

# *Categorifying the tensor product of the Kirillov-Reshetikhin crystal $B^{1,1}$ and a fundamental crystal*

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**Abstract.** We use Khovanov-Lauda-Rouquier (KLR) algebras to categorify a crystal isomorphism between a fundamental crystal and the tensor product of a Kirillov-Reshetikhin crystal and another fundamental crystal, all in affine type. The nodes of the Kirillov-Reshetikhin crystal correspond to a family of “trivial” modules. The nodes of the fundamental crystal correspond to simple modules of the corresponding cyclotomic KLR algebra. The crystal operators correspond to socle of restriction and behave compatibly with the rule for tensor product of crystal graphs.

**Résumé.** Nous utilisons Khovanov-Lauda-Rouquier (KLR) algèbres à catégorifier un cristal isomorphisme entre un cristal fondamental et le produit tensoriel d’une Kirillov-Reshetikhin cristal et un autre cristal fondamentale, le tout dans le type affine. Les noeuds du cristal Kirillov-Reshetikhin correspondre à une famille de modules “triviales”. Les noeuds du cristal fondamental correspondre à des modules simples du correspondant cyclotomique algèbre KLR. Les opérateurs de cristal correspondent à socle de la restriction et comporter la compatibilité avec la règle pour le produit de tenseur des graphes de cristal.

**Keywords.** KLR algebras, quiver Hecke algebras, crystals, categorification

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## 1 Introduction

Kang-Kashiwara [9] and Webster [23] show the cyclotomic Khovanov-Lauda-Rouquier (KLR) algebra  $R^\Lambda$  categorifies the highest weight representation  $V(\Lambda)$  in arbitrary symmetrizable type. (KLR algebras are also known in the literature as quiver Hecke algebras.) By a slight abuse of language, we will say the *combinatorial* version of this statement is that  $R^\Lambda$  categorifies the crystal  $B(\Lambda)$ , where simple modules correspond to nodes, and functors that take socle of restriction correspond to arrows, i.e. the Kashiwara crystal operators [16]. Webster [23] and Losev-Webster [18] categorify the tensor product of highest weight modules, and hence the tensor product of highest weight crystals. However, one can consider a tensor product of crystals

$$B \otimes B(\Lambda) \simeq B(\Lambda') \tag{1}$$

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where  $\Lambda, \Lambda' \in P^+$  are of level  $k$  and  $\mathcal{B}$  is a perfect crystal of level  $k$ . We (combinatorially) categorify the crystal isomorphism (1) in the case  $\Lambda = \Lambda_i$  is a fundamental weight and  $\mathcal{B} = B^{1,1}$  is a Kirillov-Reshetikhin crystal. In other words, our main theorems give a purely module-theoretic construction of this crystal isomorphism. (One must modify the form of the crystal isomorphism in the case  $B^{1,1}$  is not perfect or when  $\Lambda_i$  is not of level 1.) Each node of  $B^{1,1}$  corresponds to an infinite family of “trivial” modules, but note this does *not* give a categorification of  $\mathcal{B}$ . These “trivial” modules  $T_{p;k}$  are the KLR analogues of the nodes in highest weight crystals studied in [21].

If we apply Theorem 6.3 to iterating (1), this corresponds to constructing a simple module as a quotient of  $\text{Ind } T_{p_a;k_a} \boxtimes \cdots \boxtimes T_{p_z;k_z}$ . In type  $A$ , this is somewhat intermediate between the crystal operator construction and the Specht module construction. See [22] for details. This paper also describes how socle of restriction interacts with the construction. One can also recover the paper’s results for finite type whose Dynkin diagram is a subdiagram of that of type  $X_\ell$  studied here. For a construction of simple modules related to the crystal  $B(\infty)$  for finite type KLR algebras see [2].

This paper generalizes the theorems and constructions from [22] for type  $A$  affine.

## 2 Background and notation

### 2.1 Cartan datum

Fix an integer  $\ell \geq 2$ .  $X_\ell$  will be one of the following types:  $A_\ell^{(1)}$ ,  $C_\ell^{(1)}$ ,  $A_{2\ell}^{(2)}$ ,  $A_{2\ell}^{(2)\dagger}$ ,  $D_{\ell+1}^{(2)}$ ,  $D_\ell^{(1)}$ ,  $B_\ell^{(1)}$ , and  $A_{2\ell-1}^{(2)}$ .  $I = \{0, 1, \dots, \ell\}$  will denote the indexing set. Let  $[a_{ij}]_{i,j \in I}$  denote the associated Cartan matrix. We direct the reader to [7] for the explicit matrices. Following [7] we let  $\mathfrak{h}$  be a Cartan subalgebra,  $\Pi = \{\alpha_0, \dots, \alpha_\ell\}$  its system of simple roots,  $\Pi^\vee = \{h_0, \dots, h_\ell\}$  its simple coroots, and  $Q$  the root lattice. Then set  $Q^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ . For an element  $\nu \in Q^+$ , we define its *height*  $|\nu|$  to be the sum of the coefficients, i.e. if  $\nu = \sum_{i \in I} \nu_i \alpha_i$  then

$$|\nu| = \sum_{i \in I} \nu_i.$$

There is a canonical pairing  $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \rightarrow \mathbb{C}$  with  $a_{ij} = \langle h_i, \alpha_j \rangle$ . Using this pairing we define the fundamental weights  $\{\Lambda_i \mid i \in I\}$  via  $\langle h_i, \Lambda_j \rangle = \delta_{ij}$ . The weight lattice is  $\bigoplus_{i \in I} \mathbb{Z} \Lambda_i$  and the integral dominant weights are  $P^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i$ . We also have a symmetric bilinear form  $(\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \rightarrow \mathbb{C}$ ,

satisfying  $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$  and  $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ .

