FPSAC 2016 Vancouver, Canada DMTCS proc. **BC**, 2016, 1003–1014

Brick polytopes, lattices and Hopf algebras

Vincent Pilaud[1‡]

[1]*CNRS & LIX, École Polytechnique, France*

Abstract. Generalizing the connection between the classes of the sylvester congruence and the binary trees, we show that the classes of the congruence of the weak order on $\mathfrak{S}_n$ defined as the transitive closure of the rewriting rule $UacV_1b_1 \cdots V_kb_kW \equiv^k UcaV_1b_1 \cdots V_kb_kW$, for letters $a < b_1, \dots, b_k < c$ and words $U, V_1, \dots, V_k, W$ on $[n]$, are in bijection with acyclic k-triangulations of the $(n+2k)$-gon, or equivalently with acyclic pipe dreams for the permutation $(1, \dots, k, n+k, \dots, k+1, n+k+1, \dots, n+2k)$. It enables us to transport the known lattice and Hopf algebra structures from the congruence classes of $\equiv^k$ to these acyclic pipe dreams, and to describe the product and coproduct of this algebra in terms of pipe dreams. Moreover, it shows that the fan obtained by coarsening the Coxeter fan according to the classes of $\equiv^k$ is the normal fan of the corresponding brick polytope.

Résumé. Généralisant la connexion entre les classes de congruence sylvestre et les arbres binaires, nous montrons que les classes de la congruence de l'ordre faible sur $\mathfrak{S}_n$ définie comme cloture transitive de la règle de réécriture $UacV_1b_1 \cdots V_kb_kW \equiv^k UcaV_1b_1 \cdots V_kb_kW$, pour des lettres $a < b_1, \dots, b_k < c$ et des mots $U, V_1, \dots, V_k, W$ sur $[n]$, sont en bijection avec les k-triangulations acycliques d'un $(n+2k)$-gone, ou de manière équivalente avec des arrangements de tuyaux acycliques pour la permutation $(1, \dots, k, n+k, \dots, k+1, n+k+1, \dots, n+2k)$. Cela nous permet de transporter les structures connues de treillis et d'algèbre de Hopf des classes de congruence de $\equiv^k$ à ces arrangements de tuyaux acycliques, et de décrire le produit et le coproduit de cette algèbre en termes d'arrangements de tuyaux. Par ailleurs, cela montre que l'éventail obtenu en recollant les chambres de l'éventail de Coxeter en fonction des classes de $\equiv^k$ est l'éventail normal du polytope de briques correspondant.

Keywords. Combinatorial Hopf algebras, lattice quotients, pipe dreams, multitriangulations, brick polytopes

The motivation of this paper comes from relevant combinatorial, geometric, and algebraic structures on permutations, binary trees and binary sequences. Classical surjections from permutations to binary trees (BST insertion) and from binary trees to binary sequences (canopy) yield:

- lattice morphisms from the weak order, via the Tamari lattice, to the boolean lattice;
- normal fan coarsenings from the permutahedron, via J.-L. Loday's associahedron [Lod04], to the parallelepiped generated by the simple roots $\mathbf{e}_{i+1} - \mathbf{e}_i$;
- Hopf algebra refinements from C. Malvenuto and C. Reutenauer's algebra [MR95], via J.-L. Loday and M. Ronco's algebra [LR98], to L. Solomon's descent algebra [Sol76].

These fascinating connections were widely extended by N. Reading in his work on *Lattice congruences, fans and Hopf algebras* [Rea05]. He proves that any lattice congruence $\equiv$ of the weak order on $\mathfrak{S}_n$ defines a complete simplicial fan $\mathcal{F}_\equiv$ refined by the Coxeter fan, and he characterizes in terms of simple

‡Email: `vincent.pilaud@lix.polytechnique.fr`.
Partially supported by the French ANR grants EGOS (12 JS02 002 01) and SC3A (15 CE40 0004 01).

rewriting rules the families $(\equiv_n)_{n\in\mathbb{N}}$ of lattice congruences of the weak orders on $(\mathfrak{S}_n)_{n\in\mathbb{N}}$ which yield Hopf subalgebras of C. Malvenuto and C. Reutenauer's algebra on permutations [MR95]. His work opens two natural questions. On the geometric side, it is not clear which of the fans $\mathcal{F}_{\equiv}$ are actually normal fans of polytopes, as in the previous example of the associahedron. On the algebraic side, this construction produces a combinatorial Hopf algebra whose basis is indexed by the congruence classes of $(\equiv_n)_{n\in\mathbb{N}}$. However, the product and coproduct in this Hopf algebra are performed extrinsically: the algebra is embbeded in C. Malvenuto and C. Reutenauer's algebra on permutations and the computations are performed at that level. The remaining challenge is to realize the resulting Hopf algebra intrinsically by attaching a combinatorial object to each congruence class of $(\equiv_n)_{n\in\mathbb{N}}$ and working out the rules for product and coproduct directly on these combinatorial objects.

