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Affine type A geometric crystal structure on the Grassmannian

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Abstract. We construct a type $A_{n-1}^{(1)}$ affine geometric crystal structure on the Grassmannian $\mathrm{Gr}(k, n)$. The tropicalization of this structure recovers the combinatorics of crystal operators on semistandard Young tableaux of rectangular shape (with $n - k$ rows), including the affine crystal operator $\widetilde{e_0}$. In particular, the promotion operation on these tableaux essentially corresponds to cyclically shifting the Plücker coordinates of the Grassmannian.

Résumé. Nous munissons la grassmannienne $\mathrm{Gr}(k, n)$ d'une structure de cristal géométrique affine en type $A_{n-1}^{(1)}$. La tropicalisation de cette structure recouvre la combinatoire des opérateurs de cristal sur les tableaux de Young semistandards de forme rectangulaire (avec $n - k$ lignes), y compris l'opérateur affine $\widetilde{e_0}$. En particulier, l'opération de promotion sur ces tableaux correspond essentiellement au décalage cyclique des coordonées plückeriennes de la grassmannienne.

Keywords. crystal, geometric crystal, promotion, Kirillov-Reshetikhin module

1 Introduction

Kashiwara's theory of crystal bases provides a combinatorial model for the representation theory of semisimple Lie algebras, and more generally of Kac-Moody algebras ([Kas91]). In type A_{n-1}, this theory brings to light an intimate connection between the representation theory of $\mathfrak{sl}_n$ and the combinatorics of semistandard Young tableaux (SSYTs). The operations on tableaux that arise in this theory, such as promotion, evacuation, the "crystal operators," and the Robinson-Schensted-Knuth correspondence, are classically defined in terms of combinatorial algorithms on the individual entries in the tableau, such as bumping, sliding, or bracketing rules. Interestingly, when these operations are transferred from tableaux to Gelfand-Tsetlin patterns, they are given by piecewise-linear formulas ([Kir01], [KB96]). This suggests that there should be a way to lift all of these operations to subtraction-free rational functions on some algebraic variety, in such a way that the behavior of these rational functions parallels that of the combinatorial operations. One advantage of such a rational lift is that certain properties of the combinatorial maps may become more transparent in the rational setting. For example, this technique was used in [LP13] to study the intrinsic energy function of tensor products of one-row tableaux; in [LPS14] to compute the lengths of solitons in a discrete integrable system called the box-ball system; and in [GR14] to prove a result on the order of rowmotion.

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Berenstein and Kazhdan's work on geometric crystals shows how to lift a large part of classical crystal combinatorics to the rational setting ([BK07]). For every reductive Lie group, they define rational maps on the corresponding Schubert cells and partial flag varieties. These maps behave like rational versions of Kashiwara's crystal operators, and under a carefully chosen parametrization and tropicalization, they recover the combinatorics of Kashiwara's crystal bases.

Nakashima extended Berenstein and Kazhdan's geometric crystal theory to the Kac-Moody setting, and in particular to affine Lie algebras ([Nak05]). Affine Lie algebras have an important class of finite-dimensional representations, known as Kirillov-Reshetikhin modules. In type $A_{n-1}^{(1)}$, Kirillov-Reshetikhin modules correspond to rectangular partitions, and their crystal bases are modeled by semistandard Young tableaux of rectangular shape. In addition to the classical crystal operators defined on tableaux of all shapes, there is an extra "affine crystal operator" $\widetilde{e}_0$ defined on rectangular tableaux, corresponding to the action of the additional "affine root" of the Lie algebra $\widehat{\mathfrak{sl}_n}$ [Shi02]. In [KNO08], the authors constructed geometric crystals corresponding to the Kirillov-Reshetikhin modules for symmetric powers of the standard representation in various affine types; in type $A_{n-1}^{(1)}$, their construction provides a rational lift of the combinatorics of one-row tableaux. In [MN13], the authors constructed a geometric crystal corresponding to the second exterior power of the standard representation in type $A_{n-1}^{(1)}$, thereby giving a rational lift of the combinatorics of two-row rectangular tableaux. By duality, lifting the combinatorics of k-row tableaux is essentially equivalent to lifting the combinatorics of $(n-k)$-row tableaux. However, we do not know of any previous construction of an affine geometric crystal corresponding to k-row rectangular tableaux, for $\min(k, n-k) > 2$.

Our main result is the following: For each $k \leq n-1$, there exists a structure of $A_{n-1}^{(1)}$ geometric crystal on the Grassmannian $\mathrm{Gr}(n-k, n)$, such that the tropicalization of this geometric crystal, with respect to a "Gelfand-Tsetlin parametrization" defined below, recovers the crystal bases of Kirillov-Reshetikhin modules corresponding to rectangular partitions with k rows, and any number of columns.