For the Dynkin diagrams of the types under consideration we direct the reader to [7].

### 2.2 The tensor product of two crystals

We refer the reader to [11] for the definition of a crystal. Let  $B_1$  and  $B_2$  be two crystals with nodes  $b_1 \in B_1$  and  $b_2 \in B_2$ . We recall that the crystal structure on the tensor product  $B_1 \otimes B_2$  is given by

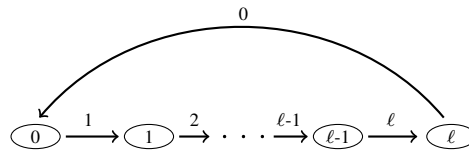
$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) > \varphi_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varepsilon_i(b_1) \leq \varphi_i(b_2) \end{cases} \quad \tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varphi_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varepsilon_i(b_1) < \varphi_i(b_2). \end{cases} \tag{2}$$

Given a crystal  $B$ , we can draw its associated crystal graph with nodes (or vertices)  $B$  and  $I$ -colored arrows (directed edges) as follows. When  $\tilde{e}_i b = a$  (so  $b = \tilde{f}_i a$ ) we draw an  $i$ -colored arrow  $a \xrightarrow{i} b$ . We also say  $b$  has an incoming  $i$ -arrow and  $a$  has an outgoing  $i$ -arrow.

### 2.3 Perfect crystals and Kirillov-Reshetikhin crystals

#### 2.3.1 Type A

In type  $A_\ell^{(1)}$ , the highest weight crystal  $B(\Lambda_i)$  has a model with nodes the  $(\ell + 1)$ -restricted partitions, i.e.  $\lambda = (\lambda_1, \dots, \lambda_t)$  such that  $\lambda_r \in \mathbb{Z}_{\geq 0}$ ,  $0 \leq \lambda_r - \lambda_{r+1} < \ell + 1$  for all  $r$ . Let  $B^{1,1}$  be the crystal graph



$B^{1,1}$  is an example of a *perfect* crystal (see [10] for the definition and important properties). One key property this level 1 perfect crystal has is that tensoring it with a fundamental (or highest weight level 1) crystal yields an isomorphism to another level 1 highest weight crystal. In particular, for  $i \in I$  there exists an isomorphism of crystals,

$$\mathcal{T} : B(\Lambda_{i+1}) \xrightarrow{\cong} B^{1,1} \otimes B(\Lambda_i).$$

The isomorphism is pictured in Figure 1 for  $i = 2$  and  $\ell = 2$ . Note the underlying graph of  $B(\Lambda_i)$  is identical to that of  $B(\Lambda_0)$ , but the colors of the arrows are obtained from those of  $B(\Lambda_0)$  by adding  $i \pmod{\ell+1}$ .

Combinatorially,  $\mathcal{T}(\lambda) = \overset{\circ}{k} \otimes \mu$  where  $k \equiv \lambda_1 + i \pmod{\ell + 1}$  and  $\mu = (\lambda_2, \dots, \lambda_t)$  if  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$ . So we obtain  $\mu$  from  $\lambda$  by removing its top row.

#### 2.3.2 General type

The perfect crystal  $B^{1,1}$  in type  $A_\ell^{(1)}$  is also an example of a Kirillov-Reshetikhin (KR) crystal. For a quantized affine algebra  $U'_q(\mathfrak{g})$ , the KR crystals  $B^{r,s}$  correspond to a special family of finite dimensional modules  $W^{r,s}$  indexed by a positive integer  $s$  and a Dynkin node  $r$  from the classical subalgebra  $\mathfrak{g}_0$  of  $\mathfrak{g}$  [14], [20]. In all of the types we consider, with the exception of  $C_\ell^{(1)}$ , the crystal  $B^{1,1}$  is perfect of level 1 [5]. When  $B^{1,1}$  is perfect and  $\Lambda_i$  is a level 1 fundamental weight for  $i \in I$ ,  $B^{1,1}$  has a unique node  $b_i$  such that  $\varepsilon(b_i) = \Lambda_i$  and  $\varphi(b_i) = \Lambda_{\sigma(i)}$  for some  $\sigma(i) \in I$ . There then exists a crystal isomorphism [10]

$$\mathcal{T} : B(\Lambda_{\sigma(i)}) \xrightarrow{\cong} B^{1,1} \otimes B(\Lambda_i). \tag{3}$$

In the interest of space we do not provide details of the case where either  $B^{1,1}$  is not perfect or  $\Lambda_i$  is not a level one weight. However, with modification the theorems and proofs in this paper hold in these cases also.

