# *Total positivity for the Lagrangian Grassmannian*

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**Abstract.** The positroid decomposition of the Grassmannian refines the well-known Schubert decomposition, and has a rich combinatorial structure. There are a number of interesting combinatorial posets which index positroid varieties, just as Young diagrams index Schubert varieties. In addition, Postnikov's boundary measurement map gives a family of parametrizations for each positroid variety. The domain of each parametrization is the space of edge weights of a weighted planar network. The positroid stratification of the Grassmannian provides an elementary example of Lusztig's theory of total nonnegativity for partial flag varieties, and has remarkable applications to particle physics. We generalize the combinatorics of positroid varieties to the Lagrangian Grassmannian, the moduli space of maximal isotropic subspaces with respect to a symplectic form.

**Résumé.** La dcomposition positroïde de la Grassmannienne raffine la dcomposition bien connue par les varits de Schubert. Plusieurs posets intéressants énumrent les variétés positroïdes, tout comme les diagrammes de Young énumèrent les variétés de Schubert. Le fonction de mesure des bords (du à Postnikov) donne, de plus, une famille de paramtrages pour chaque variété positroïde. Un tel paramétrage a comme domaine de définition l'ensemble de poids des arêtes d'un réseau planaire pondéré. Cette décomposition fournit un exemple élémentaire de la théorie de Lusztig de la nonnégativité complète pour les variétés de drapeaux, ainsi que plusieurs applications remarquables à la physique des particules. Nous généralisons la combinatoire des variétés positroïdes à la grassmannienne lagrangienne, la variété des sous-espaces isotropes d'un espace vectoriel symplectique.

**Keywords.** total positivity, Lagrangian Grassmannian, positroid, projected Richardson variety, plabic graph, partial flag variety, signed permutation

## 1 Introduction

Lusztig defined the *totally nonnegative part* of an abstract flag manifold $G/P$ and conjectured that it was made up of topological cells, a conjecture proved by Rietsch in the late 1990's (Lusztig, 1994, 1998; Rietsch, 1999). Postnikov later introduced the *positroid stratification* of the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k,n)$, and showed that his stratification was a special case of Lusztig's (Postnikov, 2006). While Lusztig's approach relied on the machinery of canonical bases, Postnikov's was more elementary. Each *positroid cell* in Postnikov's stratification was defined as the locus in $\mathrm{Gr}_{\geq 0}(k,n)$ where certain Plücker coordinates vanish.

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The positroid stratification of $\mathrm{Gr}_{\geq 0}(k,n)$ extends to a stratification of the complex Grassmannian $\mathrm{Gr}(k,n)$ of $k$-planes in $n$-space. That is, we can decompose $\mathrm{Gr}(k,n)$ into *positroid varieties* $\Pi$ which are the Zariski closures of Postnikov's totally nonnegative cells. Remarkably, these positroid varieties are the images of *Richardson varieties* in $\mathcal{F}\ell(n)$ under the natural projection

$$\pi_k : \mathcal{F}\ell(n) \to \mathrm{Gr}(k,n)$$

Knutson et al. (2013). Hence the positroid stratification of $\mathrm{Gr}(k,n)$ is a special case of the stratification of a partial flag variety $G/P$ by *projected Richardson varieties*, which was first studied by Lusztig (Lusztig, 1998).

The positroid decomposition of $\mathrm{Gr}_{\geq 0}(k,n)$ has a rich combinatorial structure. There are a several combinatorial posets which index positroid varieties, including *bounded affine permutations* and *$k$-Bruhat intervals*. In addition, there is a remarkable construction, due to Postnikov, which gives a family of parametrizations of each positroid cell $\mathring{\Pi}_{\geq 0}$. The domain of each map is the space of positive real edge weights of a weighted planar network called a *plabic graph* (Postnikov, 2006). Letting the edge weights range over $\mathbb{C}^\times$ gives a birational map onto a dense subset of the positroid variety $\Pi$ in $\mathrm{Gr}(k,n)$ corresponding to the totally nonnegative cell $\mathring{\Pi}_{\geq 0}$ (Muller and Speyer, 2014). We call these maps *parametrizations* as well. The boundary measurement map has found surprising applications to particle physics, in the theory of scattering amplitudes (Arkani-Hamed et al., 2012).

While the combinatorial theory of the positroid stratification of $\mathrm{Gr}(k,n)$ is particularly nice, the projected Richardson stratification for general $G/P$ is also of great interest. From a combinatorial standpoint, the poset of projected Richardson varieties is a shellable ball (Williams, 2007; Knutson et al., 2013). In addition, projected Richardson varieties have nice geometric properties: they are normal, Cohen-Macaulay, and have rational singularities (Knutson et al., 2013).