The present paper answers these two questions for a relevant family of lattice congruences of the weak order, generalizing the classical sylvester congruence [HNT05]. Our starting point is the world of acyclic multitriangulations [PS09, PS12]. Using classical point-line duality, it was shown in [PP12] that acyclic k-triangulations of the $(n+2k)$-gon can be equivalently modeled by acyclic pipe dreams [KM05, BB93] for the permutation $(1,\dots,k,n+k,\dots,k+1,n+k+1,\dots,n+2k)$. In this paper, we call these pipe dreams *acyclic (k,n)-twists*. On the combinatorial side (Section 1), we provide a clean description of the natural surjection ins^k from the permutations of $\mathfrak{S}_n$ to the acyclic (k,n)-twists. The fibers of this map coincide with the classes of the congruence on $\mathfrak{S}_n$ defined as the transitive closure of the rewriting rule $UacV_1b_1\cdots V_kb_kW \equiv^k UcaV_1b_1\cdots V_kb_kW$, for letters $a<b_1,\dots,b_k<c$ and words $U,V_1,\dots,V_k,W$ on $[n]$. This shows that the increasing flip poset on the acyclic (k,n)-twists is a lattice, and provides lattice homomorphisms from the weak order and to the lattice of k-recoils classes [NRT11]. On the geometric side (Section 2), we briefly recall the definition of the brick polytope, whose vertices correspond to the acyclic (k,n)-twists. The inclusions between the permutahedron, the brick polytope, and a well-chosen graphical zonotope provide a geometric interpretation of the lattice homomorphisms mentioned above. Finally, on the algebraic side (Section 3), the congruence $\equiv^k$ defines a Hopf subalgebra of C. Malvenuto and C. Reutenauer's algebra on permutations [MR95], whose product and coproduct we interpret in terms of acyclic (k,n)-twists. Interestingly, the product in this algebra can be naturally decomposed into 2^k distinct associative operations, motivating the introduction of k-twistiform operads [HP15].

The results presented here significantly extend in three independent directions: to the Cambrian setting (where the shape is not triangular anymore), to the tuple setting (where we consider simultaneously several twists, *e.g.* twin acyclic (k,n)-twists), and to the Schröder setting (where we simultaneously all faces of the brick polytope). Due to space limitations, these extensions as well as all proofs of our results are skipped in this extended abstract. For more detailed information, the reader is invited to consult the long version of this paper [Pil15].

1 Combinatorics of acyclic twists

1.1 (k,n)-twists

A *pipe dream* is a filling of a triangular shape with crosses ┼ and elbows ┘┌ so that all pipes entering on the left side exit on the top side. These objects were studied in the literature, under different names including "pipe dreams" [KM05], "RC-graphs" [BB93], or as specific "pseudoline arrangements on sorting networks" [PS12]. This paper is concerned with the following special family of pipe dreams.

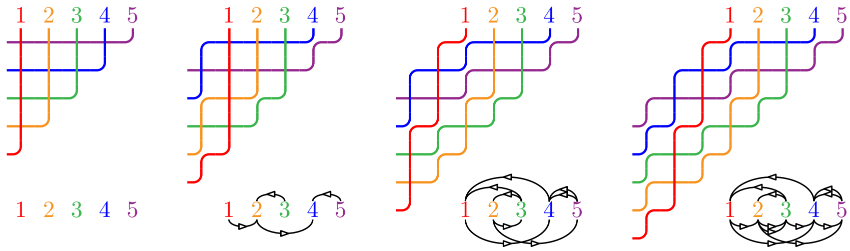


Fig. 1: $(k, 5)$-twists (top) and their contact graphs (bottom) for $k = 0, 1, 2, 3$. Only relevant pipes are represented.

Definition 1 *For $k, n \in \mathbb{N}$, a (k, n)-twist is a pipe dream with $n + 2k$ pipes such that*

- *no two pipes cross twice (the pipe dream is reduced),*
- *the pipe which enters in row i exists in column i if $k \le i \le n + k$, and $n + 2k + 1 - i$ otherwise.*

Besides the first k and last k trivial pipes, a (k, n)-twist has n relevant pipes, labeled by $[n]$ from bottom to top, or equivalently from left to right (see Figure 1). We denote by $\mathcal{T}^k(n)$ the set of (k, n)-twists.

Definition 2 *The contact graph of a (k, n)-twist T is the directed multigraph $\mathrm{T}^\#$ with vertex set $[n]$ and with an arc from the* SE*-pipe to the* WN*-pipe of each elbow in T involving two relevant pipes (see Figure 1). We say that a twist T is acyclic if its contact graph $\mathrm{T}^\#$ is (no oriented cycle), and we then denote by $\triangleleft_{\mathrm{T}}$ the transitive closure of $\mathrm{T}^\#$. We denote by $\mathcal{AT}^k(n)$ the set of acyclic (k, n)-twists.*