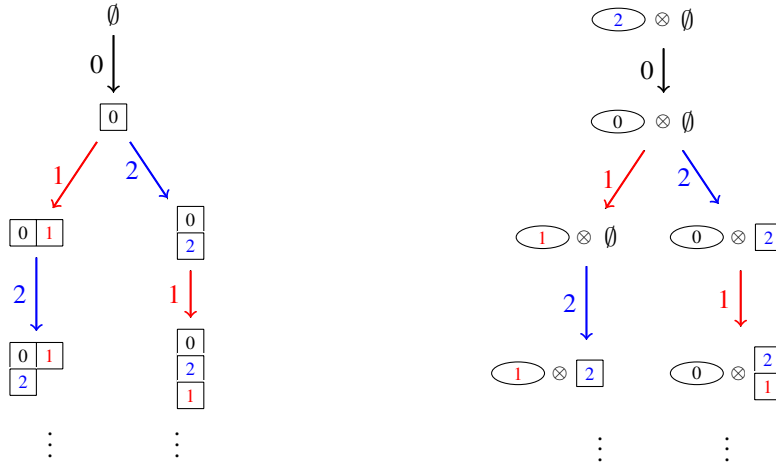


Fig. 1: The isomorphism  $B(\Lambda_0) \simeq B^{1,1} \otimes B(\Lambda_2)$  for  $\ell = 2$ .

### 3 Key definitions: class $\mathcal{A}$ , $\mathcal{B}$ , $\mathcal{D}$ nodes and cyclotomic paths

The analogues of trivial modules that we study below are constructed from crystal data given by walks on  $B^{1,1}$ . Below are definitions used to describe these walks.

**Definition 3.1** Let  $X_\ell$  be one of the affine types listed in Section 2.1 and let  $I$  be the indexing set of its Dynkin nodes. A type  $X_\ell$  path  $p$  of length  $k$ , is a function  $p : \{0, 1, \dots, k - 1\} \rightarrow I$  such that there is a directed walk in the type  $X_\ell$  crystal  $B^{1,1}$  whose  $i$ th step corresponds to a  $p(i)$ -arrow.

When  $k > 1$ , a path  $p$  of length  $k$  corresponds to a unique directed walk in  $B^{1,1}$ . In this case we use the terms “type  $X_\ell$  path” and “walk in  $B^{1,1}$ ” interchangeably.

**Definition 3.2** For a path  $p : \{0, 1, \dots, k - 1\} \rightarrow I$  we call the arrow corresponding to  $p(0)$  the tail of  $p$ , and the arrow corresponding to  $p(k - 1)$  the head of  $p$ . An extension to the tail of  $p$  by a  $j$ -arrow is a path  $p' : \{0, 1, \dots, k\} \rightarrow I$  such that  $p'(t) = p(t - 1)$  for  $1 \leq t \leq k$  and  $p'(0) = j$ . An extension to the head of  $p$  by a  $j$ -arrow, is a path  $p'' : \{0, 1, \dots, k\} \rightarrow I$  such that  $p''(t) = p(t)$  for  $0 \leq t \leq k - 1$  and  $p''(k) = j$ .

Let  $\pi(j)$  be the length 1 path  $\pi(j) : \{0\} \rightarrow I$ ,  $\pi(j)(0) = j$ . For a path  $p$ , we denote the extension of its tail by a  $j$ -arrow by  $\pi(j) \star p$  and the extension of its head by a  $j$ -arrow by  $p \star \pi(j)$ . We can think of extension as concatenation of paths. If the tail (respectively head) of  $p$  cannot be extended by a  $j$ -arrow then we set  $\pi(j) \star p = \mathbf{0}$  (respectively  $p \star \pi(j) = \mathbf{0}$ ).

The set of colors of arrows that can extend the tail of a path  $p$  of length  $k > 1$  is denoted  $\text{ext}_p^-$ ,

$$\text{ext}_p^- := \{ j \mid \pi(j) \star p \neq \mathbf{0} \}. \tag{4}$$

The set of colors of arrows that can extend the head of  $p$  is denoted  $\text{ext}_p^+$ ,

$$\text{ext}_p^+ := \{ j \mid p \star \pi(j) \neq \mathbf{0} \}. \tag{5}$$

Type	class $\mathcal{A}$	class $\mathcal{B}$	class $\mathcal{D}$ pairs
$A_\ell^{(1)}$	$0, 1, \dots, \ell - 1, \ell$		
$C_\ell^{(1)}$	$0, 1, \dots, \ell - 1, \ell$		
$A_{2\ell}^{(2)}$	$1, \dots, \ell - 1, \ell$	0	
$A_{2\ell}^{(2)\dagger}$	$0, 1, \dots, \ell - 1$	$\ell$	
$D_{\ell+1}^{(2)}$	$1, \dots, \ell - 1$	$0, \ell$	
$D_\ell^{(1)}$	$2, 3, \dots, \ell - 2$		$(0, 1), (\ell - 1, \ell)$
$B_\ell^{(1)}$	$2, \dots, \ell - 1$	$\ell$	$(0, 1)$
$A_{2\ell-1}^{(2)}$	$2, \dots, \ell - 1, \ell$		$(0, 1)$

**Tab. 1:** Classification of elements of  $I$  for each type  $X_\ell$

When  $\text{ext}_p^-$  (respectively  $\text{ext}_p^+$ ) contains a single element  $i$ , we allow ourselves the convenience of writing  $p(-1) = i$  (respectively  $p(k) = i$ ).

In Table 1 we classify the elements of  $I$  into three classes:  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{D}$ .

Below is one of our key definitions.

