The Prism tableau model for Schubert polynomials

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Abstract. The Schubert polynomials lift the Schur basis of symmetric polynomials into a basis for $\mathbb{Z}[x_1, x_2, \ldots]$. We suggest the *prism tableau model* for these polynomials. A novel aspect of this alternative to earlier results is that it directly invokes semistandard tableaux; it does so as part of a colored tableau amalgam. In the Grassmannian case, a prism tableau with colors ignored is a semistandard Young tableau. Our arguments are developed from the Gröbner geometry of matrix Schubert varieties.

Résumé. Les polynômes de Schubert prolongent la base des polynômes de Schur des polynômes symétriques en une base de $\mathbb{Z}[x_1, x_2, ...]$. Nous suggérons le modèle de *tableaux de prismes* pour ces polynômes. Un nouvel aspect de cette alternative aux résultats antérieurs est qu'elle utilise directement les tableaux semi-standards; Ces derniers apparaissent sous forme d'amalgames de tableaux colorés. Dans le cas Grassmannien, un tableau de prisme avec les couleurs omises est un tableau de Young semi-standard. Nos arguments sont basés sur la géométrie de Gröbner des variétés de matrices de Schubert.

Keywords. Schubert polynomials, Gröbner geometry, Young tableau

1 Introduction

1.1 Overview

A. Lascoux–M.-P. Schützenberger [LaSh82a] recursively defined an integral basis of $\mathsf{Pol} = \mathbb{Z}[x_1, x_2, \ldots]$ given by the **Schubert polynomials** $\{\mathfrak{S}_w : w \in S_\infty\}$. If w_0 is the longest length permutation in the symmetric group S_n then $\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$. Otherwise, $w \neq w_0$ and there exists i such that $w(i) < w(i+1)$. Now one sets $\mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i}$, where $\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}$ (since the polynomial operators ∂_i form a representation of S_n, this definition is self-consistent.) It is true that under the standard inclusion $\iota : S_n \hookrightarrow S_{n+1}$, $\mathfrak{S}_w = \mathfrak{S}_{\iota(w)}$. Thus one can refer to $\mathfrak{S}_w$ for each $w \in S_\infty = \bigcup_{n \geq 1} S_n$.

Textbook understanding of the ring Sym of symmetric polynomials centers around the basis of Schur polynomials and its successful companion, the theory of Young tableaux. Since Schur polynomials are instances of Schubert polynomials, the latter basis naturally lifts the Schur basis into Pol. Yet, it is also true that Schubert polynomials have nonnegative integer coefficients. Consequently, one has a natural problem:

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> Is there a combinatorial model for Schubert polynomials that is analogous to the semis-
> tandard tableau model for Schur polynomials?

Indeed, multiple solutions have been discovered over the years, e.g., [Ko90], [BiJoSt93], [BeBi93], [FoSt94], [FoKi96], [FoGrReSh97], [Ma98], [BeSo98, BeSo02], [BuKrTaYo04] and [CoTa13] (see also [LaSh85]). In turn, the solutions [BiJoSt93, BeBi93, FoSt94, FoKi96] have been the foundation for a vast literature at the confluence of combinatorics, representation theory and combinatorial algebraic geometry.

We wish to put forward another solution – a novel aspect of which is that it directly invokes semistandard tableaux. Both the statement and proof of our alternative model build upon ideas about the Gröbner geometry of matrix Schubert varieties X_w. We use the Gröbner degeneration of X_w and the interpretation of $\mathfrak{S}_w$ as mutidegrees of X_w [KnMi05]. Actually, a major purpose of *loc. cit.* is to establish the geometric naturality of the combinatorics of [BiJoSt93, BeBi93, FoKi96]. Our point of departure is stimulated by later work of A. Knutson on *Frobenius splitting* [Kn09, Theorem 6 and Section 7.2].

1.2   The main result

We recall some permutation combinatorics found in, e.g., in [Ma01]. The **diagram of** w is $D(w) = \{(i,j) : 1 \le i,j \le n, w(i) > j \text{ and } w^{-1}(j) > i\} \subset n \times n$. Let $\mathcal{E}ss(w) \subset D(w)$ be the **essential set** of w: the southeast-most boxes of each connected component of w. The **rank function** is $r_w(i,j) = \#\{t \le i : w(t) \le j\}$.