Remark 3 *A k-triangulation of a convex $(n + 2k)$-gon is a maximal set of diagonals of the polygon such that no $k+1$ of them are pairwise crossing. Multitriangulations appeared in the context of extremal theory for geometric graphs and were studied for their rich combinatorial structure, see references in [PS09]. Based on k-stars in k-triangulations [PS09], it was shown in [PP12] that the map sending an elbow in row i and column j of the $(n + 2k) \times (n + 2k)$ triangular shape to the diagonal $[i, j]$ of a $(n + 2k)$-gon gives a bijection between the (k, n)-twists and the k-triangulations of the $(n + 2k)$-gon (see Figure 2).*

Example 4 *For $k = 1$, a $(1, n)$-twist T corresponds to a triangulation $\mathrm{T}^\star$ of a convex $(n + 2)$-gon, and the contact graph of T coincides with the dual binary tree of $\mathrm{T}^\star$.*

Remark 5 *It is known [Jon05] that the number of k-triangulations of the $(n+2k)$-gon is the Hankel determinant of Catalan numbers $\det(C_{n+2k-i-j})_{i,j\in[k]}$, see [OEI10, A078920]. To our knowledge, there is no enumerative result on acyclic k-triangulations. See [OEI10, A263791] for the first values of $|\mathcal{AT}^k(n)|$.*

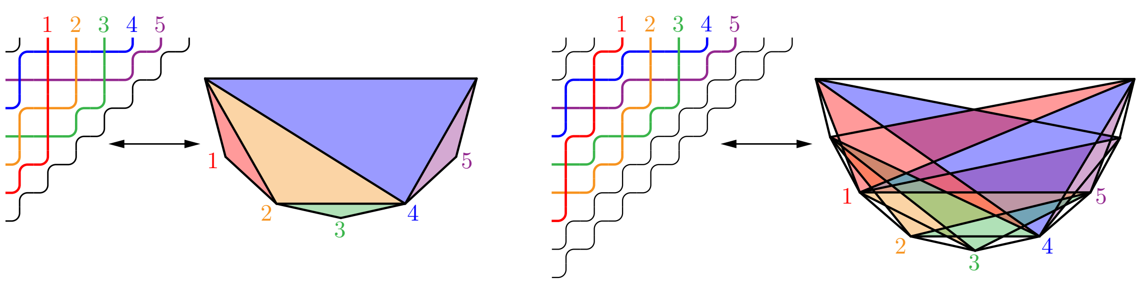


Fig. 2: Duality between (k, n)-twists and k-triangulations of the $(n + 2k)$-gon. Illustrated for $n = 5$ and $k = 1, 2$.

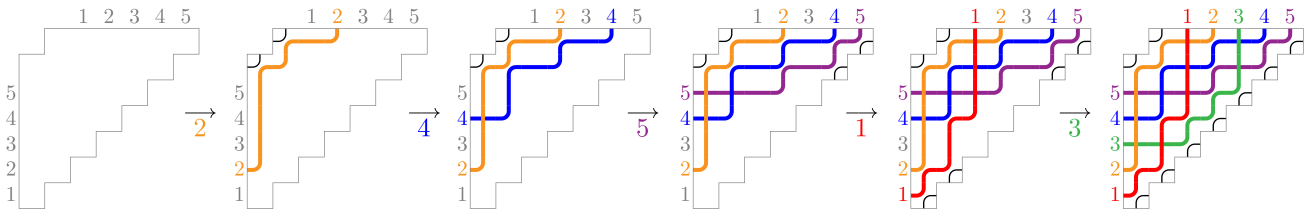


Fig. 3: Insertion of the permutation 31542 in a 2-twist.

1.2 k-twist correspondence

We present a surjection from permutations to acyclic twists. It relies on an insertion operation on twists similar to the insertion in binary search trees. Given a permutation $\tau := \tau_1 \dots \tau_n$, we start from the empty triangular shape and insert the pipes $\tau_n, \dots, \tau_1$ such that each new pipe is as northwest as possible in the space left by the pipes already inserted (see Figure 3). We denote by $\mathsf{ins}^k(\tau)$ the resulting (k,n)-twist.

Proposition 6 *For any (k,n)-twist T, the permutations $\tau \in \mathfrak{S}_n$ such that $\mathsf{ins}^k(\tau) = \mathrm{T}$ are precisely the linear extensions of the contact graph of T. In particular, ins^k is surjective from $\mathfrak{S}_n$ to $\mathcal{AT}^k(n)$.*

Remembering the insertion order, ins^k defines a bijection from permutations of $\mathfrak{S}_n$ to *leveled acyclic (k,n)-twists*, *i.e.* (k,n)-twists whose pipes have been relabeled by $[n]$ so that the label of the SE-pipe is larger than the label of the WN-pipe at each elbow. We call this bijection the *k-twist correspondence*.

Example 7 *For $k = 1$, the contact graph of the 1-twist $\mathsf{ins}^1(\tau)$ is the binary search tree obtained by the successive insertions of the entries of τ read from right to left. If we additionally remember the insertion order, we obtain the sylvester correspondence [HNT05] between permutations and leveled binary trees.*

1.3 k-twist congruence

We now characterize the fibers of ins^k as classes of a congruence $\equiv^k$ on $\mathfrak{S}_n$ defined by a simple rewriting rule, similar to the sylvester congruence [HNT05].