**Definition 3.3** Let  $p$  be a path of length  $k \geq 1$  in  $B^{1,1}$ .  $p$  is called a cyclotomic path of tail weight  $(\Lambda_{i_1}, \Lambda_{i_2})$  if the following hold:

1.  $p(0) = i_2$  and  $p(1) \neq i_2$ . If  $p$  has length 1 then  $p \star \pi(i_2) = \mathbf{0}$ , i.e. the head of  $p$  cannot be extended by an  $i_2$ -arrow.
2.  $p(0)$  and  $p(1)$  are not a class  $\mathcal{D}$  pair. If  $p$  has length 1 then there is no  $i \in I$  such that the head of  $p$  can be extended by an  $i$ -arrow and  $(p(0), i)$  is a class  $\mathcal{D}$  pair.
3.  $\text{ext}_p^- = \{i_1\}$ , i.e.  $p$  has a unique extension by an  $i_1$ -arrow to its tail.
4.  $(\pi(i_1) \star (\pi(i_1) \star p)) = \mathbf{0}$ , i.e.  $p$  cannot have its tail extended twice by  $i_1$ -arrows.

The following proposition follows from inspection of the  $B^{1,1}$  crystal graphs.

**Proposition 3.4** Let  $X_\ell$  be any type not equal to  $D_3^{(2)}$ ,  $D_4^{(1)}$  or  $B_3^{(1)}$ . Then for all  $i \in I$  there exists some  $j \in I$  such that there is a cyclotomic path  $p$  of tail weight  $(\Lambda_j, \Lambda_i)$ . When these paths exist they can be of any length  $k \in \mathbb{Z}_{\geq 1}$ .

**Definition 3.5** We refer to  $1 \in I$  in type  $D_3^{(2)}$  and  $2 \in I$  in types  $D_4^{(1)}$  and  $B_3^{(1)}$  as forbidden elements.

While paths are about  $i \in I$  and hence the arrows of  $B^{1,1}$ , informally, a cyclotomic path corresponds to a walk that starts at a ‘‘level 1 node,’’ that is a node  $b$  with  $\varepsilon(b) = \Lambda_{i_1}$  and  $\varphi(b) = \Lambda_{i_2}$ .

## 4 The KLR algebra $R$ and some functors

### 4.1 The KLR algebra $R(\nu)$

For  $\nu = \sum_{i \in I} \nu_i \alpha_i$  in  $Q^+$  with  $|\nu| = m$ , we define  $\text{Seq}(\nu)$  to be all sequences  $\mathbf{i} = (i_1, i_2, \dots, i_m)$  such that  $\alpha_{i_1} + \dots + \alpha_{i_m} = \nu$ . For  $\mathbf{i} \in \text{Seq}(\nu)$  and  $\mathbf{j} \in \text{Seq}(\mu)$ ,  $\mathbf{ij}$  will denote the concatenation of the two sequences. It follows that  $\mathbf{ij} \in \text{Seq}(\nu + \mu)$ .

For  $\nu \in Q^+$  with  $|\nu| = m$ , the *KLR algebra*  $R(\nu)$  associated with Cartan matrix  $[a_{ij}]_{i,j \in I}$  is the associative, graded, unital  $\mathbb{C}$ -algebra generated by

$$1_{\mathbf{i}} \text{ for } \mathbf{i} \in \text{Seq}(\nu), \quad x_r \text{ for } 1 \leq r \leq m, \quad \psi_r \text{ for } 1 \leq r \leq m - 1,$$

subject to relations which can be found in [12] or [16]. We set  $R = \bigoplus_{\nu \in Q^+} R(\nu)$  (which is not unital).

The elements  $1_{\mathbf{i}}$  are orthogonal idempotents since they satisfy  $1_{\mathbf{i}}1_{\mathbf{j}} = \delta_{\mathbf{i},\mathbf{j}}1_{\mathbf{i}}$  and the identity element of  $R(\nu)$  is

$$1_{\nu} = \sum_{\mathbf{i} \in \text{Seq}(\nu)} 1_{\mathbf{i}}.$$

The generators of  $R(\nu)$  are graded via  $\deg(1_{\mathbf{i}}) = 0$ ,  $\deg(x_r 1_{\mathbf{i}}) = (\alpha_{i_r}, \alpha_{i_r})$ ,  $\deg(\psi_r 1_{\mathbf{i}}) = -(\alpha_{i_r}, \alpha_{i_{r+1}})$ .

$R(\alpha_i)$  has a unique (up to grading shift) simple module  $L(i)$ . It is 1-dimensional and  $x_1 1_i$  acts as zero.  $R(n\alpha_i)$  has a unique simple module  $L(i^n)$ .

### 4.2 Induction, co-induction, and restriction

It was shown in [12] and [13] that for  $\nu, \mu \in Q^+$  there is a non-unital embedding

$$R(\nu) \otimes R(\mu) \hookrightarrow R(\nu + \mu).$$

Using this embedding one can define induction and restriction functors,

$$\begin{aligned} \text{Ind}_{\nu, \mu}^{\nu + \mu} : (R(\nu) \otimes R(\mu))\text{-mod} &\rightarrow R(\nu + \mu)\text{-mod} \\ M &\mapsto R(\nu + \mu) \otimes_{R(\nu) \otimes R(\mu)} M \end{aligned}$$

and

$$\text{Res}_{\nu, \mu}^{\nu + \mu} : R(\nu + \mu)\text{-mod} \rightarrow (R(\nu) \otimes R(\mu))\text{-mod}.$$

In the future we will write  $\text{Ind}_{\nu, \mu}^{\nu + \mu} = \text{Ind}$  and  $\text{Res}_{\nu, \mu}^{\nu + \mu} = \text{Res}$  when the algebras are understood from the context. More generally we can extend this embedding to finite tensor products

$$R(\nu^{(1)}) \otimes R(\nu^{(2)}) \otimes \dots \otimes R(\nu^{(k)}) \hookrightarrow R(\nu^{(1)} + \nu^{(2)} + \dots + \nu^{(k)}).$$

We refer to the image of this embedding as a *parabolic subalgebra* and denote it by  $R(\underline{\nu}) \subset R(\nu^{(1)} + \dots + \nu^{(k)})$ .