Define w to be **Grassmannian** if it has at most one descent, i.e., at most one index k such that $w(k) > w(k+1)$. If in addition w^{-1} is Grassmannian then w is **biGrassmannian**. For $e = (i,j) \in \mathcal{E}ss(w)$ let R_e be the $(i - r_w(i,j)) \times (j - r_w(i,j))$ rectangle with southwest corner at position $(i,1)$ of $n \times n$. The **shape** of w is $\lambda(w) = \bigcup_{e \in \mathcal{E}ss(w)} R_e$:
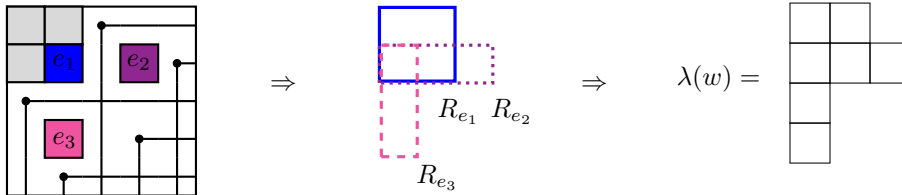

Fig. 1: The diagram of $w = 35142$ (with color coded essential set $\{e_1, e_2, e_3\}$), the overlay of $R_{e_1}, R_{e_2}, R_{e_3}$, and the shape $\lambda(w)$.

A **prism tableau** T for w fills $\lambda(w)$ with colored labels (one color for each $e \in \mathcal{E}ss(w)$) such that the labels of color e:

(S1)   sit in a box of R_e;

(S2)   weakly decrease along rows from left to right;

(S3)   strictly increase along columns from top to bottom; and

(S4)   are **flagged**: a label is no bigger than the row of the box it sits in.

Let $d_i(w)$ be the number of distinct values (ignoring color) seen on the i-th antidiagonal (i.e., the one meeting $(i, 1)$), for $i = 1, 2, \ldots, n$. We say T is **minimal** if $\sum_{i=1}^{n} d_i(w) = \ell(w)$, where $\ell(w)$ is the *Coxeter length* of w.

Let ℓ_c be a label ℓ of color c. Labels $\{\ell_c, \ell_d, \ell'_e\}$ in the same antidiagonal form an **unstable triple** if $\ell < \ell'$ and replacing the ℓ_c with ℓ'_c gives a prism tableau. See Example 1.3.

Let $\mathtt{Prism}(w)$ be the set of minimal prism tableaux with no unstable triples. Finally, set

$$\mathfrak{P}_w(x_1, \ldots, x_n) := \sum_{T \in \mathtt{Prism}(w)} \mathtt{wt}(T), \quad \text{where } \mathtt{wt}(T) = \prod_i x_i^{\text{\# of antidiagonals containing } i}.$$

Theorem 1.1. $\mathfrak{S}_w(x_1, \ldots, x_n) = \mathfrak{P}_w(x_1, \ldots, x_n)$.

Example 1.2 (Reduction to semistandard tableaux)**.** Consider the Grassmannian permutation $w = 246135$. Conflating prism tableaux with their weights, Theorem 1.1 asserts:

$$\mathfrak{S}_w = \boxed{T_1} + \boxed{T_2} + \boxed{T_3} + \boxed{T_4} + \boxed{T_5} + \boxed{T_6} + \boxed{T_7} + \boxed{T_8}.$$

Forgetting colors gives the following expansion of the Schur polynomial:

$$s_{\lambda(w)} = \boxed{T_1'} + \boxed{T_2'} + \boxed{T_3'} + \boxed{T_4'} + \boxed{T_5'} + \boxed{T_6'} + \boxed{T_7'} + \boxed{T_8'}.$$

In general, if w is Grassmannian then $\lambda(w)$ is a (French) Young diagram. Moreover, each cell of $T \in \mathtt{Prism}(w)$ uses only one number. Replacing each set in T by the common value gives a *reverse semistandard tableau*. Thus $\mathfrak{P}_w = s_\lambda(w)$ follows. $\qquad\square$

Prism tableaux provide a means to understand the RC-graphs of [BeBi93, FoKi96]. We think of the $\#\mathcal{E}ss(w)$-many semistandard tableaux of a prism tableau T as the "dispersion" of the associated RC-graph through T. See Sections 4.1 and 4.3.

Minimality and the unstable triple condition bond the tableau of each color, which is one reason why we prefer not to think of a prism tableau as merely a $\#\mathcal{E}ss(w)$-tuple:

Example 1.3 (Unstable triples)**.** Let $w = 42513$. Then $\#\mathcal{E}ss(w) = 3$. The minimal prism tableaux and their weights are:

T	$\begin{matrix}11&1&1\\22&1\\33&3\end{matrix}$	$\begin{matrix}11&1&1\\21&1\\33&3\end{matrix}$	$\begin{matrix}11&1&1\\22&1\\33&2\end{matrix}$	$\begin{matrix}11&1&1\\21&1\\33&2\end{matrix}$
$\mathtt{wt}(T)$	$x_1^3 x_2 x_3^2$	$x_1^3 x_2 x_3^2$	$x_1^3 x_2^2 x_3$	$x_1^3 x_2^2 x_3$