We also show that a "twisted" version of the cyclic shifting of Plücker coordinates (Definition 3.4) tropicalizes to the promotion map on rectangular SSYTs. Cyclic shifting on the Grassmannian has previously been applied to the combinatorics of rectangular tableaux in the study of birational rowmotion on rectangular posets ([GR14]), and in the study of cyclic sieving phenomena ([Rho10]). In fact, the map induced by cyclic shifting of Plücker coordinates maps the Kazhdan-Lusztig basis element corresponding to a rectangular tableaux T to the basis element corresponding to the promotion of T (up to a sign) ([Rho10]). This result was our inspiration for using the Grassmannian to lift promotion.

The outline of this paper is as follows. In Section 2, we review the Gelfand-Tsetlin parametrization of rectangular SSYTs, and we explain how to lift this parametrization to the Grassmannian (this lift was previously used in [NY04]). In Section 3, we review the definition of a geometric crystal, and define an affine geometric crystal structure on the Grassmannian. We state the main results of this paper (Theorems 3.6, 3.12, and 3.14). Proofs of these results will appear in a forthcoming work ([Fri]). Section 4 illustrates these results with an extended example.

2 Gelfand-Tsetlin parametrization of the Grassmannian

In this section, we recall the the Gelfand-Tsetlin parametrization of semistandard Young tableaux, and explain how to lift it to a parametrization of the Grassmannian.

2.1 Gelfand-Tsetlin patterns

Fix an integer $n \geq 2$.

Definition 2.1 *A* semistandard Young tableaux *(SSYT) is a filling of a Young diagram with numbers in* $\{1, \dots, n\}$*, so that the rows are weakly increasing, and the columns are strictly increasing. The partition corresponding to the Young diagram is the* shape *of the tableau.*

A Gelfand-Tsetlin pattern *(GT pattern) is a triangular array of nonnegative integers* $(A_{ij})_{1 \leq i \leq j \leq n}$ *satisfying the inequalities*

$$A_{i,j+1} \geq A_{ij} \geq A_{i+1,j+1} \tag{1}$$

for $1 \leq i \leq j \leq n-1$.

Gelfand-Tsetlin patterns can be represented pictorially as triangular arrays, where the jth row in the triangle lists the numbers A_{ij} for $i \leq j$. For example, if $n = 4$, then a Gelfand-Tsetlin pattern looks like:

$$\begin{array}{ccccccc} & & & A_{11} & & & \\ & & A_{12} & & A_{22} & & \\ & A_{13} & & A_{23} & & A_{33} & \\ A_{14} & & A_{24} & & A_{34} & & A_{44} \end{array}$$

There is a natural bijection between Gelfand-Tsetlin patterns and SSYTs. Given a Gelfand-Tsetlin pattern (A_{ij}), the associated tableau T is described as follows: the number of j's in the ith row of T is $A_{ij} - A_{i,j-1}$. Equivalently, the jth row of the pattern is the shape of $T_{\leq j}$, the part of T obtained by removing numbers larger than j. In particular, the nth row of the pattern is the shape of T.

Example 2.2 *Here is an example of a Gelfand-Tsetlin pattern, and the corresponding SSYT.*

$$\begin{array}{ccccccc} & & & 2 & & & \\ & & 3 & & 2 & & \\ & 5 & & 2 & & 1 & \\ 6 & & 4 & & 2 & & 1 \end{array} \quad \longleftrightarrow \quad \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 3 & 4 \\ \hline 2 & 2 & 4 & 4 & \multicolumn{2}{c}{} \\ \cline{1-4} 3 & 4 & \multicolumn{4}{c}{} \\ \cline{1-2} 4 & \multicolumn{5}{c}{} \\ \cline{1-1} \end{array}$$

In this paper, we will restrict our attention to *rectangular* SSYTs, that is, tableaux consisting of k rows of some common length L. We will typically fix the number of rows $k \leq n-1$, and allow the row length L to vary.

Definition 2.3 *A* k-row rectangular GT pattern *is an array of nonnegative integers*

$$(A_{ij}, L)_{1 \leq i \leq k, i \leq j \leq i+n-k-1}$$

subject to the inequalities (1)*, where we set* $A_{ij} = L$ *if* $i \leq k$ *and* $i + n - k \leq j \leq n$*, and* $A_{ij} = 0$ *if* $i > k$.

Clearly k-row rectangular GT patterns are in natural bijection with k-row rectangular SSYTs.