### 4.3 Crystal operators

Define the functor  $e_i : R(\nu) - \text{mod} \rightarrow R(\nu - \alpha_i) - \text{mod}$  as the restriction,

$$e_i M := \text{Res}_{R(\nu - \alpha_i)}^{R(\nu - \alpha_i) \otimes R(\alpha_i)} \circ \text{Res}_{\nu - \alpha_i, \alpha_i}^\nu M.$$

When  $M$  is simple we can further refine this functor by setting  $\tilde{e}_i M := \text{soc } e_i M$ . Recall for a module  $N$  that  $\text{soc } N$  is its largest semisimple submodule while  $\text{cosoc } N$  is its largest semisimple quotient. We measure how many times we can apply  $\tilde{e}_i$  to  $M$  by

$$\varepsilon_i(M) := \max\{n \geq 0 \mid (\tilde{e}_i)^n M \neq \mathbf{0}\}. \tag{6}$$

Let  $\tilde{f}_i : R(\nu) - \text{mod} \rightarrow R(\nu + \alpha_i) - \text{mod}$  be defined by

$$\tilde{f}_i M := \text{cosoc } \text{Ind } M \boxtimes L(i) \tag{7}$$

(still assuming  $M$  is simple). We refer the reader to [12] for the most important facts about  $e_i, \tilde{e}_i, \tilde{f}_i$ .

We define  $e_i^\vee, \tilde{e}_i^\vee, \tilde{f}_i^\vee$  by

$$e_i^\vee := \text{Res}_{R(\nu - \alpha_i)}^{R(\alpha_i) \otimes R(\nu - \alpha_i)} \circ \text{Res}_{\alpha_i, \nu - \alpha_i}^\nu, \tag{8}$$

$$\tilde{e}_i^\vee M := \text{soc } e_i^\vee M, \tag{9}$$

$$\tilde{f}_i^\vee M := \text{cosoc } \text{Ind}_{\alpha_i, \nu}^{\nu + \alpha_i} L(i) \boxtimes M, \tag{10}$$

$$\varepsilon_i^\vee(M) := \max\{m \geq 0 \mid (\tilde{e}_i^\vee)^m M \neq \mathbf{0}\}. \tag{11}$$

It is important to note that by the exactness of restriction,  $e_i, e_i^\vee$  are exact functors, while  $\tilde{e}_i$  and  $\tilde{e}_i^\vee$  are only left exact, and  $\tilde{f}_i$  and  $\tilde{f}_i^\vee$  are only right exact.

### 4.4 $\text{Rep}^\Lambda$ and the functor $\text{pr}_\Lambda$

For  $\Lambda = \sum_{i \in I} \lambda_i \Lambda_i \in P^+$  define  $\mathcal{I}_\nu^\Lambda$  to be the two-sided ideal of  $R(\nu)$  generated by the elements  $x_1^{\lambda_{i_1}} 1_{\underline{i}}$  for all  $\underline{i} \in \text{Seq}(\nu)$ . When  $\nu$  is clear from the context we write,  $\mathcal{I}_\nu^\Lambda = \mathcal{I}^\Lambda$ . The *cyclotomic KLR algebra of weight  $\Lambda$*  is then defined as

$$R^\Lambda = \bigoplus_{\nu \in Q^+} R^\Lambda(\nu) \quad \text{where} \quad R^\Lambda(\nu) := R(\nu) / \mathcal{I}_\nu^\Lambda.$$

The algebra  $R^\Lambda(\nu)$  is finite dimensional [3, 16]. The category of finite dimensional  $R$ -modules on which  $\mathcal{I}^\Lambda$  vanishes is denoted  $\text{Rep}^\Lambda$ . We construct a right-exact functor,  $\text{pr}_\Lambda : R(\nu) - \text{mod} \rightarrow R(\nu) - \text{mod}$ , via

$$\text{pr}_\Lambda M := M / \mathcal{I}^\Lambda M \tag{12}$$

and extend it to  $\text{pr}_\Lambda : R - \text{mod} \rightarrow R - \text{mod}$ .

**Proposition 4.1** [16] *Let  $\Lambda = \sum_{i \in I} \lambda_i \Lambda_i \in P^+$ ,  $\nu \in Q^+$ , and let  $M$  be a simple  $R(\nu)$ -module. Then  $\mathcal{I}^\Lambda M = \mathbf{0}$  if and only if  $\text{pr}_\Lambda M \cong M$  if and only if  $\text{pr}_\Lambda M \neq \mathbf{0}$  if and only if*

$$\varepsilon_i^\vee(M) \leq \lambda_i \quad \text{for all } i \in I.$$

*When these conditions hold,  $M \in \text{Rep}^\Lambda$ , and we may identify  $M$  with  $\text{pr}_\Lambda M$  (or as an  $R^\Lambda(\nu)$ -module).*

We will primarily consider  $\Lambda = \Lambda_i$  in which case  $\mathcal{I}_\nu^{\Lambda_i}$  is generated by  $x_1 1_{i_2 \dots i_m}$  and  $1_{j i_2 \dots i_m}, j \neq i$  ranging over  $\underline{i} \in \text{Seq}(\nu)$ .