The second and the fourth tableaux have an unstably paired label. In both tableaux, the pink 1 in the second antidiagonal is replaceable by a pink 2. So $\mathfrak{S}_w = x_1^3 x_2 x_3^2 + x_1^3 x_2^2 x_3$. $\qquad\square$

2   Main idea of the model and its proof

Let $\mathsf{G} = \mathsf{GL}_n$ and B and B^+ the Borel subgroups of lower and upper triangular matrices in G. Identify the **flag variety** with the coset space $\mathsf{B}\backslash\mathsf{G}$. Let T be the maximal torus in B. Suppose $\mathfrak{X} \subset \mathsf{B}\backslash\mathsf{G}$ is an arbitrary subvariety and $\pi : \mathsf{G} \twoheadrightarrow \mathsf{B}\backslash\mathsf{G}$ is the natural projection. Then

$$X = \overline{\pi^{-1}(\mathfrak{X})} \subseteq \mathsf{Mat}_{n \times n}$$

carries a left B action and thus the action of T. Therefore, one can speak of the equivariant cohomology class

$$[X]_T \in H_{\mathsf{T}}(\mathsf{Mat}_{n \times n}) \cong \mathbb{Z}[x_1, \ldots, x_n].$$

Moreover, the polynomial $[X]_{\mathsf{T}}$ is a coset representative under the Borel presentation of

$$[\mathfrak{X}] \in H^{\star}(\mathsf{B}\backslash\mathsf{G}, \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]/I^{S_n},$$

where I^{S_n} is the ideal generated by (non-constant) elementary symmetric polynomials (cf. [KnMi05, pg. 1280]). This is a key perspective of work of A. Knutson-E. Miller [KnMi05] when $\mathfrak{X}$ is a Schubert variety.

Let $Y \subseteq \mathsf{Mat}_{n \times n}$ be an equidimensional, reduced union of coordinate subspaces. Given $\mathcal{P} \subset n \times n$, we represent $\mathcal{P}$ visually as a collection of $+$'s in the $n \times n$ grid. We say $\mathcal{P}$ is a **plus diagram** for Y, if

$$\mathcal{L}_{\mathcal{P}} := \{M \in \mathsf{Mat}_{n \times n} : M_{i,j} = 0 \text{ if } (i,j) \in \mathcal{P}\} \subset Y.$$

Let $\mathsf{Plus}(Y)$ be the set of all such plus diagrams. Let $\mathsf{MinPlus}(Y)$ be the set of minimal plus diagrams, i.e., those $\mathcal{P}$ for which removing any $+$ would not return an element of $\mathsf{Plus}(Y)$. We refer to the union of plus diagrams as an **overlay** to emphasize whenever (i, j) is in $\mathcal{P}$ or $\mathcal{P}'$, the diagram for $\mathcal{P} \cup \mathcal{P}'$ also has a $+$ in position (i, j).

Each $\mathcal{P}$ corresponds $1 : 1$ to a face of the Stanley-Reisner complex Δ_Y. Let $\Delta_{n \times n}$ be the power set of $\{(i, j) : 1 \leq i, j \leq n\}$. Then $\Delta_Y \subseteq \Delta_{n \times n}$ and for each $\mathcal{P}$ one has the face

$$\mathcal{F}_{\mathcal{P}} = \{(i, j) : 1 \leq i, j \leq n \text{ and } (i, j) \notin \mathcal{P}\}.$$

The faces of Δ_Y are ordered by reverse containment of their plus diagrams. Thus, facets (maximal dimensional faces) of Δ_Y coincide with elements of $\mathsf{MinPlus}(Y)$. In addition, taking the overlay of $\mathcal{P} \in \mathsf{Plus}(Y)$ and $\mathcal{Q} \in \mathsf{Plus}(Z)$ corresponds to intersecting faces in the Stanley-Reisner complex:

$$\mathcal{F}_{\mathcal{P} \cup \mathcal{Q}} = \mathcal{F}_{\mathcal{P}} \cap \mathcal{F}_{\mathcal{Q}} \in \Delta_Y \cap \Delta_Z.$$

Through the interpretation of $[Y]_T$ as a *multidegree*, one may express $[Y]_T$ as a generating series over $\mathsf{MinPlus}(Y)$. That is,

$$[Y]_T = \sum_{\mathcal{P} \in \mathsf{MinPlus}(Y)} \mathtt{wt}(\mathcal{P}), \quad \text{where } \mathtt{wt}(\mathcal{P}) = \prod_{i=1}^{n} x_i^{\# \text{ of } +\text{'s in row } i}. \tag{2.1}$$

For details, the reader may consult [MiSt05]; see Chapter 1 and Chapter 8 (and its notes).