Example 2.4 *Suppose $n = 5$ and $k = 2$. Here is an example of a 2-row rectangular GT pattern, and the corresponding SSYT. The 6 in the bottom left corner is L, the length of each row. (If this number is removed from the diagram, the resulting array is a rectangle; hence the adjective "rectangular".)*

$$\begin{array}{ccccc} & & & 1 & \\ & & 4 & & 1 \\ & 4 & & 3 & \\ 6 & & 4 & & \end{array} \qquad \longleftrightarrow$$

1	2	2	2	4	4
2	3	3	4	5	5

2.2 Gelfand-Tsetlin coordinates and Plücker coordinates

In this section, we encode the Gelfand-Tsetlin coordinates of a k-row rectangular SSYT using Plücker coordinates on the Grassmannian $\mathrm{Gr}(n-k, n)$. The main result of this section is Lemma 2.10, which explains how to recover the Gelfand-Tsetlin coordinates from the Plücker coordinates.

Definition 2.5 *Let E_{ij} denote the matrix with a 1 in position (i, j), and zeros elsewhere. For $1 \leq i \leq n-1$ and $c \in \mathbb{C}^\times$, define*

$$x_{-i}(c) = cE_{ii} + c^{-1}E_{i+1,i+1} + E_{i+1,i} + \sum_{j \neq i, i+1} E_{jj}.$$

For $1 \leq i \leq n$ and $c \in \mathbb{C}^\times$, define

$$t_i(c) = cE_{ii} + \sum_{j \neq i} E_{jj}.$$

For example, if $n = 4$, then we have

$$x_{-2}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 1 & c^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad t_3(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Definition 2.6 *For $i \leq j$, define*

$$\Phi_{[i,j]}(c_i, \ldots, c_j) = x_{-i}(c_i)x_{-(i+1)}(c_{i+1}) \cdots x_{-(j-1)}(c_{j-1})t_j(c_j).$$

For $1 \leq k \leq n-1$, define the map $\Phi_k : (\mathbb{C}^\times)^{k(n-k)+1} \to GL_n$ by

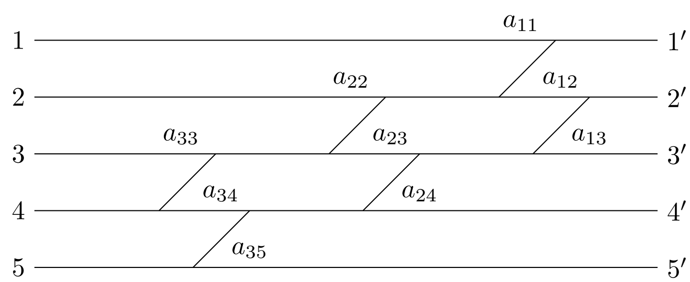
$$\Phi_k(A_{ij}, L) = \Phi_{[k,n]}(A_{kk}, \ldots, A_{k,n-1}, L)\Phi_{[k-1,n-1]}(A_{k-1,k-1}, \ldots, A_{k-1,n-2}, L) \cdots \\ \cdots \Phi_{[2,n-k+2]}(A_{22}, \ldots, A_{2,n-k+1}, L)\Phi_{[1,n-k+1]}(A_{11}, \ldots, A_{1,n-k}, L).$$

Example 2.7 *Suppose $n = 5$ and $k = 3$. Then we have*

$$\Phi_k(A_{ij}, L) = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{22} & a_{12}a_{22} & 0 & 0 & 0 \\ a_{33} & (a_{12} + a_{23})a_{33} & a_{13}a_{23}a_{33} & 0 & 0 \\ 1 & a_{12} + a_{23} + a_{34} & a_{13}(a_{23} + a_{34}) & a_{24}a_{34} & 0 \\ 0 & 1 & a_{13} & a_{24} & a_{35} \end{pmatrix}$$

where $a_{ij} = A_{ij}/A_{i,j-1}$.

Fig. 1: A network corresponding to the matrix in Example 2.7. Edges are directed from left to right, and the weight of an edge is shown directly above it. Edges with no weight label (such as the diagonal edges) are assumed to have weight 1.



Definition 2.8 *Define the map $\pi_{n-k} : GL_n \to Gr(n-k, n)$ to be the projection of an invertible matrix onto the $(n-k)$-dimensional subspace spanned by its first $n-k$ columns. Define the map*

$$\Theta_k : (\mathbb{C}^\times)^{k(n-k)+1} \to Gr(n-k, n) \times \mathbb{C}^\times \tag{2}$$

by $\Theta_k(A_{ij}, L) = (M, L)$, where $M = (\pi_{n-k} \circ \Phi_k)(A_{ij}, L)$.

