \mathrm{cl}_W(w_1 \cdots w_r), \end{aligned} \tag{1}$$

where $w_1 \cdots w_r$ is any linear extension of P_ω and $\mathrm{cl}_W(w_1 \cdots w_r)$ is its conjugacy class in W.

In what follows, we will use the notation and terminology of Coxeter groups to talk about quotients of Coxeter groups, i.e. groups generated by involutions. That is, when we speak of (W, S), all that is assumed is that the group W is generated by a finite set $S \subset W$ of involutions. The Coxeter graph Γ is defined as before, and the edge weights of Γ are $m_{i,j} := |s_i s_j|$. Other standard terms such as the set $\mathrm{C}(W)$ of Coxeter elements, reduced expressions, the length of an element, initial and terminal generators, and the acyclic orientation of a Coxeter element, easily and unambiguously carry over. Anytime we are specifically assuming that W is a Coxeter group, we will make this clear.

Any two Coxeter elements that arise as linear extensions of the same acyclic orientation are clearly equal as elements in W. Moreover, two Coxeter elements that are linear extensions of torically equivalent orientations will be conjugate as group elements. This is the "obvious" direction of the Erikssons' aforementioned theorem; the converse need not hold for non-Coxeter groups.

In particular, this means that if W is a Coxeter group, and W' a quotient of W (e.g., $W' = W_n$), we have the following commutative diagrams, where the top maps are bijections.

$$\begin{array}{ccc} \mathrm{Acyc}(\Gamma) & \xrightarrow{\cong} & \mathrm{C}(W) \\ & \searrow & \downarrow \\ & & \mathrm{C}(W') \end{array} \qquad \begin{array}{ccc} \mathrm{Acyc}(\Gamma)/\!\equiv & \xrightarrow{\cong} & \mathrm{Conj}(\mathrm{C}(W)) \\ & \searrow & \downarrow \\ & & \mathrm{Conj}(\mathrm{C}(W')) \end{array} \tag{2}$$

It could happen that two Coxeter elements that do not arise from linear extensions of the same orientation are nevertheless equal in W'. Similarly, it could be the case that two Coxeter elements arising from non-torically equivalent orientations happen to be conjugate for non-Coxeter-theoretic reasons.

Recall that when we are speaking of the toggle group W_n, we will assume that a product of toggles, such as $\tau_{i,j}\tau_{k,\ell}$, is performed right-to-left, as in function composition. The following is a direct consequence of the commutative diagrams in Eq. (2).

Proposition 3.3 *The Coxeter element in W_n defined by "toggling by columns" (left-to-right, reading each column from top-to-bottom) is the same as the Coxeter element defined by "toggling by rows" (top-to-bottom, reading each row from left-to-right). That is,*

$$\tau_{n-1,n}\tau_{n-2,n}\tau_{n-3,n}\cdots\tau_{1,4}\tau_{2,3}\tau_{1,3}\tau_{1,2} = \tau_{n-1,n}\tau_{n-2,n}\tau_{n-2,n-1}\cdots\tau_{1,5}\tau_{1,4}\tau_{1,3}\tau_{1,2}.$$

Proof: Both of these are linear extensions of the same acyclic orientation of Γ, namely the one that orients all edges east, and south, and southeast. □

Lemma 3.4 *In the toggle group W_n, Coxeter elements have length $\ell = \binom{n}{2}$.* □

Recall that conjugating a Coxeter element c by an initial generator corresponds to performing a source-to sink operation on the acyclic orientation $\omega(c)$ of Γ. We define a *c-admissible sequence* to be any sequence of generators that arises as a valid sequence of source-to-sink conversions on $\omega(c)$.