We extend the combinatorial theory of positroid varieties to the *Lagrangian Grassmannian* $\Lambda(2n)$. While the ordinary Grassmannian is the moduli space of $k$-dimensional subspaces of an $n$-dimensional complex vector space, points in the Lagrangian Grassmannian correspond to maximal isotropic subspaces of $\mathbb{C}^{2n}$ with respect to a symplectic form. Hence we may realize $\Lambda(2n)$ as a subvariety of $\mathrm{Gr}(n,2n)$. Alternatively, $\Lambda(2n)$ is the quotient of the symplectic group $Sp(2n)$ by a parabolic subgroup, and is thus a partial flag variety. By Lusztig's general theory, $\Lambda(2n)$ has a stratification by projected Richardson varieties, which are natural counterparts of positroid varieties.

The poset $\mathcal{Q}(k,n)$ of $k$-Bruhat intervals, which indexes positroid varieties in $\mathrm{Gr}(k,n)$, has a natural analog for any partial flag variety. We explicitly describe the corresponding poset $\mathcal{Q}^C(2n)$ which indexes projected Richardson varieties in $\Lambda(2n)$, and relate $\mathcal{Q}^C(2n)$ to $\mathcal{Q}(n,2n)$. We then define the natural analogs of bounded affine permutations for the Lagrangian Grassmannian. These turn out to be bounded affine permutations which satisfy a symmetry condition. Finally, we construct network parametrizations of projected Richardson varieties in the Lagrangian Grassmannian, using plabic graphs which are symmetric about a line of reflection. See Figure 1 for an example. We investigate the combinatorial theory of these *symmetric plabic graphs*, which neatly parallels that of ordinary plabic graphs.

This note is an extended abstract for the arXiv preprint (Karpman, 2015), which contains proofs of the stated results. Much of the inspiration for this project came from (Lam and Williams, 2008), where the authors generalize Postnikov's ⅃-diagrams to all *cominiscule Grassmannians*. ⅃-diagrams are tableaux filled with 0's and $+$'s which satisfy a pattern-avoidance condition. Each positroid variety in $\mathrm{Gr}(k,n)$ corresponds to a unique ⅃-diagram, which in turn encodes a particular choice of planar network (Postnikov, 2006). Hence, generalizing ⅃-diagrams is a step toward generalizing Postnikov's theory of planar networks.

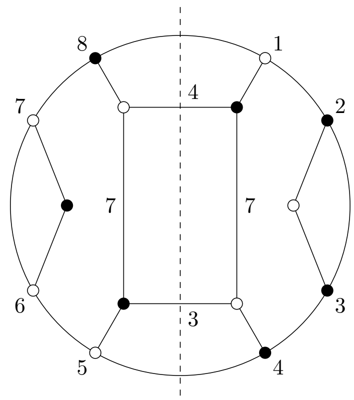



**Fig. 1:** A symmetric weighting of a symmetric plabic graph. All unlabeled edges have weight 1.

# 2 Background

## 2.1 *Flag varieties, Schubert varieties and Richardson varietes*

We recall some basic facts about flag varieties, as stated for example in (Knutson et al., 2014). Let $G$ be a semisimple Lie group over $\mathbb{C}$, and let $B_+$ and $B_-$ be a pair of opposite Borel subgroups of $G$. Then $G/B_+$ is a flag variety, and there is a set-wise inclusion of the Weyl group $W$ of $G$ into $G/B_+$. The flag variety $G/B_+$ has a stratification by Schubert cells, indexed by elements of $W$. For $w \in W$, the *Schubert cell* $\mathring{X}_w$ is $B_- w B_+/B_+$ and the *Schubert variety* $X_w$ is the closure of $\mathring{X}_w$. Similarly, the *opposite Schubert cell* $\mathring{Y}_w$ is $B_+ w B_+/B_+$ and the *opposite Schubert variety* $Y_w$ is the closure of $\mathring{Y}_w$. Both Schubert and opposite Schubert cells give stratifications of $G/B_+$.

For $u, w \in W$, the *Richardson variety* $\mathring{R}_{u,w}$ is the transverse intersection $\mathring{R}_{u,w} = \mathring{X}_u \cap \mathring{Y}_w$. This is empty unless $u \leq w$, in which case it has dimension $\ell(w) - \ell(u)$. Open Richardson varieties form a stratification of $G/B_+$ which refines the Schubert and opposite Schubert stratifications.

We use the superscripts $A$ and $C$ to indicate subvarieties of flag varieties of types $A$ and $C$, respectively. For example, the Schubert cell in the type $A$ flag variety corresponding to a Weyl group element $w$ is denoted $\mathring{X}^A_w$.

### 2.1.1 *Type* $A$

Let $G = \mathrm{SL}(n)$, a semisimple Lie group of type $A_{n-1}$. Let $B_+$ be the subgroup of upper-triangular matrices, and let $B_-$ be the subgroup of lower-triangular matrices. Let $\mathcal{F}\ell(n)$ be the quotient of $\mathrm{SL}(n)$ by the right action of $B_+$. Then $\mathcal{F}\ell(n)$ is an algebraic variety whose points correspond to flags

$$V_\bullet = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n\} \tag{1}$$

where $V_i$ is a subspace of $\mathbb{C}^n$ of dimension $i$.