Definition 8 *The k-twist congruence $\equiv^k$ on $\mathfrak{S}_n$ is obtained as the transitive closure of the rewriting rule $UacV_1b_1V_2b_2 \cdots V_kb_kW \equiv^k UcaV_1b_1V_2b_2 \cdots V_kb_kW$, where $a, b_1, \dots, b_k, c$ are elements of $[n]$ such that $a < b_i < c$ for all $i \in [k]$, while $U, V_1, \dots, V_k, W$ are (possibly empty) words on $[n]$.*

Proposition 9 *For any $\tau, \tau' \in \mathfrak{S}_n$, we have $\tau \equiv^k \tau' \iff \mathsf{ins}^k(\tau) = \mathsf{ins}^k(\tau')$.*

Example 10 *For $k = 1$, the 1-twist congruence is precisely the sylvester congruence defined in [HNT05].*

Remember that the (right) weak order on $\mathfrak{S}_n$ is defined as the inclusion order of coinversions, where a coinversion of $\tau \in \mathfrak{S}_n$ is a pair of values $i, j \in \mathbb{N}$ such that $i < j$ while $\tau^{-1}(i) > \tau^{-1}(j)$. Let us also recall that a *lattice congruence* is an equivalence relation $\equiv$ on a lattice $(L, \wedge, \vee)$ compatible with meets and joins: for any $x \equiv x'$ and $y \equiv y'$, we have $x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$. The congruence classes $L/\!\equiv$ then inherit a lattice structure: the order relation is defined by $X \le Y$ in $L/\!\equiv$ if and only if there exists representatives $x \in X$ and $y \in Y$ such that $x \le y$ in L, and the meet $X \wedge Y$ (resp. the join $X \vee Y$) of two congruence classes X and Y is the congruence classe of $x \wedge y$ (resp. of $x \vee y$) for arbitrary representatives $x \in X$ and $y \in Y$.

Proposition 11 *The k-twist congruence $\equiv^k$ is a lattice congruence of the weak order.*

1.4 Increasing flip lattice

We now recall the notion of flips in pipe dreams and study the graph of increasing flips in acyclic k-twists.

Definition 12 *A* flip *in a k-twist is the exchange of an elbow ⌟⌜ between two relevant pipes with the unique crossing ┼ between them. The flip is* increasing *if the initial elbow is located (largely) south-west of the final elbow. The* increasing flip order *is the transitive closure of the increasing flip graph on $\mathcal{AT}^k(n)$.*

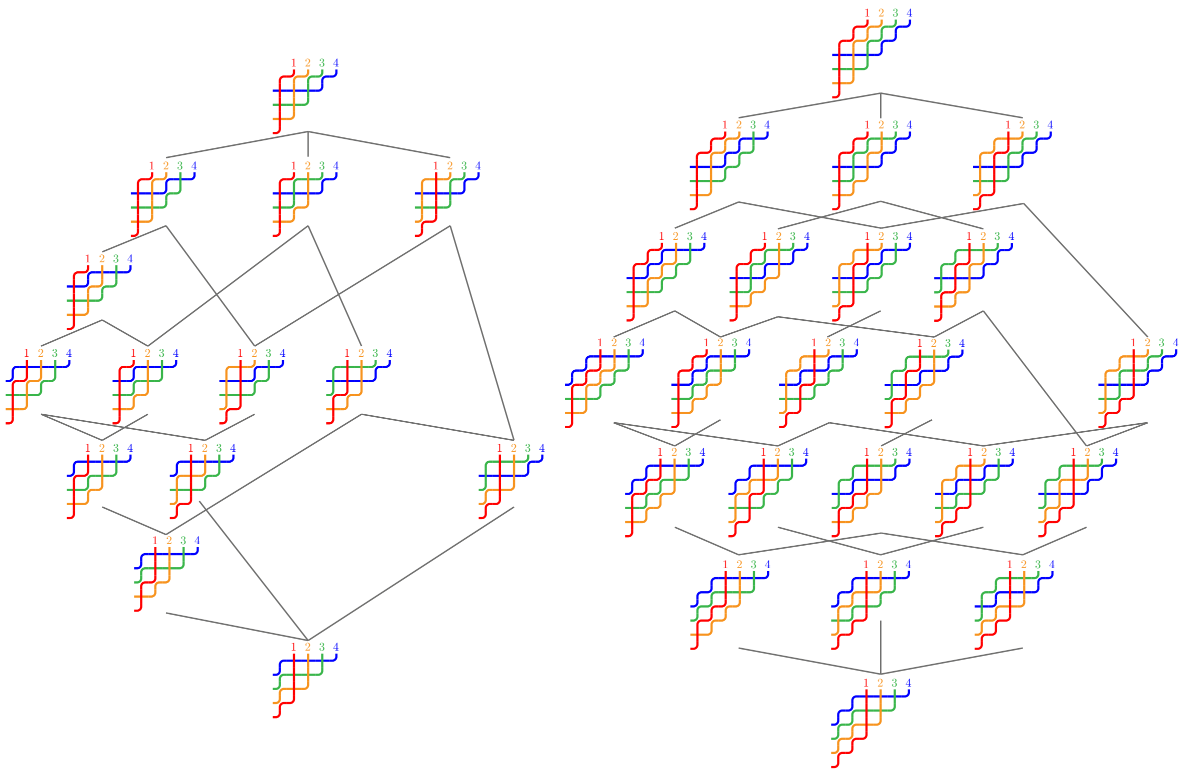


Fig. 4: The increasing flip order on $(k, 4)$-twists for $k = 1$ (left) and $k = 2$ (right).