Definition 2.9 *Let M be a matrix representative for a point in the Grassmannian $Gr(n-k, n)$. We use the convention that M is an $n \times (n-k)$ matrix. For each subset $J \subset \{1, \ldots, n\}$ of size $n-k$, let P_J denote the maximal minor of M using the rows in J. The P_J are the* Plücker coordinates *of M.*

Lemma 2.10 *The map Θ_k is an open embedding of $(\mathbb{C}^\times)^{k(n-k)+1}$ into $Gr(n-k, n) \times \mathbb{C}^\times$. The (rational) inverse is given by $(M, L) \to (A_{ij}, L)$, where*

$$A_{ij} = \frac{P_{[i,j] \cup [k+j-i+2, n]}(M)}{P_{[i+1,j] \cup [k+j-i+1, n]}(M)} \tag{3}$$

for $i \leq k$ and $i \leq j \leq i + n - k - 1$. Here $[i, j]$ denotes the interval $\{i, i+1, \ldots, j\}$ if $i \leq j$, and the empty set if $i > j$.

We call Θ_k the *Gelfand-Tsetlin paramaterization* of $\mathrm{Gr}(n-k, n) \times \mathbb{C}^\times$. See Section 4 for an explicit example of the maps Θ_k and Θ_k^{-1}.

Remark 2.11 The matrix in Example 2.7 is the "weight matrix" of the planar network shown in Figure 1. That is, the matrix entry in position (i, j) is equal to

$$\sum_{\gamma : i \to j'} \mathrm{wt}(\gamma)$$

where the sum is over directed paths γ starting at i and ending at j', and the weight of a path is the product of the weights of the edges in the path. In general, the matrix $\Phi_k(A_{ij}, L)$ has such a network representation, and one can apply the Lindström-Gessel-Viennot Lemma(i) to the network to obtain formula (3) for Θ_k^{-1}, and to show that all the Plücker coordinates of $\Theta_k(A_{ij}, L)$ are given by subtraction-free rational expressions in the variables A_{ij}, L.

(i) See [FZ00] for a nice exposition of this lemma and some of its applications.

3 Geometric crystal structure and tropicalization

3.1 Geometric crystals

In this section we define a type $A_{n-1}^{(1)}$ geometric crystal structure on the variety $\mathrm{Gr}(k,n) \times \mathbb{C}^{\times}$. First, we recall the definition of a type $A_{n-1}^{(1)}$ geometric crystal ([BK07], [Nak05]).

Definition 3.1 *A type $A_{n-1}^{(1)}$* geometric crystal *consists of*

- *an irreducible complex algebraic variety X*
- *a rational map $\gamma : X \to (\mathbb{C}^{\times})^n$*
- *for each $i \in \mathbb{Z}/n\mathbb{Z}$, a rational map $\phi_i : X \to \mathbb{C}^{\times}$*
- *for each $i \in \mathbb{Z}/n\mathbb{Z}$, a rational unital action $e_i^{\cdot}$ of $\mathbb{C}^{\times}$ on X, where we denote the action of $c \in \mathbb{C}^{\times}$ on $x \in X$ by $e_i^c(x)$.*

These data must satisfy the following three properties:

1. *For $i \in \mathbb{Z}/n\mathbb{Z}, x \in X$ and $c \in \mathbb{C}^{\times}$, we have $\gamma(e_i^c(x)) = \alpha_i^{\vee}(c)\gamma(x)$ where*
$$\alpha_i^{\vee}(c) = (1, \ldots, c, c^{-1}, \ldots, 1)$$
with c in the ith component and c^{-1} in the $(i+1)$st component (mod n).
2. *For $i \in \mathbb{Z}/n\mathbb{Z}, x \in X$ and all $c \in \mathbb{C}^{\times}$, we have $\phi_i(e_i^c(x)) = c^{-1}\phi_i(x)$.*
3. *If $n \geq 3$, then for each pair $i, j \in \mathbb{Z}/n\mathbb{Z}$ and $c_1, c_2 \in \mathbb{C}^{\times}$, the maps $e_i^{\cdot}, e_j^{\cdot}$ satisfy*
$$\begin{array}{ll} e_i^{c_1} e_j^{c_2} = e_j^{c_2} e_1^{c_1} & \text{if } |i-j| > 1 \\ e_i^{c_1} e_j^{c_1 c_2} e_i^{c_2} = e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1} & \text{if } |i-j| = 1. \end{array}$$

A geometric crystal is *decorated* if there is a rational function $f : X \to \mathbb{C}$ such that
$$f(e_i^c(x)) = f(x) + \frac{c-1}{\phi_i(x)} + \frac{c^{-1}-1}{\phi_i(x)\alpha_i(\gamma(x))}$$
for $x \in X$ and $i \in \mathbb{Z}/n\mathbb{Z}$, where $\alpha_i(a_1, \ldots, a_n) = \dfrac{a_i}{a_{i+1}}$, with indices taken mod n.