The Weyl group of type $A$ is the symmetric group $S_n$. We may represent a permutation $w$ in $\mathcal{F}\ell(n)$ by any matrix $\overline{w}$ with $\pm 1$'s in positions $(w(i), i)$, and 0's everywhere else, where the number of $-1$'s is odd or even, depending on the parity of $w$. Then the Schubert cell $\mathring{X}_w^A$ corresponding to $w$ is given by $B_-\overline{w}B_+$, while the opposite Schubert cells $\mathring{Y}_w^A$ is $B_+\bar{w}B_+$.

Let $P$ denote the parabolic subgroup of $\mathrm{SL}(n)$ consisting of block-diagonal matrices of the form

$$\begin{bmatrix} C & D \\ 0 & E \end{bmatrix}$$

where the block $C$ is a $k \times k$ square, and $E$ is an $(n-k) \times (n-k)$ square. Then $\mathrm{SL}(n)/P$ is the Grassmannian $\mathrm{Gr}(k, n)$ of $k$-dimensional linear subspaces of the vector space $\mathbb{C}^n$.

Alternatively, we may realize $\mathrm{Gr}(k, n)$ as the space of full-rank $k \times n$ matrices modulo row operations; a matrix $M$ represents the span of its rows. The natural projection $\pi_k : \mathcal{F}\ell(n) \to \mathrm{Gr}(k, n)$, carries a flag $V_\bullet$ to the $k$-plane $V_k$. If $V$ is a representative matrix for $V_\bullet$ then transposing the first $k$ columns of $V$ gives a representative matrix $M$ for $V_k$.

The *Plücker embedding*, which we denote $p$, maps $\mathrm{Gr}(k, n)$ into the projective space $\mathbb{P}^{\binom{n}{k}-1}$ with homogeneous coordinates $x_J$ indexed by the elements of $\binom{[n]}{k}$. For $J \in \binom{[n]}{k}$ let $\Delta_J$ denote the minor with columns indexed by $J$. Let $V$ be an $k$-dimensional subspace of $\mathbb{C}^n$ with representative matrix $M$. Then $p(V)$ is the point defined by $x_J = \Delta_J(M)$. This map embeds $\mathrm{Gr}(k, n)$ as a smooth projective variety in $\mathbb{P}^{\binom{n}{k}-1}$. The homogeneous coordinates $\Delta_J$ are known as *Plücker coordinates* on $\mathrm{Gr}(k, n)$. The *totally nonnegative Grassmannian*, denoted $\mathrm{Gr}_{\geq 0}(k, n)$, is the subset of $\mathrm{Gr}(k, n)$ whose Plücker coordinates are all nonnegative real numbers, up to multiplication by a common scalar.

### 2.1.2 Type $C$

We now outline the same story in type $C$. Our discussion follows (Billey and Lakshmibai, 2000, Chapter 3). However, we use the bilinear form given in (Makisumi, 2011). Let $V$ be the complex vector space $\mathbb{C}^{2n}$ with standard basis $e_1, \ldots, e_{2n}$. Let $\langle \cdot, \cdot \rangle$ denote the non-degenerate, skew-symmetric form defined by

$$\langle e_i, e_j \rangle = \begin{cases} (-1)^j & \text{if } j = 2n + 1 - i \\ 0 & \text{otherwise} \end{cases}$$

A subspace $U \subseteq V$ is *isotropic* with respect to the form $\langle \cdot, \cdot \rangle$ if $\langle u, v \rangle = 0$ for all $u, v \in V$.

The *symplectic group* $Sp(2n)$ is the group of matrices $A \in \mathrm{SL}(2n)$ which leave the form $\langle \cdot, \cdot \rangle$ invariant. It is a semi-simple Lie group of type $C_n$. Let $B_+$, $B_-$ and $P$ be the subgroups of $\mathrm{SL}(2n)$ given above, where $k = n$. Let $B_+^\sigma$, $B_-^\sigma$, and $P^\sigma$ denote the intersection of $B_+$, $B_-$ and $P$ respectively with $Sp(2n)$. Then $B_+^\sigma$ and $B_-^\sigma$ are a pair of opposite Borel subgroups of $Sp(2n)$, while $P^\sigma$ is a parabolic subgroup of $Sp(n)$.

The generalized flag variety $Sp(2n)/B_+^\sigma$ embeds in $\mathcal{F}\ell(2n)$ in the obvious way, and $Sp(2n)/P^\sigma$ embeds in $\mathrm{Gr}(n, 2n)$. The image of $Sp(2n)/P^\sigma$ is precisely the subset of $\mathrm{Gr}(n, 2n)$ corresponding to maximal isotropic subspaces; that is, the Lagrangian Grassmannian $\Lambda(2n)$.