Definition 3.5 *Let W be a group generated by a set S of involutions, and $c \in W$ a fixed Coxeter element. A* c**-admissible sequence** *is any sequence of generators $s_{x_1}, \ldots, s_{x_m}$ such that s_{x_1} is a source of $\omega(c)$, s_{x_2} is a source of $\omega(s_{x_1} c s_{x_1})$, s_{x_3} is a source of $\omega(s_{x_2} s_{x_1} c s_{x_1} s_{x_2})$, and so on.*

Every c-admissible sequence defines a canonical group element $a = s_{x_1} \cdots s_{x_m}$ in W, and we say that $a^{-1}ca$ is an **admissible conjugation** *of c.*

The main theorem of Speyer (2009) is that if W is an infinite irreducible (that is, Γ is connected) Coxeter group, and $s_{x_1}, \ldots, s_{x_m}$ is a c-admissible sequence, then $s_{x_1} \cdots s_{x_m}$ is reduced in W. This was the first proof that powers of Coxeter elements are reduced. The utility of c-admissible sequences in this paper is that they preserve the number of times a particular arc appears in an orbit. As a result, the homomesies that we establish in this paper are preserved under conjugation by c-admissible sequences.

4 Homomesy

Definition 4.1 *Let X be a finite set, A a $\mathbb{Q}$-vector space (frequently a field such as $\mathbb{R}$), $f : X \to A$ a function or "statistic," and $T : X \to X$ a bijection. We say that the triple (X, T, f) exhibits* **homomesy** *if there exists some $c \in A$ so that, for any T-orbit $\mathcal{O}$,*

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

We call c the **mean**, *and we say that the function f is* **homomesic** *with average c (or c-***mesic** *for short) under the action of T.*

Even though we use c to denote both a Coxeter element and the mean of a homomesic function, it should always be clear from the context to which we are referring.

Definition 4.2 *The* **arc count statistic** *$\alpha(P)$ of a noncrossing partition $P \in \mathrm{NC}(n)$ is the number of pairs (i, j) with $1 \leq i < j \leq n$ appearing in P.*

The **block count statistic** $\beta(P)$ can be defined similarly: $\beta(P) = |\pi|$ where π is the block representation of the noncrossing partition P. Since a block with k elements contains $k - 1$ arcs, it follows that $\alpha(P) + \beta(P) = n$. Next, we need to define a certain action on $\mathrm{NC}(n)$.

Definition 4.3 *An element $w \in W_n$ is called a* **partial Coxeter element** *if it can be written as $w = \tau_{a_k}\tau_{a_{k-1}} \cdots \tau_{a_1}$, where each τ_{a_i} is a toggle at some arc, and $a_i \neq a_j$ if $i \neq j$. In other words, each arc appears as a toggle in w at most once, but some might not appear in w at all.*

Remark 4.4 *Much of the theory of Coxeter elements also makes sense for partial Coxeter elements. For instance, a partial Coxeter element c determines an acyclic orientation of the subgraph of Γ_n consisting of all edges corresponding to the involutions contained in c. We may also talk about admissible conjugations of partial Coxeter elements; it should be obvious how these are defined.*

The main point of this paper is to understand the distribution of α and its variants in w-orbits of $\mathrm{NC}(n)$ for partial Coxeter elements $w \in W_n$. In particular, we show the following:

Theorem 4.5 *Let $w \in W_n$ be any partial Coxeter element that contains every toggle of the form $\tau_{i,i+1}$. Then α is $\frac{n-1}{2}$-mesic on w-orbits of $\mathrm{NC}(n)$.*

Example 4.6 *Figure 3 shows the five orbits of $w = \tau_{3,4}\tau_{1,2}\tau_{2,3}\tau_{1,4}$ on $\mathrm{NC}(4)$. Note that w satisfies the necessary conditions in Theorem 4.5 but is not a Coxeter element, since it does not contain $\tau_{1,3}$ or $\tau_{2,4}$. The figure shows that α is $\frac{3}{2}$-mesic on w-orbits of $\mathrm{NC}(4)$.*

The following corollary is a special case of Theorem 4.5.