Suppose $\prec$ is any term order on $\mathbb{C}[\mathsf{Mat}_{n \times n}]$ and $X' := \mathtt{init}_{\prec} X$. Since X is T-stable the same is true of X'; thus $[X']_{\mathsf{T}}$ is defined. Gröbner degeneration preserves the T-equivariant class, so $[X]_{\mathsf{T}} = [X']_{\mathsf{T}}$.

Suppose X' is reduced, and hence a reduced union of coordinate subspaces. Since $\mathfrak{X}$ was assumed to be irreducible, then X is irreducible. So by [KaSt95, Theorem 1] the Stanley-Reisner complex $\Delta_{X'}$ of X' is equidimensional. Hence we may apply the discussion above using $Y = X'$ to compute $[X']_\mathsf{T} = [X]_\mathsf{T}$.

We are interested in understanding $\Delta_{X'}$ under certain hypotheses on X. Assume that we have a collection of varieties $X, X_1, \ldots, X_m \subseteq V \cong \mathbb{C}^N$ such that

$$X = X_1 \cap X_2 \cap \cdots \cap X_k. \tag{2.2}$$

Assume $\prec$ is a term order on $\mathbb{C}[V]$ that defines a Gröbner degeneration of these varieties so that each Gröbner limit

$$X' := \mathtt{init}_\prec X, \ X_i' := \mathtt{init}_\prec X_i \ \text{(for } i = 1, 2, \ldots, k) \text{ is reduced.} \tag{2.3}$$

Finally, suppose

$$X' = X_1' \cap X_2' \cap \cdots \cap X_k'. \tag{2.4}$$

Call $\{X_i\}$ a $\prec$-**spectrum** for X.

To construct a cheap example, pick any Gröbner basis $G = \{g_1, \ldots, g_M\}$ with square-free lead terms to define X. Now partition $G = G_1 \cup G_2 \cup \cdots \cup G_s$ and set X_k to be cut out by G_k. On the other hand, a motivating example is A. Knutson [Kn09, Theorem 6]: given a term order $\prec$ (satisfying a hypothesis), there is a stratification of V into a poset of varieties (ordered by inclusion) with the additional feature that each stratum X admits a $\prec$-spectrum using higher strata.

How can a $\prec$-spectrum be used to understand the combinatorics of $[X']_\mathsf{T}$? Here is a simple observation:

Lemma 2.1. *Let* $\{X_i\}$ *be a $\prec$-spectrum for X. Then*

(I) $\mathtt{Plus}(X') = \{\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_k : \mathcal{P}_i \in \mathtt{Plus}(X_i')\}$.

(II) $\mathtt{MinPlus}(X') \subseteq \{\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_k : \mathcal{P}_i \in \mathtt{MinPlus}(X_i')\}$.

Proof. (I): Let $\mathcal{P} \in \mathtt{Plus}(X')$. Then $\mathcal{L}_\mathcal{P} \subseteq X' \subseteq X_i'$ for all i. Therefore $\mathcal{P} \in \mathtt{Plus}(X_i')$ and trivially $\mathcal{P} = \mathcal{P} \cup \ldots \cup \mathcal{P}$, proving "$\subseteq$". For the other containment, suppose $\mathcal{P}_i \in \mathtt{Plus}(X_i')$ for $1 \leq i \leq k$ and let $\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_k$. Then $\mathcal{L}_\mathcal{P} = \mathcal{L}_{\mathcal{P}_1} \cap \ldots \cap \mathcal{L}_{\mathcal{P}_k}$ and hence $\mathcal{L}_\mathcal{P} \subseteq \mathcal{L}_{\mathcal{P}_i} \subseteq X_i'$. So $\mathcal{P} \in \mathtt{Plus}(X_i')$ for each i, which implies $\mathcal{P} \in \mathtt{Plus}(X')$.

(II): Let $\mathcal{P} \in \mathtt{MinPlus}(X')$. By (I), $\mathcal{P} \in \mathtt{Plus}(X_i')$ for each i. Then there exists $\mathcal{P}_i \in \mathtt{MinPlus}(X_i')$ so that $\mathcal{P}_i \subseteq \mathcal{P}$. Then $\mathcal{P} \supseteq \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_k \in \mathtt{Plus}(X')$ by (I). As $\mathcal{P}$ is minimal, this is an equality. $\square$

Our point is that in good cases, the plus diagrams of X_i' are "simpler" to understand than those of X. Lemma 2.1(II) says that one can think of each $\mathcal{P} \in \mathtt{MinPlus}(X)$ as an overlay $\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_k$ of these simpler $\mathcal{P}_i$. Of course, this representation is not unique in general, so one can make a *choice* of representation for each $\mathcal{P}$. The hope is to transfer understanding of the combinatorics of $\mathtt{MinPlus}(X_i)$ to the combinatorics of $\mathtt{MinPlus}(X)$.