Remark 13 *An elbow flip in a k-twist T corresponds to a diagonal flip in its dual k-triangulation $T^\star$ [PS09]. The polygon in Figure 2 is chosen so that increasing flips in T correspond to slope increasing flips in $T^\star$.*

Example 14 *For $k = 1$, the increasing flip order is isomorphic to the Tamari lattice on binary trees.*

The k-twist correspondence transports the lattice structure on $(\mathfrak{S}_n, \leq)/\equiv^k$ to acyclic k-twists.

Proposition 15 *The following posets are all isomorphic:*

- *the increasing flip order on acyclic k-twists,*
- *the quotient lattice of the weak order by the k-twist congruence $\equiv^k$,*
- *the subposet of the weak order induced by the permutations of $\mathfrak{S}_n$ avoiding the generalized pattern $1(k+2)$—(σ_1+1)—$\cdots$—(σ_k+1) for any $\sigma \in \mathfrak{S}_k$.*

1.5 k-recoils and k-canopy schemes

To prepare the definition of the canopy of acyclic k-twists, we first recall the notion of k-recoil schemes of permutations already considered in [NRT11]. We use a description in terms of acyclic orientations as it is closer to the description of the vertices of the zonotope that we will use later in Section 2. We consider the graph $\mathrm{G}^k(n)$ with vertex set $[n]$ and edge set $\big\{\{i,j\} \in [n]^2 \;\big|\; i < j \le i + k\big\}$ (see Figure 5). We denote by $\mathcal{AO}^k(n)$ the set of acyclic orientations of $\mathrm{G}^k(n)$, *i.e.* with no oriented cycle.

Fig. 5: The graphs $\mathrm{G}^k(5)$ for $k = 0, 1, 2, 3$.

Definition 16 *The k-recoil scheme of a permutation $\tau \in \mathfrak{S}_n$ is the orientation $\mathsf{rec}^k(\tau) \in \mathcal{AO}^k(n)$ with an edge $i \to j$ for all $i, j \in [n]$ such that $|i-j| \le k$ and $\tau^{-1}(i) < \tau^{-1}(j)$. The map $\mathsf{rec}^k : \mathfrak{S}_n \to \mathcal{AO}^k(n)$ is called k-recoil map. The k-recoil congruence $\approx^k$ on $\mathfrak{S}_n$ is the transitive closure of the rewriting rule $UijV \approx^k UjiV$, where $i, j \in [n]$ are such that $i + k < j$ while U, V are (possibly empty) words on $[n]$.*

Example 17 *For $k = 1$, the graph $G^1(n)$ is just the n-path. The 1-recoil scheme of $\tau \in \mathfrak{S}_n$ thus stores the recoil positions in τ, i.e. the positions $i \in [n-1]$ such that $\tau^{-1}(i) > \tau^{-1}(i+1)$.*

Proposition 18 ([NRT11]) *For any $\tau, \tau' \in \mathfrak{S}_n$, we have $\tau \approx^k \tau' \iff \mathsf{rec}^k(\tau) = \mathsf{rec}^k(\tau')$.*

Definition 19 *A flip in an acyclic orientation of $\mathrm{G}^k(n)$ is the switch of the direction of an edge of $\mathrm{G}^k(n)$. The flip is increasing if the initial direction was increasing. The increasing flip order on $\mathcal{AO}^k(n)$ is the transitive closure of the increasing flip graph on $\mathcal{AO}^k(n)$.*

Theorem 20 *The k-recoil congruence $\approx^k$ is a lattice congruence of the weak order. The k-recoil map rec^k defines an isomorphism from the quotient lattice of the weak order by the k-recoil congruence $\approx^k$ to the increasing flip lattice on the acyclic orientations of $\mathrm{G}^k(n)$.*

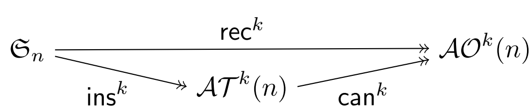
We can now generalize the notion of canopy of binary trees to acyclic twists. To ensure that Definition 22 is valid, we need the following observation on comparisons of closed pipes in a k-twist.