Corollary 4.7 *If $w \in W_n$ is a Coxeter element, then α is $\frac{n-1}{2}$-mesic on w-orbits of $\mathrm{NC}(n)$.* □

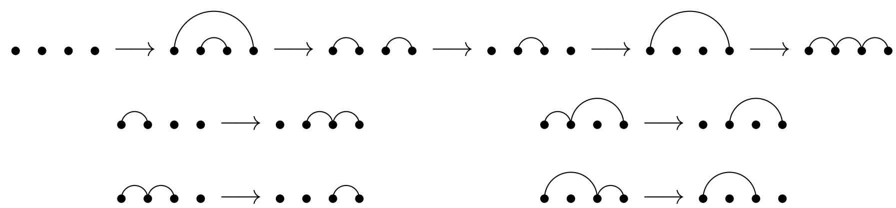

Fig. 3: The five orbits of $w = \tau_{3,4}\tau_{1,2}\tau_{2,3}\tau_{1,4}$ on $\mathrm{NC}(4)$. Notice that in any orbit, the average of the arc count is $\frac{3}{2}$.

Hence the arc count statistic is *simultaneously* homomesic for all Coxeter elements and partial Coxeter elements that contain every $\tau_{i,i+1}$. We will also show that there are more refined statistics that are homomesic for certain partial Coxeter elements. Another consequence of Theorem 4.5 is the following.

Corollary 4.8 *Let n be even and $w \in W_n$ be any partial Coxeter element that contains every toggle of the form $\tau_{i,i+1}$. Then each w-orbit of $\mathrm{NC}(n)$ contains an even number of noncrossing partitions.* □

This gives an example of how homomesy can be used to prove statements that neither mention homomesy nor the statistic that is homomesic.

5 Kreweras complementation and the Simion-Ullman involution

The action of the Coxeter element in Proposition 3.3 on $\mathrm{NC}(n)$ is actually the inverse of a well-studied action called *Kreweras complementation* introduced in Kreweras (1972) and been further investigated in Heitsch. It also arises in the new field of *Catalan combinatorics*, where Armstrong et al. (2013) generalize it to different finite Coxeter groups and use it to construct a uniform bijection between noncrossing partitions and nonnesting partitions (antichains in the positive root poset).

Definition 5.1 *Let $\pi \in \mathrm{NC}(n)$. Draw π on a circle, as shown on the right side of Figure 1, and insert a new point i' immediately clockwise from i along the circle. The* **Kreweras complement** *$\kappa(\pi)$ is the coarsest noncrossing partition of the primed numbers in the complement of π.*

See Figure 4 for a pictorial example of Kreweras complementation. The next results shows how it is closely related to the action of toggling by rows (equivalently, by columns) described in Proposition 3.3.

Theorem 5.2 *Let $\pi \in \mathrm{NC}(n)$, and denote by $\kappa(\pi)'$ the partition obtained from $\kappa(\pi)$ by replacing i with $i+1$ for each $1 \le i \le n$, such that n is replaced by 1. Then*

$$\kappa(\pi)' = \tau_{n-1,n}\tau_{n-2,n}\tau_{n-2,n-1}\cdots\tau_{1,5}\tau_{1,4}\tau_{1,3}\tau_{1,2}(\pi).$$

Refer to the full version of this paper Einstein et al. (2015) for the proof of Theorem 5.2. □

Remark 5.3 *Using the notation of Theorem 5.2, $\kappa(\pi)' = \kappa^{-1}(\pi)$. That is, κ is the inverse of the map*

$$\tau_{n-1,n}\tau_{n-2,n}\tau_{n-2,n-1}\cdots\tau_{1,5}\tau_{1,4}\tau_{1,3}\tau_{1,2}.$$

Lemma 5.4 *Applying the Kreweras complement twice to a noncrossing partition π rotates the convex hulls of the blocks counterclockwise by $2\pi/n$ radians. In particular, the order of κ divides $2n$.* □

The **Simion-Ullman involution** is $\lambda = \eta \circ \kappa$, where η is the relabeling map that replaces i by $n-i$ for $1 \le i < n$ and leaves n fixed.