3 Proof of the Theorem 1.1

We now carry out the ideas described in Section 2 in the case of Schubert varieties.

3.1   Matrix Schubert varieties and Schubert polynomials

The flag variety $B\backslash G$ decomposes into **Schubert cells** $\mathfrak{X}_w^\circ := B\backslash BwB^+$ indexed by $w \in S_n$. The **Schubert variety** is the Zariski-closure $\mathfrak{X}_w := \overline{\mathfrak{X}_w^\circ}$. The **matrix Schubert variety** is

$$X_w := \overline{\pi^{-1}(\mathfrak{X}_w)} \subset \mathsf{Mat}_{n\times n}.$$

Let $Z = (z_{ij})_{1 \le i,j \le n}$ be the generic $n \times n$ matrix. The **Schubert determinantal ideal** is

$$I_w = \langle r_w(i,j) + 1 \text{ minors of the the northwest } i \times j \text{ submatrix of } Z \rangle \subset \mathbb{C}[\mathsf{Mat}_{n\times n}].$$

In [Fu91, Lemma 3.10] it is proved that I_w cuts out X_w scheme-theoretically. Moreover in *loc. cit.* it is shown that I_w is generated by the smaller set of generators coming from those $(i,j) \in \mathcal{E}ss(w)$.

By [KnMi05, Theorem A],

$$[X_w]_{\mathsf{T}} = \mathfrak{S}_w(x_1,\ldots,x_n) \in H_{\mathsf{T}}(\mathsf{Mat}_{n\times n}).$$

Moreover, let $\prec_{\mathtt{anti}}$ be any **antidiagonal term order** on $\mathbb{C}[\mathsf{Mat}_{n\times n}]$, i.e., one that picks off the antidiagonal term of any minor of Z. In [KnMi05, Theorem B] it is shown that $\mathtt{MinPlus}(X_w')$ are in a transparent bijection with the RC-graphs of [BeBi93] (cf. [FoKi96]).

For each $e \in \mathcal{E}ss(w)$, there is a unique biGrassmannian permutation u_e such that $r_{u_e}(e) = r_w(e)$ and $\mathcal{E}ss(u_e) = \{e\}$ [LaSh96]. Let

$$\mathtt{biGrass}(w) := \{u_e : e \in \mathcal{E}ss(w)\} = \{u_1,\ldots,u_k\}.$$

Call $\{X_{u_1},\ldots,X_{u_k}\}$ the **biGrassmannian $\prec_{\mathtt{anti}}$-spectrum** for X_w. By [Kn09, Section 7.2], $\{X_{u_i}\}$ indeed gives a $\prec_{\mathtt{anti}}$-spectrum for X_w over $\mathbb{Q}$. This result can also be readily obtained (over $\mathbb{Z}$) if one assumes the Gröbner basis result [KnMi05, Theorem B]. (It should be emphasized that one of the points of [Kn09, Section 7.2] is to reprove said Gröbner basis theorem more easily.)

3.2   Multi-plus diagrams

The technical core of our proof is to analyze the combinatorics of overlays of plus diagrams for the biGrassmannian $\prec_{\mathtt{anti}}$-spectrum $\{X_{u_1},\ldots,X_{u_k}\}$. Let

$$\mathtt{Multi}(w) = \prod_{i=1}^{k} \mathtt{MinPlus}(X_{u_i}')$$

be the set of **multi-plus diagrams** for w: we represent $(\mathcal{P}_1,\ldots,\mathcal{P}_k) \in \mathtt{Multi}(w)$ as a placement of colored +'s in a single $n \times n$ grid, where (a,b) has a + of color u_i if $(a,b) \in \mathcal{P}_i$.

By Lemma 2.1(I), there is a map

$$\mathtt{supp} : \mathtt{Multi}(w) \to \mathtt{Plus}(X_w')$$

given by $(\mathcal{P}_1,\ldots,\mathcal{P}_k) \mapsto \mathcal{P}_1 \cup \ldots \cup \mathcal{P}_k$. Call $\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_k$ the **support** of $(\mathcal{P}_1,\ldots,\mathcal{P}_k)$. Central to our study is

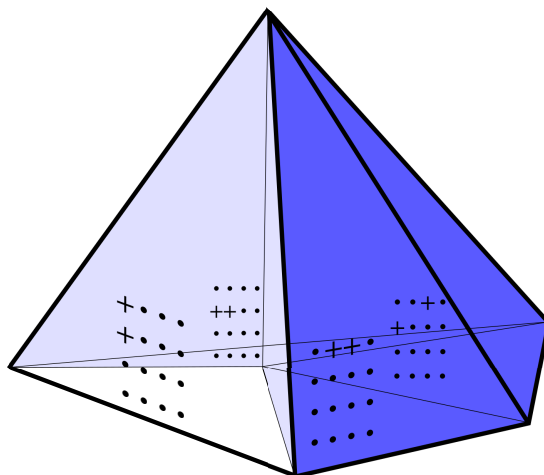$$\mathtt{Multi}(\mathcal{P}) := \mathtt{supp}^{-1}(\mathcal{P}).$$

Fig. 2: The Stanley-Reisner complexes for X'_{1423} and X'_{2314} intersect to give the complex for X'_{2413}. These complexes are a multicone over the depicted complex.