Lemma 21 *If $|i-j| \le k$, the ith and jth pipes in an acyclic k-twist T are comparable for $\triangleleft_{\mathrm{T}}$.*

Definition 22 *The k-canopy scheme of a (k,n)-twist T is the orientation $\mathsf{can}^k(\mathrm{T}) \in \mathcal{AO}^k(n)$ with an edge $i \to j$ for all $i, j \in [n]$ such that $|i-j| \le k$ and $i \triangleleft_{\mathrm{T}} j$. It indeed defines an acyclic orientation of $\mathrm{G}^k(n)$ by Lemma 21. We call k-canopy the map $\mathsf{can}^k : \mathcal{AT}^k(n) \to \mathcal{AO}^k(n)$.*

Example 23 *For $k = 1$, the 1-canopy scheme of a $(1,n)$-twist T is the classical canopy of $\mathrm{T}^{\#}$, i.e. the sign vector $\mathsf{can}(\mathrm{T}) \in \{-,+\}^{n-1}$ defined by $\mathsf{can}(\mathrm{T})_i = -$ if the node i of $\mathrm{T}^{\#}$ is above the node $i+1$ of $\mathrm{T}^{\#}$ and $\mathsf{can}(\mathrm{T})_i = +$ otherwise. This map was already used by J.-L. Loday in [LR98, Lod04], but the name "canopy" was coined by X. Viennot [Vie07]. The binary search tree insertion map and the canopy map factorize the recoil map: $\mathsf{can} \circ \mathsf{ins} = \mathsf{rec}$.*

Proposition 24 *The maps ins^k, can^k, and rec^k define a commutative diagram of lattice homomorphisms:*



2 Geometry of acyclic twists

This section is devoted to the polyhedral geometry of permutations of $\mathfrak{S}_n$, acyclic twists of $\mathcal{AT}^k(n)$, and acyclic orientations of $\mathcal{AO}^k(n)$. It is mainly based on properties of brick polytopes of sorting networks, defined and studied by V. Pilaud and F. Santos in [PS12].

2.1 Permutahedra, brick polytopes, and zonotopes

We denote by $(\mathbf{e}_i)_{i\in[n]}$ the canonical basis of $\mathbb{R}^n$ and let $\mathbb{1} = \sum_{i\in[n]} \mathbf{e}_i$. We recall the definition of three families of polytopes, which are illustrated in Figure 6:

Permutahedron. The *permutahedron* $\mathsf{Perm}(n)$ is the $(n-1)$-dimensional polytope defined as the convex hull $\mathsf{Perm}(n) := \operatorname{conv}\big\{ \sum_{i\in[n]} i\mathbf{e}_{\tau(i)} \mid \tau \in \mathfrak{S}_n \big\}$. Its vertices correspond to the permutations of $\mathfrak{S}_n$.

Brick polytope. We call *bricks* the squares $[i, i+1] \times [j, j+1]$ of the triangular shape. The *brick area* of a pipe p is the number of bricks located below p but inside the axis-parallel rectangle defined by the two endpoints of p. The *brick vector* of a k-twist T is the vector of $\mathbb{R}^n$ whose ith coordinate is the brick area of the ith pipe of T. The *brick polytope* $\mathsf{Brick}^k(n)$ is the $(n-1)$-dimensional polytope defined as the convex hull of the brick vectors of all (k, n)-twists. Its vertices correspond to the acyclic (k, n)-twists of $\mathcal{AT}^k(n)$.

Zonotope. The *graphical zonotope* $\mathsf{Zono}(\mathrm{G})$ of a graph G is the Minkowski sum of the segments $[\mathbf{e}_i, \mathbf{e}_j]$ for all edges $\{i, j\}$ of G. We consider the $(n-1)$-dimensional zonotope $\mathsf{Zono}^k(n) := \mathsf{Zono}(\mathrm{G}^k(n))$. Its vertices correspond to the acyclic orientations of $\mathcal{AO}^k(n)$.

Example 25 *For $k = 1$, the brick polytope $\mathsf{Brick}^1(n)$ coincides (up to translation) with J.-L. Loday's associahedron [Lod04] and the zonotope $\mathsf{Zono}^1(n)$ coincides with the parallelotope generated by the simple roots $\mathbf{e}_{i+1} - \mathbf{e}_i$.*

2.2 The geometry of the surjections ins^k, can^k, and rec^k

The main geometric connection between the three polytopes $\mathsf{Perm}(n)$, $\mathsf{Brick}^k(n)$ and $\mathsf{Zono}^k(n)$ is given by their normal fans. Remember that a *polyhedral fan* is a collection of polyhedral cones of $\mathbb{R}^n$ closed under faces and which intersect pairwise along faces. The (outer) *normal cone* of a face F of a polytope P is the cone generated by the outer normal vectors of the facets of P containing F. Finally, the (outer) *normal fan* of P is the collection of the (outer) normal cones of all its faces.