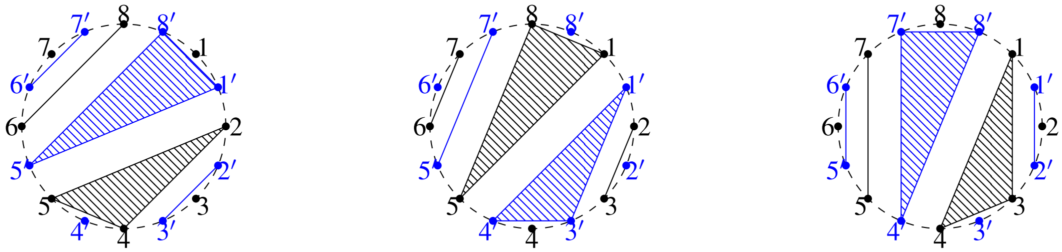

Fig. 4: Applying the Kreweras complement κ to the noncrossing partition $\pi = \{(2,4),(4,5),(6,8)\}$ shown at left yields $\kappa(\pi) = \{(1,5),(2,3),(5,8),(6,7)\}$, which is the blue noncrossing partition on the left, and the black one in the middle. Applying κ twice yields $\kappa^2(\pi) = \{(1,3),(3,4),(5,7)\}$, shown at right. The convex hulls of $\kappa^2(\pi)$ and π differ by a counterclockwise rotation of $2\pi/8$ radians.

6 Proof of Theorem 4.5

When searching for homomesies, it helps to define simple indicator function statistics, and then determine which linear combinations of these are homomesic.

Definition 6.1 *The* **indicator function** *of the arc* (i,j) *is* $\chi_{i,j} : \mathrm{NC}(n) \to \{0,1\}$*, where* $\chi_{i,j}(P)$ *is 1 if* $(i,j) \in P$ *and 0 if* $(i,j) \notin P$*.*

If two Coxeter elements $w, w' \in W_n$ are conjugate, then there is a bijection of W_n that sends w-orbits to w'-orbits.

Lemma 6.2 *Given an admissible conjugation* $w' = a^{-1}wa$ *of a Coxeter element, there is a natural bijection between* w*-orbits of* $\mathrm{NC}(n)$ *and* w'*-orbits of* $\mathrm{NC}(n)$*, given by* $\mathcal{O} \mapsto a^{-1}\mathcal{O}$*.*

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\tau_w} & \mathcal{O} \\ \downarrow & & \downarrow \\ a^{-1}\mathcal{O} & \xrightarrow{\tau_{w'}} & a^{-1}\mathcal{O} \end{array}$$

Moreover, this bijection preserves the size of the orbits. □

There is no reason to expect conjugations of Coxeter elements to preserve statistics or homomesy, because the contents of the orbits are generally scattered. However, in certain cases it surprisingly does.

Lemma 6.3 *Let* w *be a partial Coxeter element, and let* $w' = a^{-1}wa$ *be an admissible conjugate of* w*. Let* $\mathcal{O}$ *be a* w*-orbit in* $\mathrm{NC}(n)$*, and let* $\mathcal{O}' = a^{-1}\mathcal{O}$ *be the corresponding orbit of* w'*. Then*

$$\sum_{P \in \mathcal{O}} \chi_{i,j}(P) = \sum_{P' \in \mathcal{O}'} \chi_{i,j}(P').$$

Remark 6.4 *The indicator function* $\chi_{i,j}$ *is not homomesic. For example, consider the action* $\tau_{1,3} \circ \tau_{2,3} \circ \tau_{1,2}$ *on* $\mathrm{NC}(3)$ *which forms two orbits —* $\chi_{1,3}$ *is* 0 *on one orbit and nonzero on the other.*

Before proving Theorem 4.5, we first define some other statistics.