Example 3.1. Let $w = 42513$. Then $\texttt{biGrass}(w) = \{41235, 23415, 14523\}$. Now,

$$\mathcal{P} = \begin{bmatrix} + & + & + & \cdot & \cdot \\ + & \cdot & + & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \in \texttt{MinPlus}(X'_w).$$

One can check that

$$\texttt{Multi}(\mathcal{P}) = \left\{ \begin{bmatrix} ++ & ++ & ++ & \cdot & \cdot \\ + & \cdot & + & \cdot & \cdot \\ ++ & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} ++ & + & ++ & \cdot & \cdot \\ ++ & \cdot & + & \cdot & \cdot \\ ++ & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \right\}.$$

$\square$

3.3 Local moves on plus diagrams

A **southwest move** is the following local operation on a plus diagram:

$$\begin{bmatrix} \cdot & + \\ \cdot & \cdot \end{bmatrix} \mapsto \begin{bmatrix} \cdot & \cdot \\ + & \cdot \end{bmatrix}. \tag{3.1}$$

The inverse operation is a **northeast move**:

$$\begin{bmatrix} \cdot & \cdot \\ + & \cdot \end{bmatrix} \mapsto \begin{bmatrix} \cdot & + \\ \cdot & \cdot \end{bmatrix}. \tag{3.2}$$

Suppose $\mathcal{E}ss(u) = \{(i,j)\}$. Define $\mathcal{D}_{\mathtt{bot}}(u) \in \mathtt{MinPlus}(X'_u)$ as the $(i - r_u(i,j)) \times (j - r_u(i,j))$ rectangle of $+$'s, with southwest corner in row i and column 1. The following is well-known, and is a consequence (by specialization) of the chute and ladder moves of [BeBi93, Theorem 3.7]:

Lemma 3.2. *Let $u \in S_n$ be biGrassmannian.*

(I) $\mathtt{MinPlus}(X'_u)$ *is connected and closed under the moves (3.1) and (3.2).*

(II) *Each $\mathcal{P} \in \mathtt{MinPlus}(X'_u)$ can be obtained from $\mathcal{D}_{\mathtt{bot}}(u)$ using only the moves (3.2).*

Define a partial order on $\mathtt{MinPlus}(X'_u)$ by taking the transitive closure of the covering relation $\mathcal{P} < \mathcal{P}'$ if $\mathcal{P}'$ is obtained from $\mathcal{P}$ by a northeast local move (3.2). Let $<'$ be the partial order on $\mathtt{Multi}(w)$ defined as the $\mathcal{E}ss(w)$-factor Cartesian product of $<$. That is $(\mathcal{P}_1, \ldots, \mathcal{P}_k) <' (\mathcal{Q}_1, \ldots, \mathcal{Q}_k)$ if and only if $\mathcal{P}_i < \mathcal{Q}_i$ for each i. Then $<'$ induces a partial order on $\mathtt{Multi}(\mathcal{P}) \subseteq \mathtt{Multi}(w)$.

Given $(\mathcal{P}_1, \ldots, \mathcal{P}_m) \in \mathtt{Multi}(w)$, a **long move** is a repeated application of (3.1) (respectively, (3.2)) to a single $+$ appearing in one of the $\mathcal{P}_i$'s. Recall that a **lattice** is a partially ordered set in which every two elements x and y have a least upper bound $x \vee y$ (join) and a unique greatest lower bound $x \wedge y$ (meet). It is basic that a Cartesian product of lattices is a lattice.

Theorem 3.3. *Let $w \in S_n$ and $\mathcal{P} \in \mathtt{MinPlus}(X'_w)$.*

(I) $\mathtt{Multi}(\mathcal{P})$ *is connected by long moves.*

(II) *Each $(\mathtt{MinPlus}(X'_{u_i}), <)$ is a lattice. Consequently, $(\mathtt{Multi}(w), <')$ is a lattice.*

(III) $(\mathtt{Multi}(\mathcal{P}), <')$ *is a sublattice of $(\mathtt{Multi}(w), <')$.*

To prove Theorem 3.3, we construct a path of long moves between any two $\mathcal{Q}, \mathcal{Q}' \in \mathtt{Multi}(\mathcal{P})$, noting that this path contains the meet of $\mathcal{Q}$ and $\mathcal{Q}'$ in $(\mathtt{Multi}(w), <')$, and thus is the meet for $(\mathtt{Multi}(\mathcal{P}), <')$. A symmetric argument produces the join. To prove the main theorem, we show there is a bijection between the set of prism tableau for w and $\mathtt{Multi}(w)$. We then identify each $\mathcal{P} \in \mathtt{MinPlus}(X'_w)$ with the prism tableau that corresponds to the minimum element of $\mathtt{Multi}(\mathcal{P})$ and show this is a bijection.