The Weyl group $S_n^C$ of type $C_n$ is the subgroup of $S_{2n}$ consisting of permutations $\sigma$ which satisfy

$$\sigma(2n + 1 - a) = 2n + 1 - \sigma(a)$$

for all $a \in [2n]$. Under the embedding $Sp(2n)/B_+^\sigma \hookrightarrow \mathcal{F}\ell(2n)$ given above, we have:

$$\mathring{X}_w^C = \mathring{X}_w^A \cap (Sp(2n)/B_+^\sigma)$$
$$\mathring{Y}_w^C = \mathring{Y}_w^A \cap (Sp(2n)/B_+^\sigma)$$
$$\mathring{R}_{u,w}^C = \mathring{R}_{u,w}^A \cap (Sp(2n)/B_+^\sigma).$$

## 2.2 Bruhat intervals and projected Richardson varieties

Let $G$ be a reductive group with a Borel subgroup $B_+$ and a parabolic subgroup $P$ containing $B_+$. Let $W$ be the Weyl group of $G$, and let $W_P \subseteq W$ be the subgroup corresponding to $P$. Let $\ell$ denote the standard length function on $W$. Let $\mathcal{Q}(W_P, W)$ be the set of *Bruhat intervals* $[u, w]$ where $u \leq w$ in the Bruhat order on $W$, and $w$ is of minimal length in its left coset of $W_P$. For $[u, w] \in \mathcal{Q}(W_P, W)$, the projection $\pi_P : G/B \to G/P$ is an isomorphism when restricted to $\mathring{R}_{u,w}$. The variety $\mathring{\Pi}_{u,w} = \pi_P(\mathring{R}_{u,w})$ is a *projected Richardson variety*, and each of Lusztig's projected Richardson varieties arises uniquely in this way (Rietsch, 2006).

The set $\mathcal{Q}(W, W_P)$ has a poset structure, due to Rietsch (Rietsch, 2006). We say $[u, w] \leq [u', w']$ in $\mathcal{Q}(W, W_P)$ if there exists $\sigma \in W_P$ such that the following hold:

1. $\ell(u'\sigma) = \ell(u') + \ell(\sigma)$.
2. $u \leq u'\sigma \leq w'\sigma \leq w$.

The partial order on $\mathcal{Q}(W, W_P)$ gives the *reverse* closure order on projected Richardson varieties in $G/P$. For $G/P = \mathrm{Gr}(k, n)$, we denote $\mathcal{Q}(W, W_P)$ by $\mathcal{Q}(k, n)$. In this case $W = S_n$, $W_P$ is the Young subgroup $S_k \times S_{n-k}$, and the minimal-length left coset representatives are $W/W_P$ are the permutations $w$ which satisfy

$$w(1) < w(2) < \cdots < w(k)$$
$$w(k+1) < w(k+2) < \cdots < w(n).$$

We say such a permutation is *Grassmannian* of type $(k, n)$.

For $\Lambda(2n)$, we denote the corresponding poset by $\mathcal{Q}^C(2n)$. The Weyl group $W_P$ is the subgroup of $S_{2n}$ given by $S_n^C \cap (S_n \times S_n)$. The minimal-length coset representatives are the permutations in $S_n^C \subseteq S_{2n}$ which are Grassmannian of type $(n, 2n)$.

## 2.3 Positroid varieties

Let $V \in \mathrm{Gr}_{\geq 0}(k, n)$. The indices of the non-vanishing Plücker coordinates of $V$ give a set $\mathcal{J} \subseteq \binom{[n]}{k}$ called the *matroid* of $V$. We define the *matroid cell* $\mathcal{M}_\mathcal{J}$ as the locus of points $V \in \mathrm{Gr}_{\geq 0}(k, n)$ with matroid $\mathcal{J}$. The nonempty matroid cells in $\mathrm{Gr}_{\geq 0}(k, n)$ are the *positroid cells* defined by Postnikov, and the corresponding matroids are called *positroids*. Positroid cells form a stratification of $\mathrm{Gr}_{\geq 0}(k, n)$, and each cell is homeomorphic to $(\mathbb{R}^+)^d$ for some $d$ (Postnikov, 2006, Theorem 3.5).

The positroid stratification of $\mathrm{Gr}_{\geq 0}(k, n)$ extends to the complex Grassmannian $\mathrm{Gr}(k, n)$. Taking the Zariski closure of a positroid cell of $\mathrm{Gr}_{\geq 0}(k, n)$ in $\mathrm{Gr}(k, n)$ gives a *closed positroid variety*. For a closed positroid variety $\Pi^A \subseteq \mathrm{Gr}(k, n)$, we define the *open positroid variety* $\mathring{\Pi}^A \subset \Pi^A$ by taking the complement in $\Pi^A$ of all lower-dimensional positroid varieties. The open positroid varieties give a stratification of $\mathrm{Gr}(k,n)$ (Knutson et al., 2013). Throughout the remainder of this abstract, the phrase *positroid variety* will mean an *open* positroid variety.

Positroid varieties in $\mathrm{Gr}(k,n)$ may be defined in numerous other ways. There is a beautiful description of positroid varieties as intersections of cyclically permuted Schubert varieties. In particular, the positroid stratification refines the well-known Schubert stratification of $\mathrm{Gr}(k,n)$ (Knutson et al., 2013).