### 4.5 Module-theoretic model of $B(\Lambda)$

Let  $M$  be a simple  $R(\nu)$ -module. Set

$$\text{wt}(M) = -\nu \quad \text{and} \quad \text{wt}_i(M) = -\langle h_i, \nu \rangle. \tag{13}$$

Let  $\text{Irr } R$  be the set of isomorphism classes of simple  $R$ -modules and  $\text{Irr } R^\Lambda$  be the set of isomorphism classes of simple modules in  $\text{Rep}^\Lambda$ . For  $M \in \text{Irr } R$  set  $\varphi_i(M) = \varepsilon_i(M) + \text{wt}_i(M)$ . For  $M \in \text{Irr } R^\Lambda$  set

$$\varphi_i^\Lambda(M) = \lambda_i + \varepsilon_i(M) + \text{wt}_i(M). \tag{14}$$

In [16] it was shown that the tuple  $(\text{Irr } R, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \text{wt})$  defines a crystal isomorphic to  $B(\infty)$  and  $(\text{Irr } R^\Lambda, \varepsilon_i, \varphi_i^\Lambda, \tilde{e}_i, \tilde{f}_i, \text{wt})$  defines a crystal isomorphic to the highest weight crystal  $B(\Lambda)$ .

**Proposition 4.2** [16] *Let  $M$  be a simple  $R(\nu)$ -module with  $\text{pr}_\Lambda M \neq \mathbf{0}$ . Then*

$$\varphi_j^\Lambda(M) = \max\{k \in \mathbb{Z} \mid \text{pr}_\Lambda \tilde{f}_j^k M \neq \mathbf{0}\}.$$

We mimic the conventions usually used in the theory of crystals and define

$$\varphi^\Lambda(M) := \sum_{i \in I} \varphi_i^\Lambda(M) \Lambda_i, \quad \varepsilon(M) := \sum_{i \in I} \varepsilon_i(M) \Lambda_i, \quad \varepsilon^\vee(M) := \sum_{i \in I} \varepsilon_i^\vee(M) \Lambda_i. \tag{15}$$

## 5 The family of modules $T_{p;k}$

Denote by  $\mathbb{1}$  the trivial  $R(0)$ -module.

**Definition 5.1** *For any type  $X_\ell$  and for a fixed path  $p$  of length  $k$  in  $B^{1,1}$ , define*

$$\tilde{f}_{p(k-1)} \tilde{f}_{p(k-2)} \cdots \tilde{f}_{p(0)} \mathbb{1} \cong T_{p;k} \tag{16}$$

and  $\gamma_{p;k} := \sum_{i=0}^{k-1} \alpha_{p(i)}$  so that  $T_{p;k}$  is an  $R(\gamma_{p;k})$ -module.

When  $T_{p;k} \notin \text{Rep}^{\Lambda_{p(0)}}$  then  $T_{p;k} \in \text{Rep}^{\Lambda_{p(0)} + \Lambda_{p(1)}}$ . However, such  $p$  will not ever arise in our main theorems. If  $p$  is a cyclotomic path of tail weight  $(\Lambda_{i_1}, \Lambda_{i_2})$  then the modules  $T_{p;k}$  belong to  $\text{Rep}^{\Lambda_{i_2}}$  for any  $k \geq 0$ . This is part of the motivation for Definition 3.3 of cyclotomic path.

Note that this definition implies that if  $j \in \text{ext}_p^+$  then  $\tilde{f}_j T_{p;k} \cong T_{p \star \pi(j); k+1}$ . Observe  $T_{p;0} = \mathbb{1}$ .

For  $k > 1$ , we define  $\varphi_j^-(p; k)$  and  $\varphi_j^+(p; k)$  as in Table 2, so that

$$\varphi_j(p; k) = \varphi_j^-(p; k) + \varphi_j^+(p; k).$$

We also define,

$$\varphi^-(p; k) := \sum_{i \in I} \varphi_i^-(p; k) \Lambda_i \quad \varphi^+(p; k) := \sum_{i \in I} \varphi_i^+(p; k) \Lambda_i \quad \varphi(p; k) = \varphi^-(p; k) + \varphi^+(p; k). \tag{17}$$

**Remark 5.2** *Recall that if  $p$  is a cyclotomic path of tail weight  $(\Lambda_{i_1}, \Lambda_{i_2})$  then  $p$  has a unique extension to its tail by an  $i_1$ -arrow, i.e.  $\text{ext}_p^- = \{i_1\}$ .*



Class	$\varphi_j^-(p; k)$	$\varphi_j^+(p; k)$
class $\mathcal{A}$	$\delta_{j, \text{ext}_p^-}$	$\delta_{j, \text{ext}_p^+}$
class $\mathcal{B}$	$2\delta_{j, \text{ext}_p^-} - \delta_{j, p(0)}$	$\delta_{j, \text{ext}_p^+}(2 - \delta_{j, p(k-1)})$
class $\mathcal{D}$	$\delta_{j, \text{ext}_p^-} + (\delta_{j, \text{ext}_p^-} - 1)\delta_{j', p(0)}$	$\delta_{j, \text{ext}_p^+}$

**Tab. 2:** Values of  $\varphi_j^-(p; k)$  and  $\varphi_j^+(p; k)$ ,  $k > 1$ , by class of  $j \in I$ . When  $j$  is Class  $\mathcal{D}$ ,  $(j, j')$  is a Class  $\mathcal{D}$  pair.