4   Further discussion

4.1   Comparisons to the literature

Ultimately, the evaluation of any model for Schubert polynomial rests on its success towards the *Schubert problem*, i.e., finding a generalized Littlewood-Richardson rule for Schubert polynomials. Due to the analogy with Sym, one hopes that a solution will not only provide *merely* a rule, but rather lead to an entire companion combinatorial theory. This would presumably enrich our understanding of Pol and its role in mathematics just as the Young tableau theory does for Sym.

That the prism model manifestly uses Young tableaux is our impetus for ongoing investigations that fundamental tableaux algorithms might admit prism-generalizations.

The first rule for Schubert polynomials was conjectured by [Ko90]. This rule begins with the diagram of w and evolves other subsets of $n \times n$ by a simple move, the Schubert polynomial is a generating series over these subsets. A proof is presented in [Wi99, Wi02]. Arguably, this rule is the most handy of all known rules, even though the set of Kohnert diagrams does not have a closed description.

Probably the most well-known and utilized formula is given by [BiJoSt93], which expresses the Schubert polynomial in terms of reduced decompositions of w. This rule is made graphical by the RC-graphs of [BeBi93] (cf. [FoKi96]). One can obtain any RC-graph for w from any other by the *chute* and *ladder* moves of [BeBi93].

While neither of the above rules transparently reduces to the tableau rule for Schur polynomials, it is not too difficult to show in either case, that the objects involved do biject with semistandard tableaux, see [Ko90] and [Ko00] respectively.

We are not aware of any published bijection between the Kohnert rule and any other model for Schubert polynomials. On the other hand, there is a map between the prism tableaux and RC-graphs: the labels on the i-th antidiagonal indicate the row position of the $+$'s on the same antidiagonal in the associated RC-graph. We treat each RC-graph as a *specific* overlay of RC-graphs for bigrassmannian permutations. The latter RC-graphs are in bijection with semistandard tableaux of rectangular shape. This is the reason for the "dispersion" remark of the introduction.

The work of [FoGrReSh97] gives a tableau rule for Schubert polynomials of a different flavor. This rule treats $\mathfrak{S}_w$ as a generating series for **balanced fillings** of the diagram of w. The reduction to semistandard tableaux for Grassmannian w seems non-trivial.

In [BuKrTaYo04], a formula is given for a Schubert polynomial as a nonnegative integer linear combination of sum of products of Schur functions in disjoint sets of variables (with nontrivial coefficients). This is also in some sense a tableau formula for $\mathfrak{S}_w$. In [Le04] this result is rederived as a consequence of the crystal graph structure on RC-graphs developed there.

4.2 Stable Schubert polynomials

The **stable Schubert polynomial** (also known as the *Stanley symmetric polynomial*) is the generating series defined by

$$F_w(x_1, x_2, \ldots) := \lim_{m \to \infty} \mathfrak{S}_{1^m \times w},$$

where if $w \in S_n$ then $1^m \times w$ is the permutation in S_{m+n} defined by

$$(1^m \times w)(i) = i \text{ for } 1 \leq i \leq m \text{ and } (1^m \times w)(m + i) = m + w(i) \text{ for } 1 \leq i \leq n.$$

It is true that
$$F_w(x_1, x_2, \ldots, x_m, 0, 0, \ldots) = \mathfrak{S}_{1^m \times w}(x_1, \ldots, x_m, 0, 0, \ldots).$$

Now, notice that $\lambda(1^m \times w)$ and $\lambda(w)$ are the same shape, but the former is shifted down m steps in the grid relative to $\lambda(w)$. Therefore it follows that

$$F_w(x_1, x_2, \ldots, x_m, 0, 0, \ldots) = \sum_T \mathtt{wt}(T),$$

where the sum is over all *unflagged* (i.e., exclude (S4)) minimal prism tableaux of shape $\lambda(w)$ that use the labels $1, 2, \ldots, m$. In the limit, this argument implies the generating series $F_w(x_1, x_2, \ldots)$ is given by the same formula, except we allow all labels from $\mathbb{N}$.