Remarkably, positroid varieties in $\mathrm{Gr}(k,n)$ coincide with projected Richardson varieties (Knutson et al., 2013, Section 5.4). Since positroid varieties are projected Richardson varieties, we have an isomorphism between $\mathcal{Q}(k,n)$ and the poset of positroid varieties, ordered by *reverse* inclusion (Knutson et al., 2013, Section 5.4). We denote the positroid variety corresponding to $\langle u,w\rangle_k$ by $\mathring{\Pi}^A_{u,w}$.

## 2.4 Bounded affine permutations and Bruhat intervals in type $A$

**Definition 1** *An* ***bounded affine permutation*** *of type $(k,n)$ is a bijection $f:\mathbb{Z}\to\mathbb{Z}$ which satisfies the following criteria:*

1. $f(i+n)=f(i)+n$ *for all* $i\in\mathbb{Z}$.
2. $\frac{1}{n}\sum_{i=1}^{n}(f(i)-i)=k$.
3. $i\le f(i)\le i+n$ *for all* $i\in\mathbb{Z}$.

*We write* $\mathrm{Bound}(k,n)$ *for the set of all bounded affine permutations of type* $(k,n)$.

**Definition 2** *For $k\in\mathbb{Z}$, an affine permutation of order $n$ has type $(k,n)$ if*

$$\frac{1}{n}\sum_{i=1}^{n}(f(i)-i)=k. \tag{2}$$

We may realize $\mathrm{Bound}(k,n)$ as a subset of the extended affine Weyl group $\widehat{W}_n$ of type $\mathrm{GL}(n)$ , and the Bruhat order on $\widehat{W}_n$ gives a partial order on $\mathrm{Bound}(k,n)$ (Knutson et al., 2013). The poset $\mathrm{Bound}(k,n)$ is anti-isomorphic to the poset of *decorated permutations*, which Postnikov introduced as an indexing set for positroid varieties (Postnikov, 2006).

We define the *translation element* $t_k\in\mathrm{Bound}(k,n)$ by setting

$$t_k(i)=\begin{cases} i+n & 1\le i\le k\\ i & k+1\le i\le n\end{cases} \tag{3}$$

and extending periodically in accordance with Condition (1) above.

Let $\langle u,w\rangle_k\in\mathcal{Q}(k,n)$. The function

$$f_{u,w}=ut_{[k]}w^{-1} \tag{4}$$

is a bounded affine is a bounded affine permutation of type $(k,n)$, and the map $\langle u,w\rangle_k\to f_{u,w}$ gives an isomorphism from $\mathcal{Q}(k,n)$ to $\mathrm{Bound}(k,n)$. In particular, the poset $\mathrm{Bound}(k,n)$ indexes positroid varieties in $\mathrm{Gr}(k,n)$ (Knutson et al., 2013).

## 2.5 Plabic graphs

A *plabic graph* $G$ is a planar bicolored graph, embedded in a disk, which satisfies some combinatorial conditions. In particular, $G$ has $n$ distinguished vertices on the boundary of the disk, numbered $1, 2, \ldots, n$ in clockwise order. All boundary vertices have degree one, and no edge connects two boundary vertices. We assume that $G$ has no leaves, except perhaps some adjacent to boundary vertices, and that each connected component of $G$ contains at least one boundary vertex. Finally, following the conventions of (Lam, 2013), we require $G$ to be bipartite, with black and white vertices forming the partite sets.

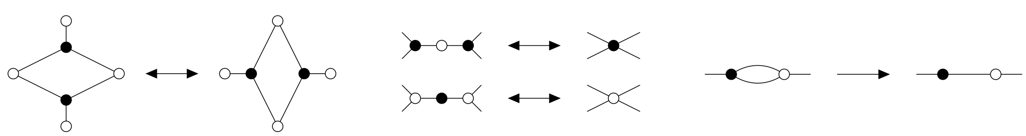
We define a collection of directed paths in $G$, called *trips*, as follows. Begin at a boundary vertex $a$, and traverse the unique edge $\{a, b\}$ incident at $a$ in the direction $a \to b$. Proceed according to the rules of the road: turn (maximally) left at every white internal vertex, and (maximally) right at every black internal vertex. Continuing in this fashion, we eventually reach a boundary vertex. The resulting directed path is a *trip* in $G$. Repeat this process for every boundary vertex. We define the trip permutation $\sigma_G$ of by setting $\sigma_G(a) = b$ if the trip that starts at boundary vertex $a$ ends at boundary vertex $b$ (Postnikov, 2006).

Postnikov gave a system of *local moves* and *reductions* defined for plabic graphs. The moves reversible, and preserve the trip permutation. Reductions are not reversible, and change the trip permutation. With our conventions, we have two moves M1 and M2, and one reductions M1, defined as follows.

(M1) The *square move*. We may transform the portion of a plabic graph shown at left in Figure 2(a) into the portion shown at right, and vice versa.