Definition 6.5 *Given $k \in [n-1]$ and $P \in \mathrm{NC}(n)$, define the statistic $\psi_k : \mathrm{NC}(n) \to \mathbb{Z}$ as*

$$\psi_k(P) = 2\chi_{k,k+1}(P) + \sum_{1 \le i \le k-1} \chi_{i,k+1}(P) + \sum_{k+2 \le j \le n} \chi_{k,j}(P) = \sum_{1 \le i \le k} \chi_{i,k+1}(P) + \sum_{k+1 \le j \le n} \chi_{k,j}(P) .$$

Due to the restrictions on arcs with a common left or right endpoint, and arcs that cross, any noncrossing partition can only contain at most one arc of the form $(i, k+1)$ or (k, j), where k is fixed and i and j are allowed to vary. Thus, for any $P \in \mathrm{NC}(n)$, $\psi_k(P) \in \{0, 1, 2\}$. Also, $\psi_k(P)$ is fully determined by the following three cases:

- $\psi_k(P) = 0$ if and only if P does not contain any arcs of the form $(i, k+1)$ or (k, j);
- $\psi_k(P) = 1$ if and only if P contains an arc of the form $(i, k+1)$ or (k, j) that is not $(k, k+1)$;
- $\psi_k(P) = 2$ if and only if P contains the arc $(k, k+1)$.

Theorem 6.6 *Given $k \in [n-1]$, let T be either a Coxeter word, or a partial Coxeter word that contains $\tau_{k,k+1}$. Then the statistic ψ_k is 1-mesic on orbits of T.*

The proof of Theorem 6.6 is done by establishing the following equivalent statement: In any orbit $\mathcal{O}$,

$$\#\{P \in \mathcal{O} : \psi_k(P) = 0\} = \#\{P \in \mathcal{O} : \psi_k(P) = 2\}.$$

In other words, in any orbit, the number of partitions that do not contain any arcs of the form $(i, k+1)$ or (k, j) is equal to the number of partitions that contain the arc $(k, k+1)$.

Note that the arc count statistic is $\sum_{i<j} \chi_{i,j}$. Given any $i < j$ with $j - i \ge 2$, the coefficient of $\chi_{i,j}$ in ψ_i and ψ_{j-1} is 1, and the coefficient of $\chi_{i,j}$ in all other ψ_k is 0. For any i, the coefficient of $\chi_{i,i+1}$ in ψ_i is 2, and the coefficient of $\chi_{i,i+1}$ in all other ψ_k is 0. Therefore, the arc count statistic is equal to $\frac{1}{2} \sum_{k=1}^{n-1} \psi_k$. Theorem 4.5 now follows.

Proof of Theorem 4.5: By Theorem 6.6, ψ_k is 1-mesic on orbits of T, for every $k \in [n-1]$. Therefore, the arc count statistic $\alpha = \frac{1}{2} \sum_{k=1}^{n-1} \psi_k$ is $\frac{n-1}{2}$-mesic. □

7 Toggling independent sets

In this section we generalize our main result to the toggle operations on independent sets.

Definition 7.1 *Let $G = (V, E)$ be a simple graph. For $v \in V$, we denote by $N(v)$ the set of neighbors of v. A set $W \subset V$ of vertices is* **independent** *if no two vertices in W are adjacent. We denote by $\beta(W)$ the cardinality of W and $\mathrm{Ind}(G)$ the set of all independent sets of V.*

Definition 7.2 *Let $G = (V, E)$ be a simple graph and let $v \in V$. The* **toggle operation** *τ_v on $\mathrm{Ind}(G)$ is*

$$\tau_v(W) = \begin{cases} W \cup \{v\} & v \notin W \text{ and } W \cup \{v\} \in \mathrm{Ind}(G), \\ W \setminus \{v\} & v \in W, \\ W & \text{otherwise.} \end{cases}$$

The **toggle group** *W_G is the subgroup of the permutation group $S_{\mathrm{Ind}(G)}$ generated by $\{\tau_v : v \in V\}$.*

Definition 7.3 *An element $a \in W_G$ is called a* **partial Coxeter element** *if it can be written as $a = \tau_{v_1}\tau_{v_2}\cdots\tau_{v_k}$, where $v_i \neq v_j$ for $i \neq j$, and $\{v_1,\ldots,v_k\} \subset V$. We call it a* **Coxeter element** *if $\{v_1,\ldots,v_k\} = V$.*

The toggle operations on $\mathrm{NC}(n)$ are a special case of Definition 7.2. Define $G = (V, E)$ to be the graph whose vertices represent the arcs in $\mathrm{NC}(n)$, and two vertices are connected by an edge if the corresponding pair cannot appear together in a noncrossing partition (i.e., crossing or half-nesting). By viewing the elements in $\mathrm{NC}(n)$ as collections of arcs, we see that $W \subset V$ is an independent set if and only if $W \in \mathrm{NC}(n)$. Note that G is just the base graph Γ_n (see Figure 2). Thus, the action of the group W_n on $\mathrm{NC}(n)$ is isomorphic to the action of the group W_{Γ_n} on Γ_n.