Fix $k \leq n-1$. We will now define a type $A_{n-1}^{(1)}$ geometric crystal structure on the variety $\mathrm{Gr}(k,n) \times \mathbb{C}^{\times}$. Let M be a matrix representative for a point in $\mathrm{Gr}(k,n)$. As in the previous section, we use the convention that M is an $n \times k$ matrix, and we denote the Plücker coordinates of M by P_J.

Definition 3.2 *For $i \in \mathbb{Z}/n\mathbb{Z}$, let $\langle i \rangle$ denote the cyclic interval $\{i, i-1, \ldots, i-k+1\} \subset \mathbb{Z}/n\mathbb{Z}$.*

Definition 3.3 *The data for our (decorated) affine geometric crystal structure on $Gr(k,n) \times \mathbb{C}^{\times}$ are defined as follows:*

1. *The map* $\gamma : Gr(k,n) \times \mathbb{C}^\times \to (\mathbb{C}^\times)^n$ *is defined by* $\gamma(M,L) = (\gamma_1, \ldots, \gamma_n)$*, where*

$$\gamma_i = \gamma_i(M,L) = \begin{cases} \dfrac{P_{\langle i \rangle}}{P_{\langle i-1 \rangle}} & \text{if } 1 \le i \le k \\ \\ L\dfrac{P_{\langle i \rangle}}{P_{\langle i-1 \rangle}} & \text{if } k+1 \le i \le n. \end{cases}$$

2. *The map* $\phi_i : Gr(k,n) \times \mathbb{C}^\times \to \mathbb{C}^\times$ *is defined by*

$$\phi_i(M,L) = \begin{cases} \dfrac{P_{(\langle i \rangle \setminus \{i\}) \cup \{i+1\}}}{P_{\langle i \rangle}} & \text{if } i \neq 0 \\ \dfrac{1}{L}\dfrac{P_{(\langle 0 \rangle \setminus \{0\}) \cup \{1\}}}{P_{\langle 0 \rangle}} & \text{if } i = 0. \end{cases}$$

3. *The map* $e_i^c : Gr(k,n) \times \mathbb{C}^\times \to Gr(k,n) \times \mathbb{C}^\times$ *is defined by* $e_i^c(M,L) = (M', L)$*, where*

$$M' = \begin{cases} x_i\left(\dfrac{c-1}{\phi_i(M,L)}\right) \cdot M & \text{if } i \neq 0 \\ x_0\left(\dfrac{(-1)^{k-1}}{L} \cdot \dfrac{c-1}{\phi_0(M,L)}\right) \cdot M & \text{if } i = 0. \end{cases}$$

Here $x_i(a)$ *is the* $n \times n$ *matrix with 1's on the diagonal,* a *in position* $(i, i+1)$*, and 0's elsewhere. Note that* $x_0(a)$ *has an* a *in position* $(n, 1)$*.*

4. *The decoration* $f : Gr(k,n) \times \mathbb{C}^\times \to \mathbb{C}$ *is defined by*

$$f(M,L) = \sum_{i \neq k} \frac{P_{(\langle i \rangle \setminus \{i-k+1\}) \cup \{i-k\}}}{P_{\langle i \rangle}} + L\frac{P_{(\langle k \rangle \setminus \{1\}) \cup \{n\}}}{P_{\langle k \rangle}}.$$

Definition 3.4 *Define a map* $\mathrm{PR} : Gr(k,n) \times \mathbb{C}^\times \to Gr(k,n) \times \mathbb{C}^\times$ *by* $\mathrm{PR}(M,L) = (M', L)$*, where* M' *is obtained by shifting the rows in* M *down by one (mod* n*), and multiplying the new first row by* $(-1)^{k-1}L$*. For example, we have*

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \xrightarrow{\mathrm{PR}} \begin{pmatrix} -La_{41} & -La_{42} \\ a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$

Although this map is not part of the data for a geometric crystal, it is an essential tool for studying the affine geometric crystal structure, due to the following lemma.

Lemma 3.5 *For each* $i \in \mathbb{Z}/n\mathbb{Z}$*, we have* $\phi_i \circ \mathrm{PR} = \phi_{i-1}$ *and* $\mathrm{PR}^{-1} \circ e_i^c \circ \mathrm{PR} = e_{i-1}^c$*. We also have* $(\gamma \circ \mathrm{PR})(M,L) = (\gamma_n, \gamma_1, \ldots, \gamma_{n-1})$*, where* $\gamma_i = \gamma_i(M,L)$ *are defined in Definition 3.3.*

Theorem 3.6 *The data $(Gr(k,n) \times \mathbb{C}^\times, \gamma, \phi_i, e_i^\cdot, f)$ define a decorated geometric crystal of type $A_{n-1}^{(1)}$.*

For an explicit computation of some of the maps defined above, see Section 4.