(M2) Edge-contraction moves. If a vertex $v$ has degree 2, we may contract the incident edges $(u, v)$ and $(v, u')$ to a single vertex, as in Figure 2(b).

(R1) Multiple edges with the same endpoints may be replaced by a single edge. See Figure 2(c).



(a) A square move. (b) Edge-contraction moves. (c) A reduction.

**Fig. 2:** Moves and reductions for plabic graphs.

A plabic graph $G$ is *reduced* if it cannot be transformed using the local moves M1-M2 into a plabic graph $G'$ on which one may perform a a reduction. If $G$ is a reduced graph, each fixed point of $\sigma_G$ corresponds to a leaf attached to a boundary vertex (Postnikov, 2006, Section 13). Suppose $G$ has $n$ boundary vertices. Then we can construct a bounded affine permutation $f_G$ corresponding to $G$ by setting

$$f_G(a) = \begin{cases} \sigma_G(a) & \sigma_G(a) > a \text{ or } G \text{ has a black boundary leaf at } a \\ \sigma_G(a) + n & \sigma_G(a) < a \text{ or } G \text{ has a white boundary leaf at } a \end{cases} \tag{5}$$

Thus we have have a correspondence between reduced plabic graphs and positroid varieties: to a reduced plabic graph $G$, we associate the positroid $\Pi_G^A$ corresponding to $f_G$. This coincides with the bijection $f_{u,w} \mapsto \mathring{\Pi}_{u,w}$ given earlier (Knutson et al., 2013).

**Theorem 1** *(Postnikov, 2006). Two reduced plabic graphs $G$ and $G'$ have the same bounded affine permutation if and only if we can transform $G$ into $G'$ using a sequence of local moves M1 and M2*

## 2.6 Parametrizations from plabic graphs

Let $G$ be a reduced plabic graph, and suppose $f_G \in \mathrm{Bound}(k, n)$. An *almost perfect matching* of $G$ is a collection of edges which uses each internal vertex exactly once. For $P$ an almost perfect matching on a plabic graph $G$ let

$$\partial(P) = \{\text{black boundary vertices used in } P\} \cup \{\text{white boundary vertices } \textit{not} \text{ used in } P\} \tag{6}$$

Then $|\partial(P)| = k$ (Lam, 2013).

Suppose $G$ has $e$ edges, and weights each edge with a positive real parameter $t_i$. The *boundary measurement map*

$$\partial_G : \mathbb{C}^e \to \mathbb{P}^{\binom{n}{k}-1} \tag{7}$$

sends $(t_1, \ldots, t_e)$ to the point with homogeneous coordinates

$$x_J = \sum_{\partial(\omega)=J} t^\omega \tag{8}$$

where the sum is over all matchings $\omega$ of $G$, and $t^\omega$ is the product of the weights of all edges used in $P$. This map is surjective onto the totally nonnegative part of $\mathring{\Pi}_G^A$. If instead we let the parameters $t_i$ range over nonzero complex values, we obtain a well-defined map whose image is a dense subset of $\Pi_G^A$ (Muller and Speyer, 2014).

The boundary measurement map was first constructed by Postnikov for real edge weights, using the language of directed networks (Postnikov, 2006, Section 11.5). Postnikov, Speyer and Williams re-cast this construction in terms of matchings (Postnikov et al., 2009, Section 4-5), an approach developed further by Lam (Lam, 2013). Using Lam's construction, Muller and Speyer showed that we can extend the boundary measurement map to nonzero complex edge weights (Muller and Speyer, 2014). The definition of the boundary measurement map given above is from (Lam, 2013).

The boundary measurement map is typically not injective, due to the action of the *gauge group*. Let $V$ denote the set of internal vertices of $G$. The gauge group is a copy of $\mathbb{C}^{|V|}$ with coordinates $x_v$ indexed by internal vertices of $v$. For $\mu \in \mathbb{C}^{|V|}$ and $v \in V$, the action of $\mu$ multiplies the weights of each edge incident to $v$ by $\mu_v$. It is easy to see that the action of $\mathbb{G}^V$ preserves the boundary measurement map, so $\partial_G$ descends to a well-defined map

$$\mathbb{D}_G : \mathbb{C}^e / \mathbb{C}^{|V|} \to \mathring{\Pi}_G^A.$$

The map $\mathbb{D}_G$ is birational onto a dense open subset of $\mathring{\Pi}_G$ (Muller and Speyer, 2014).

Analogous statements hold for positive real edge weights, where the action is by positive real gauge transformations; in this setting, taking the quotient by the gauge group gives an isomorphism onto the positroid cell corresponding to $G$ (Postnikov, 2006). We will abuse terminology slightly, and refer to both $\partial_G$ and $\mathbb{D}_G$ as the *boundary measurement map*; it should be clear from context which map is meant.