Consider the following analogue of the statistic from Definition 6.5.

Definition 7.4 *Given $G = (V, E)$ and $v \in V$, define the statistic $\psi_v : \mathrm{Ind}(G) \to \mathbb{Z}$ in the following way:*

$$\psi_v(W) \quad = \quad 2\chi_v(W) + \sum_{u \in N(v)} \chi_u(W)$$

where χ_v is the indicator function of the vertex v.

For $G = \Gamma_n$ and $v = (k, k+1)$ this coincides with Definition 6.5. The following is a generalization of Theorem 6.6.

Theorem 7.5 *Let $G = (V, E)$ be a simple graph. Given $v \in V$, let T be a partial Coxeter element that contains τ_v. If $N(v)$ forms a clique in G, then the statistic ψ_v is 1-mesic on orbits of T.*

The proof of Theorem 7.5 is analogous to that of Theorem 6.6. We can now generalize Theorem 4.5.

Theorem 7.6 *Let $G = (V, E)$ be a simple graph with maximal independent set U of vertices that satisfies the following two properties (in which case we say G is* **2-cliquish***):*

1. *For any $u \in U$, the set of vertices $N(u)$ forms a clique in G.*
2. *Any vertex in $V \setminus U$ has exactly two neighbors in U.*

Let T be a partial Coxeter element containing all toggles τ_u for $u \in U$. Then the cardinality function β is $\frac{A}{2}$-mesic on T-orbits of $\mathrm{Ind}(G)$, where A is the cardinality of U.

Proof: We have $|U| = A$, so by Theorem 7.5 $\sum_{u \in U} \psi_u$ is A-mesic on orbits of T. On the other hand, property (2) implies that $\sum_{u \in U} \psi_u = 2 \sum_{v \in V} \chi_v = 2\beta$. Therefore, β is $\frac{A}{2}$-mesic. □

Corollary 7.7 *If T is a Coxeter element, then β is $\frac{A}{2}$-mesic on T-orbits.* □

Example 7.8 *The following examples of 2-cliquish graphs appear in Figure 5 at left, middle, and right, respectively.*

- *A complete graph with a single edge removed is 2-cliquish with a maximal independent set of size 2. The two vertices without an edge connecting them form the maximal independent set.*
- *Given any graph G, construct a new graph as follows: Start with G, and for each vertex v, add two new vertices and connect them to v. This graph is 2-cliquish and the new vertices form a maximal independent set.*

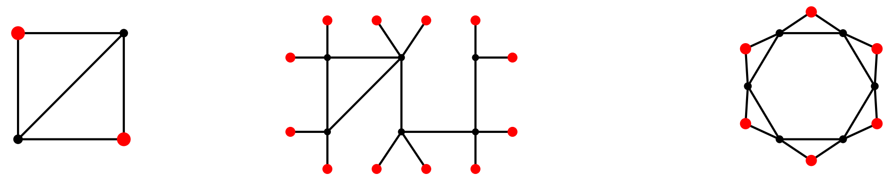

Fig. 5: Three example constructions of 2-cliquish graphs. The maximal independent sets are the large red vertices.

- *Let C_n be the cycle graph on n vertices. For each edge e, add a new vertex v_e and two new edges connecting it to the endpoints of e. This graph is 2-cliquish. The new vertices form a maximal independent set.*

We refer the reader to the full version of this paper Einstein et al. (2015) for more discussion of 2-cliquish graphs, including how to generate all 2-cliquish graphs.

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