4.3 An overlay interpretation of chute and ladder moves

In [BeBi93], **chute moves** were defined for pipe dreams. These moves are locally of the form

$$\mathcal{P} = \begin{matrix} \cdot & + & + & + & \cdots & + & + & \cdot \\ + & + & + & + & \cdots & + & + & \cdot \end{matrix} \quad \rightarrow \quad \begin{matrix} \cdot & + & + & + & \cdots & + & + & + \\ \cdot & + & + & + & \cdots & + & + & \cdot \end{matrix} = \mathcal{Q} \qquad (4.1)$$

Suppose $\mathcal{P} \in \mathtt{MinPlus}(X'_w)$, $\mathtt{biGrass}(w) = \{u_1, \ldots, u_k\}$ and $\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_k$, where $\mathcal{P}_i \in \mathtt{MinPlus}(X'_{u_i})$. The following claims show:

> *The chute move's "long jump" of a single $+$ may be interpreted as a sequence of the northeast local moves (3.1) applied to the $\mathcal{P}_i$'s.*

Example 4.1. Let $w = 1432$. Now, $\mathtt{biGrass}(w) = \{u_1 = 1423, u_2 = 1342\}$. Consider the following sequence of northeast moves

$$\begin{bmatrix} \cdot & + & \cdot & \cdot \\ + & ++ & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \rightarrow \begin{bmatrix} \cdot & + & + & \cdot \\ + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \rightarrow \begin{bmatrix} \cdot & ++ & + & \cdot \\ \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Let the support of the first and third plus diagrams be $\mathcal{P}$ and $\mathcal{Q}$, respectively. We have $\mathcal{P}, \mathcal{Q} \in \mathtt{MinPlus}(X'_w)$. $\mathcal{P}$ and $\mathcal{Q}$ differ by a chute move. At the level of the overlays, one sees this transition as an application of (3.1) to each blue $+$ in the second row.   □

Example 4.1 indicates the general pattern. Let (i, j) be the position of the southwest $+$ of $\mathcal{P}$ in (4.1) and $(i - 1, j')$ the position of the northeast $+$ of $\mathcal{Q}$ in (4.1). Without loss of generality, we may assume each $\mathcal{P}_1, \ldots, \mathcal{P}_t$ contains a $+$ at (i, j) while all other $\mathcal{P}_h$ do not.

Claim 4.2. *Consider the interval of consecutive $+$'s in row i of $\mathcal{P}_h$ ($1 \leq h \leq t$) starting at the left with the $+$ in position (i, j). One can apply the move (3.2) (in the right to left order) to each of these $+$'s to obtain $\mathcal{P}'_1, \ldots, \mathcal{P}'_t$.*

Claim 4.3. $\mathcal{P}'_h \subseteq \mathcal{Q}$ *for* $1 \leq h \leq t$ *and* $\mathcal{P}_h \subseteq \mathcal{Q}$ *for* $t + 1 \leq h \leq k$.

Claim 4.4. $\mathcal{Q} = \mathcal{P}'_1 \cup \cdots \cup \mathcal{P}'_t \cup \mathcal{P}_{t+1} \cup \cdots \cup \mathcal{P}_k$.

A similar discussion applies to the ladder moves.

4.4 Future work

It is straightforward to assign weights to prism tableau in order to give a formula for double Schubert polynomials.

A generalization to Grothendieck polynomials requires a deeper control of the overlay procedure. In investigating this, one is led to some results of possibly independent interest.

Specifically, for Theorem 1.1, we have used the fact that the facets of $\Delta_{X'_w}$ are intersections of facets of those associated to $\mathtt{biGrass}(w)$. One can make a similar conjecture for all *interior* faces w's complex. Each $\Delta_{X'_w}$ is a ball or sphere [KnMi04, Theorem 3.7]. Hence one can refer to the interior faces of this complex. Let

$$\mathtt{IntPlus}(w) = \{\mathcal{P} : \mathcal{P} \in \mathtt{Plus}(w) \text{ and } \mathcal{F}_\mathcal{P} \text{ is an interior face of } \Delta_{X'_w}\}.$$

Conjecture 4.5. $\texttt{IntPlus}(w) \subseteq \{\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_k : \mathcal{P}_i \in \texttt{IntPlus}(u_i),\ \textit{for}\ u_i \in \texttt{biGrass}(w)\}$.

Conjecture 4.5 has been exhaustively computer checked for all $n \leq 6$.

As part of an intended proof of Conjecture 4.5, one defines K-theoretic analogues of the chute and ladder moves of [BeBi93]: that is if $\mathcal{P} \to \mathcal{Q}$ by a chute move (respectively, ladder move) then $\mathcal{P} \to \mathcal{P} \cup \mathcal{Q}$ is a K-chute (respectively, K-ladder move). Whereas not all interior plus diagrams are connected by the original chute and ladder moves, it is true that they are connected once one allows the extended moves.

The first author plans to address these and related issues elsewhere.

Acknowledgements

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