Remark 3.7 In [BK07], the authors construct a type A_{n-1} geometric crystal on the variety $X_P^- = UZ(L_P)\overline{w_P}U \cap B^- \subset GL_n$, for each parabolic subgroup P. Let P_k be the maximal parabolic subgroup in type A_{n-1} corresponding to the subset of Dynkin nodes $\{1,\ldots,k-1,k+1,\ldots,n-1\}$. There is a rational isomorphism from $X_{P_k}^-$ to $\mathrm{Gr}(n-k,n) \times \mathbb{C}^\times$, essentially given by projecting from GL_n to the first $n-k$ columns, as in Section 2.2. This isomorphism commutes with the classical part of the crystal structure defined above (i.e., everything other than ϕ_0 and e_0^c). One advantage of projecting to the Grassmannian is that it brings out the cyclic symmetry which is "invisible" in $X_{P_k}^-$, making it clear how to define promotion and the affine operator e_0^c.

The Gelfand-Tsetlin parametrization used in this paper is closely related to the family of parametrizations given in [BK07], although it is not a member of that family.

3.2 Combinatorial crystals

Here we list several important properties of the combinatorial maps on tableaux that show up in Kashiwara's crystal theory in type $A_{n-1}^{(1)}$. Let $B^{k,L}$ be the set of rectangular semistandard Young tableaux with k rows and L columns (and entries in $\{1,\ldots,n\}$). We will identify $B^{k,L}$ with the set of k-row rectangular GT patterns with length parameter L.

Fact 3.8

1. *There is a weight map $\widetilde{\gamma} : B^{k,L} \to \mathbb{Z}^n$, which maps a tableau to the vector $(a_1,\ldots,a_n)$, where a_i is the number of i's in the tableau.*
2. *For each $i \in \mathbb{Z}/n\mathbb{Z}$, there is a map $\widetilde{\phi_i} : B^{k,L} \to \mathbb{Z}$, and a crystal operator $\widetilde{e_i} : B^{k,L} \to B^{k,L} \cup \{0\}$.*
3. *There is a promotion operation $\mathrm{pr} : B^{k,L} \to B^{k,L}$, which satisfies the following properties.*
   - *(a) If $\widetilde{\gamma}(T) = (a_1,\ldots,a_n)$, then $\widetilde{\gamma}(\mathrm{pr}(T)) = (a_n,a_1,\ldots,a_{n-1})$.*
   - *(b) For $i \in \mathbb{Z}/n\mathbb{Z}$, we have $\widetilde{\phi_i} \circ \mathrm{pr} = \widetilde{\phi_{i-1}}$ and $\mathrm{pr}^{-1} \circ \widetilde{e_i} \circ \mathrm{pr} = \widetilde{e_{i-1}}$, where subscripts are taken mod n.*
   - *(c) $\mathrm{pr}^n(T) = T$ for all $T \in B^{k,L}$.*

Proofs of these facts, along with combinatorial descriptions of the maps involved, can be found in [Shi02].

3.3 Tropicalization

Tropicalization is the procedure which transforms a subtraction-free rational function into a piecewise-linear function by replacing multiplication by addition, division by subtraction, and addition by the operation min.[ii] If $g : \mathbb{R}^n \to \mathbb{R}^m$ is a subtraction-free rational map, then we denote by $\mathrm{Trop}(g)$ its tropicalization, i.e., the map from $\mathbb{R}^n$ to $\mathbb{R}^m$ obtained by tropicalizing each component function of g.

Example 3.9 *Suppose $g = \dfrac{x_1^2x_2 + x_3}{x_2^5 + x_1x_3}$. Then* $\mathrm{Trop}(g) = \min(2x_1 + x_2, x_3) - \min(5x_2, x_1 + x_3)$.

[ii] For a more careful definition, see e.g. [BK07].

Recall the parametrization $\Theta_k : (\mathbb{C}^\times)^{k(n-k)+1} \to \mathrm{Gr}(n-k,n) \times \mathbb{C}^\times$ (Definition 2.8).