Suppose $G$ and $G'$ are related to each other by a local move. Then the maps $\partial_G$ and $\partial_{G'}$ are related by a birational change of variables. For degree-two vertex removal, we may simply assume that both edges adjacent to the degree-two vertex are fixed to 1, so the change of variables is trivial. For the square move, assume all unlabeled edges in Figure 3 are gauge-fixed to 1. Then we have the transformation shown in Figure 3 where

$$a' = \frac{a}{ac+bd},\ b' = \frac{b}{ac+bd},\ c' = \frac{c}{ac+bd},\ d' = \frac{d}{ac+bd}.$$

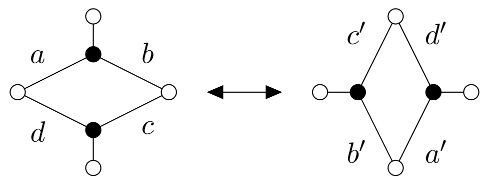



**Fig. 3:** A square move gives a change of coordinates.

# 3 Bruhat intervals and bounded affine permutations in type $C$

We give a concrete description of the poset $\mathcal{Q}^C(2n)$, and show that it is isomorphic to a subposet of $\mathcal{Q}(n, 2n)$. This results allows us to relate projected Richardson varieties in $\Lambda(2n)$ to positroid varieties in $\mathrm{Gr}(n, 2n)$.

**Theorem 2** *The poset* $\mathcal{Q}^C(n, 2n)$ *may be realized as the induced subposet of* $\mathcal{Q}(n, 2n)$ *consisting of Bruhat intervals* $\langle u, w\rangle_n$ *where* $u, w \in S_n^C$.

**Proposition 1** *The projected Richardson varieties in* $\Lambda(2n)$ *are precisely the set-theoretic intersections*

$$\mathring{\Pi}^C_{u,w} = \mathring{\Pi}^A_{u,w} \cap \Lambda(2n)$$

*for* $\langle u, w\rangle_n \in \mathcal{Q}^C(2n)$*, where we embed* $\mathcal{Q}^C(2n)$ *in* $\mathcal{Q}(n, 2n)$*. The closure partial order on projected Richardson varieties in* $\Lambda(2n)$ *is induced by the closure partial order on positroid varieties in the obvious way.*

**Definition 3** *The set* $\mathrm{Bound}^C(2n)$ *of* type C bounded affine permutations *is the image of* $\mathcal{Q}^C(2n)$ *under the map* $\mathcal{Q}(n, 2n) \to \mathrm{Bound}(n, 2n)$.

**Proposition 2** *For* $f \in \mathrm{Bound}(n, 2n)$*, we have* $f \in \mathrm{Bound}^C(2n)$ *if and only if, for all* $a \in [2n]$*, we have*

$$f(2n + a - 1) = 4n + 1 - f(a).$$

In (Lam and Williams, 2008), the authors introduced the poset of *type* $B$ *decorated permutations* of order $2n$, denoted $\mathcal{D}_n^B$, and showed that it indexes projected Richardson varieties in both the odd orthogonal Grassmannian OG$(n, 2n + 1)$ (a flag variety of type $B_n$) and the Lagrangian Grassmannian $\Lambda(2n)$. The correspondence between decorated permutations and bounded affine permutations maps $\mathcal{D}_n^B$ isomorphically to $\mathrm{Bound}^C(2n)$. Lam and Williams also give an isomorphism from $\mathcal{Q}^C(2n)$ to $\mathcal{D}_n^B$ which is equivalent to our isomorphism from $\mathcal{Q}^C(2n)$ to $\mathrm{Bound}^C(2n)$. What is new here is the realization of $\mathcal{Q}^C(2n)$ as an induced subposet $\mathcal{Q}(n, 2n)$, and the geometric interpretation of this result.

# 4 Symmetric plabic graphs

We now recall symmetric plabic graphs, first introduced in (Karpman and Su, 2015). See Figure 1 for an example.

**Definition 4** *A symmetric plabic graph $G$ is a plabic graph with $2n$ boundary vertices, which has a distinguished diameter $d$ such that the following hold:*

1. *The diameter $d$ has one endpoint between $2n$ and $1$, and the other between $n$ and $n+1$.*
2. *No vertex of $G$ lies on $d$, although some edges may cross $d$.*
3. *Reflecting the graph $G$ through the diameter $d$ gives a graph $G'$ which is identical to $G$, but with the colors of vertices reversed.*

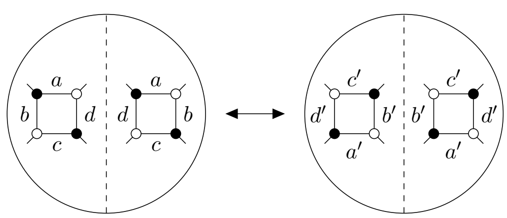
**Lemma 1 (Karpman and Su (2015))** *Let $G$ be a symmetric plabic graph. Then $f_G \in \mathrm{Bound}^C(2n)$. Conversely, for every $g \in \mathrm{Bound}^C(2n)$, there is a symmetric plabic graph $G$ such that $f_G = g$.*

Let $G$ be a symmetric plabic graph with vertex set $V$. We define a map $r : V \to V$ which maps each vertex $v \in V$ to its image under reflection through $d$.