### 6 Main theorems

The proofs of our main theorems rely on induction on  $k$ , the length of the path  $p$ , and the lemmas below lay out several key algebraic facts that let us pass from  $k$  to  $k + 1$ .

Below it is clear our induction must terminate since the  $k$  in question is bounded by  $|\nu|$ .

**Lemma 6.1** *Let  $p$  be a cyclotomic path of tail weight  $(\Lambda_{p(-1)}, \Lambda_{p(0)})$  so that  $\varphi^-(p; k) = \Lambda_{p(-1)}$ . When  $k \geq 1$  then the following hold:*

1.  $T_{p;k} \in \text{Rep}^{\Lambda_{p(0)}}$ .
2. If  $j \neq p(-1)$  and  $\varphi_j^+(p; k) = 0$ , then  $\text{pr}_{\Lambda_{p(0)}} \text{Ind } T_{p;k} \boxtimes L(j) = \mathbf{0}$ .
3. If  $j \neq p(-1)$  but  $\varphi_j^+(p; k) \geq 1$  then  $\text{pr}_{\Lambda_{p(0)}} \text{Ind } T_{p;k} \boxtimes L(j) \cong T_{p \star \pi(j); k+1}$ .
4. When  $j = p(-1)$  and  $\varphi_j^+(p; k) \geq 1$ , there is a surjection  $\text{pr}_{\Lambda_{p(0)}} \text{Ind } T_{p;k} \boxtimes L(j) \twoheadrightarrow T_{p \star \pi(j); k+1}$ , and for all composition factors  $K$  of  $\text{Ind } T_{p;k} \boxtimes L(j)$  such that  $K \not\cong T_{p \star \pi(j); k+1}$  we have that  $\varphi_j^{\Lambda_{p(0)}}(K) \leq \varphi_j(p; k) - 2$ .

**Lemma 6.2** *Suppose that  $A \in \text{Rep}^{\Lambda_i}$  is a simple  $R(\nu)$ -module,  $D$  is a simple  $R(\nu - \gamma_{p;k})$ -module,  $p$  is a cyclotomic path of length  $k$  and tail weight  $(\Lambda_{p(-1)}, \Lambda_i)$ , and there is a surjection*

$$\text{Ind } T_{p;k} \boxtimes D \twoheadrightarrow A.$$

Then  $D \in \text{Rep}^{\Lambda_{p(-1)} + \varphi^+(p;k)}$ .

**Theorem 6.3** *Let  $A \in \text{Rep}^{\Lambda_i}$  be a simple  $R(\nu)$ -module with  $|\nu| \geq 1$  and  $i \in I$  not forbidden.*

1. There exists a cyclotomic path  $p : \{0, 1, \dots, k - 1\} \rightarrow I$  with  $p(0) = i$  of tail weight  $(\Lambda_{p(-1)}, \Lambda_i)$  and length  $k$  such that  $\tilde{e}_{p(k-1)}^\vee \cdots \tilde{e}_{p(1)}^\vee \tilde{e}_{p(0)}^\vee A$  is a simple  $R(\nu - \gamma_{p;k})$ -module in  $\text{Rep}^{\Lambda_{p(-1)}}$ .
2. Let  $r(A) = k$  be the minimal  $k$  such that statement (1) holds and let  $\mathcal{R}(A) = \tilde{e}_{p(k-1)}^\vee \cdots \tilde{e}_{p(0)}^\vee A$ . Then there exists a surjection

$$\text{pr}_{\Lambda_i} \text{Ind } T_{p;k} \boxtimes \mathcal{R}(A) \twoheadrightarrow A.$$

While the proof is too technical to give in full here, an outline of it follows. Set  $\mathcal{R}_0(A) := A$ , and

$$\mathcal{R}_t(A) := \tilde{e}_{p_t(t-1)}^\vee \tilde{e}_{p_t(t-2)}^\vee \cdots \tilde{e}_{p_t(1)}^\vee \tilde{e}_{p_t(0)}^\vee A. \tag{18}$$

We show inductively that there exist surjections

$$\mathrm{pr}_{\Lambda_i} \mathrm{Ind} T_{p_t;t} \boxtimes \mathcal{R}_t(A) \rightarrow A. \tag{19}$$

For each of these surjections Lemma 6.2 implies that  $\mathcal{R}_t(A) \in \mathrm{Rep}^{\Lambda_{p_t(-1)} + \varphi^+(p_t;t)}$ . The induction will end at the smallest  $k$  such that  $\mathcal{R}_k(A) \in \mathrm{Rep}^{\Lambda_{p(-1)}}$ . Then we set  $p = p_k$ ,  $r(A) = k$ , and  $\mathcal{R}(A) = \mathcal{R}_k(A)$ .

We further conjecture the following.