Lemma 3.10 *Each of $\gamma \circ \Theta_k, \phi_i \circ \Theta_k, \Theta_k^{-1} \circ e_i^c \circ \Theta_k, f \circ \Theta_k$, and $\Theta_k^{-1} \circ \mathrm{PR} \circ \Theta_k$ is a subtraction-free rational map in the coordinates A_{ij}, L. Thus, their tropicalizations are defined.*

Definition 3.11 *Set $\widehat{g} := \mathrm{Trop}(g \circ \Theta_k)$ if g is any of the functions γ, ϕ_i, f. Also, set $\widehat{\mathrm{PR}} := \mathrm{Trop}(\Theta_k^{-1} \circ \mathrm{PR} \circ \Theta_k)$, and $\widehat{e_i} := (\mathrm{Trop}(\Theta_k^{-1} \circ e_i^c \circ \Theta_k))|_{c=1}$.*

Theorem 3.12 *Suppose (A_{ij}, L) is a k-row rectangular GT pattern. Then we have*

$$\widehat{\mathrm{PR}}(A_{ij}, L) = \mathrm{pr}(A_{ij}, L)$$

where pr *is the (combinatorial) promotion operation.*

The proof of this result (which will appear in [Fri]) is based on Kirillov and Berenstein's piecewise-linear description of Bender-Knuth moves in [KB96].

Proposition 3.13 *We have*

$$\widehat{f}(A_{ij}, L) \geq 0$$

if and only if (A_{ij}, L) is a k-row rectangular GT pattern.

Theorem 3.14 *The following statements hold for k-row rectangular GT patterns (A_{ij}, L).*

1. $\widehat{\gamma}(A_{ij}, L) = \widetilde{\gamma}(A_{ij}, L)$.
2. $\widehat{\phi}_i(A_{ij}, L) = -\widetilde{\phi}(A_{ij}, L)$.
3. *If $\widehat{f}(\widehat{e_i}(A_{ij}, L)) \geq 0$, then $\widetilde{e_i}(A_{ij}, L) \neq 0$, and $\widehat{e_i}(A_{ij}, L) = \widetilde{e_i}(A_{ij}, L)$.*
   If $\widehat{f}(\widehat{e_i}(A_{ij}, L)) < 0$, then $\widetilde{e_i}(A_{ij}, L) = 0$.

Proof (Sketch): The first statement is easy. For the second and third statements, we compute directly from the definitions that they hold when $i = 1$. Then we apply PR and invoke Fact 3.8(3b) and Theorem 3.12 to prove the statements for all i. □

Remark 3.15 A result similar to Theorem 3.14 is shown to hold for a large class of the geometric crystals considered in [BK07]. These results can be viewed as a precise formulation of the statement that geometric crystals tropicalize to combinatorial crystals, or that geometric crystals are a rational lift of combinatorial crystals.

4 An example

In this section, we explicitly write down the maps e_0^c and PR, as well as their tropicalizations, in the case $n = 4, k = 2$. Define M by $\Theta_k(A_{11}, A_{12}, A_{22}, A_{23}, L) = (M, L)$. Then we have

$$M = \begin{pmatrix} A_{11} & 0 \\ A_{22} & \dfrac{A_{12}}{A_{11}} A_{22} \\ 1 & \dfrac{A_{12}}{A_{11}} + \dfrac{A_{23}}{A_{22}} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{array}{ll} A_{11} = \dfrac{P_{14}}{P_{34}} & A_{12} = \dfrac{P_{12}}{P_{24}} \\ A_{22} = \dfrac{P_{24}}{P_{34}} & A_{23} = \dfrac{P_{23}}{P_{34}}. \end{array} \tag{4}$$

In terms of A_{ij}, the Plücker coordinates of M are given by:

$$\begin{array}{lll} P_{12} = A_{12}A_{22} & P_{13} = \dfrac{A_{11}A_{23} + A_{12}A_{22}}{A_{22}} & P_{14} = A_{11} \\ P_{23} = A_{23} & P_{24} = A_{22} & P_{34} = 1. \end{array} \tag{5}$$

Now we compute e_0^c. We have $\phi_0(M,L) = \dfrac{1}{L}\dfrac{P_{13}}{P_{34}} = \dfrac{A_{11}A_{23} + A_{12}A_{22}}{LA_{22}}$, and

$$e_0^c(M,L) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -(c-1)\dfrac{A_{22}}{X} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ A_{22} & \dfrac{A_{12}}{A_{11}}A_{22} \\ 1 & \dfrac{A_{12}}{A_{11}} + \dfrac{A_{23}}{A_{22}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{22} & \dfrac{A_{12}}{A_{11}}A_{22} \\ 1 & \dfrac{A_{12}}{A_{11}} + \dfrac{A_{23}}{A_{22}} \\ -(c-1)\dfrac{A_{11}A_{22}}{X} & 1 \end{pmatrix}$$

where $X = A_{11}A_{23} + A_{12}A_{22}$. Observe that e_0^c changes the A_{ij} as follows:

$$\begin{array}{ll} A_{11} = \dfrac{P_{14}}{P_{34}} \mapsto \dfrac{A_{11}}{c} & A_{12} = \dfrac{P_{12}}{P_{24}} \mapsto A_{12}\dfrac{A_{11}A_{23} + A_{12}A_{22}}{A_{11}A_{23} + cA_{12}A_{22}} \\ A_{22} = \dfrac{P_{24}}{P_{34}} \mapsto A_{22}\dfrac{c^{-1}A_{11}A_{23} + A_{12}A_{22}}{A_{11}A_{23} + A_{12}A_{22}} & A_{23} = \dfrac{P_{23}}{P_{34}} \mapsto \dfrac{A_{23}}{c} \end{array}$$