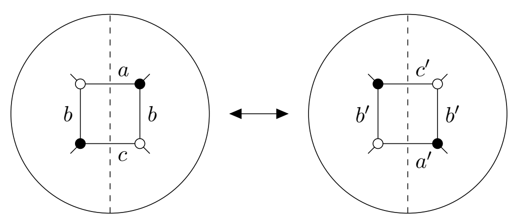
**Definition 5** *A weighting $\omega$ of a symmetric plabic graph $G$ is* symmetric *if $\omega$ assigns the same weight to $(u, v)$ and $(r(u), r(v))$ for each edge $(u, v)$ of $G$.*

# 5 Local moves for symmetric plabic graphs

We define moves for symmetric plabic graphs, analogous to Postnikov's local moves for ordinary plabic graphs. There are two types of symmetric moves. First, performing a pair of local moves which are mirror images of each other gives a symmetric move, as shown in Figure 4(a). Second, a single local move may preserve symmetry, as in Figure 4(b). Symmetric reductions may be defined similarly. Applying Postnikov's change-of-coordinates formulas, we see that symmetric moves preserve the property of being a symmetric weighting. For a complete list of symmetric moves and reductions, see (Karpman, 2015).




(a) Two of Postnikov's moves combine to give a symmetric local move.




(b) A single local move that preserves symmetry

**Fig. 4:** Two types of symmetric local moves.

**Theorem 3** *Any symmetric plabic graph may be transformed into a reduced symmetric plabic graph by a sequence of symmetric moves and reductions.*

**Theorem 4** *Let* $G$ *and* $G'$ *be two reduced symmetric plabic graphs. Then* $G$ *and* $G'$ *have the same bounded affine permutation* $f \in \mathrm{Bound}^C(2n)$ *if and only if we can transform* $G$ *into* $G'$ *by a series of symmetric moves. Hence move-equivalence classes of reduced symmetric plabic graphs are in bijection with projected Richardson varieties in* $\Lambda(2n)$*.*

Hence the combinatorics of symmetric plabic graph neatly parallels that of ordinary plabic graphs. It remains to make the link between combinatorics and geometry.

# 6 Network parametrizations for projected Richardson varieties in $\Lambda(2n)$.

We construct an analogue of the boundary measurement map for $\Lambda(2n)$. Let $G$ be a reduced symmetric plabic graph with edge set $E$, and let $\Pi_G^A$ denote the corresponding positroid variety in $\mathrm{Gr}(n, 2n)$. Let $\mathbb{G}_C^E$ be the space of symmetric weightings of $G$, and let $\mathbb{G}_C^V$ denote the group of symmetric gauge transformations; that is, gauge transformations which act by the same value on $v$ and $r(v)$ for each internal vertex $v$ of $G$.

**Definition 6** *The* Lagrangian boundary measurement map $\partial_G^C$ *is the restriction of* $\partial_G$ *to the space* $\mathbb{G}_C^E/G_C^V$ *of symmetric edge weightings of* $G$ *modulo symmetric gauge transformations.*

**Theorem 5** *The Lagrangian boundary measurement map* $\mathbb{D}^C$ *takes* $\mathbb{G}_C^E/\mathbb{G}_C^V$ *birationally to a dense subset of* $\Pi_G^C$*.*

Hence symmetric plabic graphs parametrize projected Richardson varieties in $\Lambda(2n)$, just as ordinary plabic graphs parametrize positroid varieties in $\mathrm{Gr}(n, 2n)$, and we have extended the boundary measurement map to $\Lambda(2n)$.

# 7 Total nonnegativity for $\Lambda(2n)$

In the Grassmannian case, nonnegativity of Plücker coordinates coincides with Lusztig's notion of total nonnegativity for partial flag manifolds. Moreover, plabic graphs with positive real edge weights parametrize totally nonnegative cells in $\mathrm{Gr}(k, n)$. We prove that analogous statements hold for $\Lambda(2n)$.

**Proposition 3** *Let* $\mathring{\Pi}^C$ *be a projected Richardson variety in* $\Lambda(2n)$*. Then we have*

$$\mathring{\Pi}^C_{\geq 0} = \mathring{\Pi}^C \cap \mathrm{Gr}_{\geq 0}(k, n).$$

**Theorem 6** *For* $G$ *a symmetric plabic graph, restricting the map* $\mathbb{D}_G^C$ *to the space* $(\mathbb{G}_C^E)_{\geq 0}/(\mathbb{G}_C^V)_{\geq 0}$ *of symmetric positive edge weightings of* $G$ *gives an isomorphism of real semi-algebraic sets*

$$(\mathbb{G}_C^E)_{\geq 0}/(\mathbb{G}_C^V)_{\geq 0} \cong (\Pi_G^C)_{\geq 0}.$$

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