The *incidence cone* $\mathrm{C}(\lhd)$ and the *braid cone* $\mathrm{C}^\diamond(\lhd)$ of an poset $\lhd$ are the polyhedral cones defined by $\mathrm{C}(\lhd) := \operatorname{cone}\{\mathbf{e}_i - \mathbf{e}_j \mid \text{for all } i \lhd j\}$ and $\mathrm{C}^\diamond(\lhd) := \big\{\mathbf{x} \in \mathbb{1}^\perp \;\big|\; x_i \le x_j \text{ for all } i \lhd j\big\}$. These two cones lie in the space $\mathbb{1}^\perp := \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x} | \mathbb{1} \rangle = 0\}$ and are polar to each other. For a permutation $\tau \in \mathfrak{S}_n$ (resp. a twist $\mathrm{T} \in \mathcal{AT}^k(n)$, resp. an orientation $\mathrm{O} \in \mathcal{AO}^k(n)$), we slightly abuse notation to write $\mathrm{C}(\tau)$ (resp. $\mathrm{C}(\mathrm{T})$, resp. $\mathrm{C}(\mathrm{O})$) for the incidence cone of the chain $\tau_1 \lhd \cdots \lhd \tau_n$ (resp. of the transitive closure $\lhd$ of the contact graph $\mathrm{T}^\#$, resp. of the transitive closure of O). We define similarly the braid cone $\mathrm{C}^\diamond(\tau)$ (resp. $\mathrm{C}^\diamond(\mathrm{T})$, resp. $\mathrm{C}^\diamond(\mathrm{O})$).

Proposition 26 *The cones $\big\{\mathrm{C}^\diamond(\tau) \mid \tau \in \mathfrak{S}_n\big\}$, $\big\{\mathrm{C}^\diamond(\mathrm{T}) \mid \mathrm{T} \in \mathcal{AT}^k(n)\big\}$, and $\big\{\mathrm{C}^\diamond(\mathrm{O}) \mid \mathrm{O} \in \mathcal{AO}^k(n)\big\}$, together with all their faces, respectively form the normal fans of the permutahedron $\mathsf{Perm}(n)$, of the brick polytope $\mathsf{Brick}^k(n)$ and of the zonotope $\mathsf{Zono}^k(n)$.*