Now we tropicalize these rational expressions for e_0^c. We have

$$\widehat{\phi_0}(A_{ij}, L) = \min(A_{11} + A_{23}, A_{12} + A_{22}) - A_{22} - L$$

and $\widehat{e_0}$ acts on the A_{ij} as follows:

$$\begin{array}{l} A_{11} \mapsto A_{11} - 1 \\ A_{12} \mapsto A_{12} + \min(A_{11} + A_{23}, A_{12} + A_{22}) - \min(A_{11} + A_{23}, 1 + A_{12} + A_{22}) \\ A_{22} \mapsto A_{22} + \min(A_{11} + A_{23} - 1, A_{12} + A_{22}) - \min(A_{11} + A_{23}, A_{12} + A_{22}) \\ A_{23} \mapsto A_{23} - 1. \end{array} \tag{6}$$

We verify that these piecewise-linear formulas agree with $\widetilde{e_0}$ for a particular tableau. Consider the following 2-row tableau T, and its corresponding rectangular GT pattern:

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & 4 & 4 & 4 \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{array}{ccccc} & & 2 & & \\ & 5 & & 1 & \\ 6 & & 3 & & \end{array} \tag{7}$$

The image of T under $\widetilde{e_0}$ (and its corresponding GT pattern) are:

$$\widetilde{e_0}(T) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 2 & 2 & 3 \\ \hline 3 & 3 & 4 & 4 & 4 & 4 \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{array}{ccccc} & & 1 & & \\ & 5 & & 0 & \\ 6 & & 2 & & \end{array} \tag{8}$$

One can easily verify that the second GT pattern is obtained from the first according to the piecewise-linear formulas (6). For example, substituting the numbers in the GT pattern for T into the piecewise-linear formula for A_{22} in (6), we compute that the value of A_{22} after applying $\widehat{e_0}$ is

$$1 + \min(4, 6) - \min(5, 6) = 1 + 4 - 5 = 0,$$

which agrees with the value computed combinatorially by $\widetilde{e_0}$.

Now we write down the map PR in terms of the coordinates A_{ij} and L. By definition, PR acts on Plücker coordinates as follows:

$$\begin{array}{lll} P_{12} \mapsto LP_{14} & P_{13} \mapsto LP_{24} & P_{14} \mapsto LP_{34} \\ P_{23} \mapsto P_{12} & P_{24} \mapsto P_{13} & P_{34} \mapsto P_{23}. \end{array}$$

Substituting into the equations for A_{ij} in (4), we get

$$\begin{array}{ll} A_{11} \mapsto \dfrac{LP_{34}}{P_{23}} = \dfrac{L}{A_{23}} & A_{12} \mapsto \dfrac{LP_{14}}{P_{13}} = \dfrac{A_{11}A_{22}}{A_{11}A_{23} + A_{12}A_{22}} \\ A_{22} \mapsto \dfrac{P_{13}}{P_{23}} = \dfrac{A_{11}A_{23} + A_{12}A_{22}}{A_{22}A_{23}} & A_{23} \mapsto \dfrac{P_{12}}{P_{23}} = \dfrac{A_{12}A_{22}}{A_{23}}. \end{array}$$

Tropicalizing these formulas, we get the following piecewise-linear formulas for the action of $\widehat{\mathrm{PR}}$ on the GT coordinates A_{ij}:

$$\begin{array}{ll} A_{11} \mapsto L - A_{23} & A_{12} \mapsto L + A_{11} + A_{22} - \min(A_{11} + A_{23}, A_{12} + A_{22}) \\ A_{22} \mapsto \min(A_{11} + A_{23}, A_{12} + A_{22}) - A_{22} - A_{23} & A_{23} \mapsto A_{12} + A_{22} - A_{23}. \end{array}$$

The (combinatorial) promotion of the tableau T in (7) is

$$\mathrm{pr}(T) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 3 & 4 & 4 & 4 \\ \hline \end{array} \qquad \longleftrightarrow \qquad \begin{array}{ccccc} & & 3 & & \\ & 4 & & 1 & \\ 6 & & 3 & & \end{array}$$

We leave it to the reader to verify that the GT pattern for $\mathrm{pr}(T)$ agrees with the output of the piecewise-linear formulas for the action of $\widehat{\mathrm{PR}}$ on the A_{ij}, in accordance with Theorem 3.12.

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