**Conjecture 6.4** *Under the same hypotheses as Theorem 6.3*

$$A \cong \mathrm{cosoc} \mathrm{pr}_{\Lambda_i} \mathrm{Ind} T_{p;k} \boxtimes \mathcal{R}(A). \tag{20}$$

**Remark 6.5** *In some types, having chosen  $p(0) = i$ , the choice of  $p(1)$  and consequently  $p_t$ ,  $t < r(A)$  is forced upon us. In other types, such as  $C_\ell^{(1)}$ , there can be 2 choices for  $p(1)$  (and hence  $p(-1)$ ). This choice is mirrored by the combinatorial structure of  $B^{1,1} \otimes B(\Lambda_i)$ .*

Note that by Section 4.5, the crystal-theoretic consequence of Theorem 6.3 is a map

$$B(\Lambda_i) \rightarrow \bigoplus_j B^{1,1} \otimes B(\Lambda_j) \quad \text{given by} \quad [A] \mapsto \overset{p(k-1)}{\circlearrowleft} \otimes [\mathcal{R}(A)],$$

and where  $j$  runs over all possibilities for  $p(-1)$ . (See Remark 6.5.) By abuse of notation, we let  $\overset{p(k-1)}{\circlearrowleft}$  be the node in  $B^{1,1}$  that the path  $p$  ends at. In this way, each node of  $B^{1,1}$  corresponds to the infinite collection of paths  $p$  that end at that node, and in turn to the collection of modules  $T_{p;k}$ . (As remarked in the introduction, this is not a categorification, but it is a useful correspondence.) If we choose to specify  $B(\Lambda_i)$  as above, that further specifies that we consider the  $p$  with  $p(0) = i$  from that collection.

To recover the crystal isomorphism (3) from Theorem 6.3 one must actually fix  $p(-1)$  and let  $i$  vary (whereas the theorem fixes  $i$ ). In many types  $X_\ell$ ,  $i \in I$ , specifying  $p(0) = i$  determines  $p(-1)$  (in particular when  $\Lambda_i$  is of level 1 and  $B^{1,1}$  is perfect). In type  $A$  the relationship between (3) and (2) is transparent. [22] discusses the crystal isomorphism in type  $A$  in more depth. At the moment, the above theorem just gives us a map of nodes. Our second main theorem in Section 6.1 below will show that it is a morphism of crystals.

### 6.1 The action of the crystal operators

Next we study the action of the crystal operators  $\tilde{e}_j$  and  $\tilde{f}_j$  on (2) to show that the map in part 2 of Theorem 6.3 categorifies our crystal isomorphism  $\mathcal{T}$ .

Compare the theorems below with the crystal-theoretic statement (2). As in [16] simple modules in  $\mathrm{Rep}^{\Lambda_i}$  correspond to nodes in the highest weight crystal  $B(\Lambda_i)$ . Each node  $b$  of the KR crystal  $B^{1,1}$  corresponds to an infinite family of  $R(\gamma_{p;k})$ -modules  $T_{p;k}$ ,  $k \in \mathbb{Z}_{\geq 0}$  that satisfy  $\varepsilon(T_{p;k}) = \varepsilon(b)$ . It is in this manner that the main theorems of this paper give a categorification of the crystal isomorphism  $\mathcal{T}$ .

**Theorem 6.6** Let  $A \in \text{Rep}^{\Lambda_i}$  be a simple  $R(\nu)$ -module and  $j \in I$  be such that  $\tilde{e}_j A \neq \mathbf{0}$ . When  $i$  is class  $B$  we furthermore require that  $|\nu| > 1$ . Let  $p$  be a cyclotomic path of tail weight  $(\Lambda_{p(-1)}, \Lambda_i)$  and length  $k = r(A)$ , and  $\mathcal{R}(A) = \tilde{e}_{p(k-1)}^\vee \cdots \tilde{e}_{p(1)}^\vee \tilde{e}_{p(0)}^\vee A$  as constructed by the algorithm in Theorem 6.3. Then there exists a surjection

$$\text{Ind } \tilde{e}_j T_{p;k} \boxtimes \mathcal{R}(A) \rightarrow \tilde{e}_j A \quad \text{if } \varepsilon_j(T_{p;k}) > \varphi_j^{\Lambda_{p(-1)}}(\mathcal{R}(A)), \quad (21)$$

$$\text{Ind } T_{p;k} \boxtimes \tilde{e}_j \mathcal{R}(A) \rightarrow \tilde{e}_j A \quad \text{if } \varepsilon_j(T_{p;k}) \leq \varphi_j^{\Lambda_{p(-1)}}(\mathcal{R}(A)). \quad (22)$$

**Theorem 6.7** Let  $i, p, k, A$ , and  $\mathcal{R}(A)$  be as in Theorem 6.3. Let  $j \in I$  be such that  $\text{pr}_{\Lambda_i} \tilde{f}_j A \neq \mathbf{0}$ . Then there exists a surjection

$$\text{Ind } \tilde{f}_j T_{p;k} \boxtimes \mathcal{R}(A) \rightarrow \tilde{f}_j A \quad \text{if } \varepsilon_j(T_{p;k}) \geq \varphi_j^{\Lambda_{p(-1)}}(\mathcal{R}(A)), \quad (23)$$

$$\text{Ind } T_{p;k} \boxtimes \tilde{f}_j \mathcal{R}(A) \rightarrow \tilde{f}_j A \quad \text{if } \varepsilon_j(T_{p;k}) < \varphi_j^{\Lambda_{p(-1)}}(\mathcal{R}(A)). \quad (24)$$

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