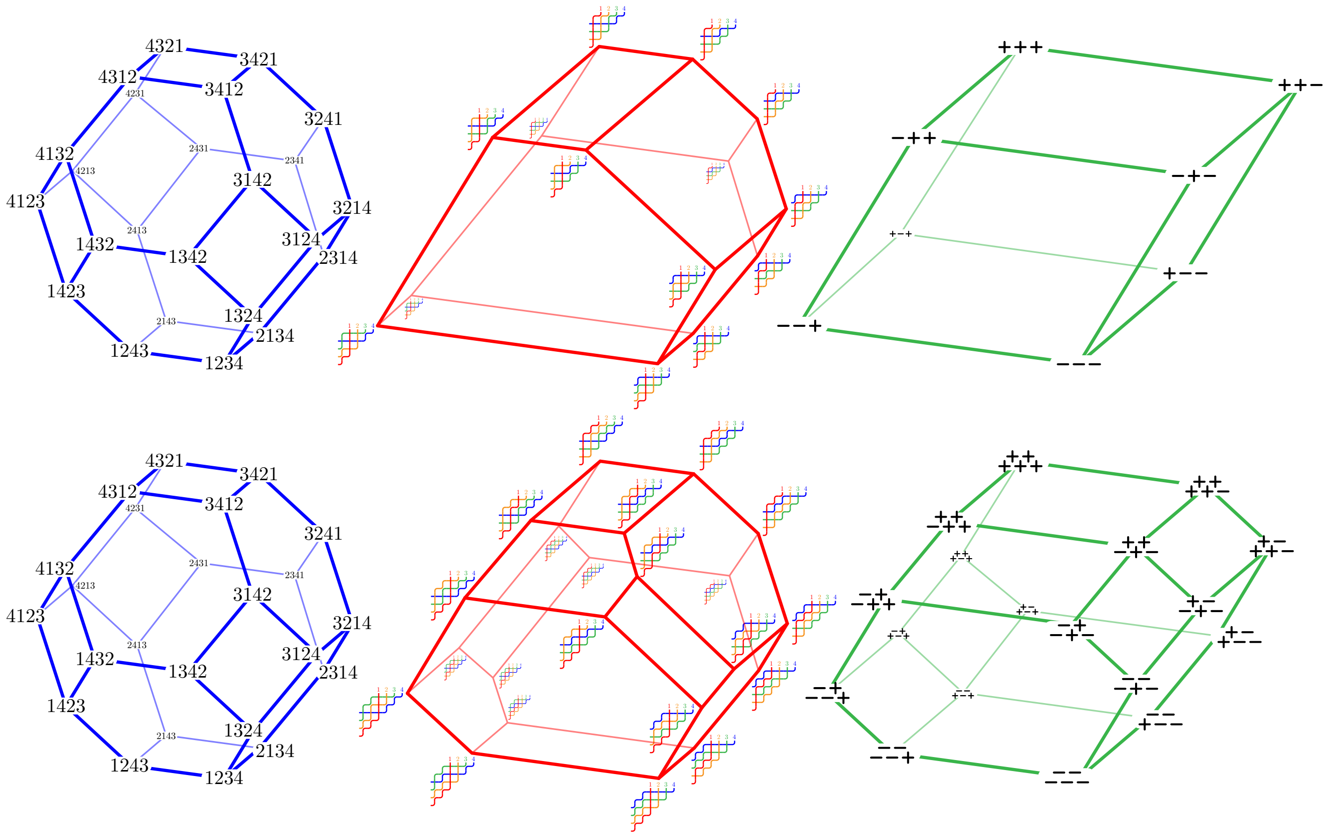


Fig. 6: The permutahedron $\mathsf{Perm}(4)$ (left), the brick polytope $\mathsf{Brick}^k(4)$ (middle) and the zonotope $\mathsf{Zono}^k(4)$ (right) for $k = 1$ (top) and $k = 2$ (bottom). For readibility, we represent orientations of $\mathrm{G}^k(n)$ by pyramids of signs.

Using these normal fans, one can interpret geometrically the maps ins^k, can^k, and rec^k as follows.

Proposition 27 *The insertion map* $\mathsf{ins}^k : \mathfrak{S}_n \to \mathcal{AT}^k(n)$*, the* k*-canopy* $\mathsf{can}^k : \mathcal{AT}^k(n) \to \mathcal{AO}^k(n)$ *and the* k*-recoil map* $\mathsf{rec}^k : \mathfrak{S}_n \to \mathcal{AO}^k(n)$ *are characterized by*

$$\mathrm{T} = \mathsf{ins}^k(\tau) \iff \mathrm{C}(\mathrm{T}) \subseteq \mathrm{C}(\tau) \iff \mathrm{C}^\diamondsuit(\mathrm{T}) \supseteq \mathrm{C}^\diamondsuit(\tau),$$
$$\mathrm{O} = \mathsf{can}^k(\mathrm{T}) \iff \mathrm{C}(\mathrm{O}) \subseteq \mathrm{C}(\mathrm{T}) \iff \mathrm{C}^\diamondsuit(\mathrm{O}) \supseteq \mathrm{C}^\diamondsuit(\mathrm{T}),$$
$$\mathrm{O} = \mathsf{rec}^k(\tau) \iff \mathrm{C}(\mathrm{O}) \subseteq \mathrm{C}(\tau) \iff \mathrm{C}^\diamondsuit(\mathrm{O}) \supseteq \mathrm{C}^\diamondsuit(\tau).$$

Finally, the lattices studied in Section 1 also appear naturally in the geometry of these polytopes.

Proposition 28 *When oriented in the direction* $U := (n, \dots, 1) - (1, \dots, n) = \sum_{i \in [n]} (n + 1 - 2i)\, \mathbf{e}_i$*, the* 1*-skeleta of the permutahedron* $\mathsf{Perm}(n)$ *(resp. of the brick polytope* $\mathsf{Brick}^k(n)$*, resp. of the zonotope* $\mathsf{Zono}^k(n)$*) is the Hasse diagram of the weak order on permutations (resp. of the increasing flip lattice on acyclic* (k, n)*-twists, resp. of the increasing flip lattice on acyclic orientations of* $\mathrm{G}^k(n)$*).*

3 Algebra of acyclic twists

3.1 *The Malvenuto-Reutenauer Hopf algebra* FQSym *on permutations*

We briefly recall here the definition and some elementary properties of C. Malvenuto and C. Reutenauer's Hopf algebra on permutations [MR95], that we denote by FQSym. Let $\mathfrak{S} := \bigsqcup_{n\in\mathbb{N}} \mathfrak{S}_n$ denote the set of all permutations, of arbitrary size.

For $n, n' \in \mathbb{N}$, let $\mathfrak{S}^{(n,n')} := \{\tau \in \mathfrak{S}_{n+n'} \mid \tau_1 < \cdots < \tau_n \text{ and } \tau_{n+1} < \cdots < \tau_{n+n'}\}$ denote the set of permutations of $\mathfrak{S}_{n+n'}$ with at most one descent, at position n. The *shifted concatenation* $\tau\bar{\tau}'$, the *shifted shuffle* $\tau \,\bar{\sqcup\!\sqcup}\, \tau'$, and the *convolution* $\tau \star \tau'$ of two permutations $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$ are defined by

$$\tau\bar{\tau}' := [\tau_1, \dots, \tau_n, \tau'_1 + n, \dots, \tau'_{n'} + n] \in \mathfrak{S}_{n+n'},$$
$$\tau \,\bar{\sqcup\!\sqcup}\, \tau' := \big\{ (\tau\bar{\tau}') \circ \pi^{-1} \mid \pi \in \mathfrak{S}^{(n,n')} \big\} \qquad \text{and} \qquad \tau \star \tau' := \big\{ \pi \circ (\tau\bar{\tau}') \mid \pi \in \mathfrak{S}^{(n,n')} \big\}.$$

For example,

$$12 \,\bar{\sqcup\!\sqcup}\, 231 = \{12453, 14253, 14523, 14532, 41253, 41523, 41532, 45123, 45132, 45312\},$$
$$12 \star 231 = \{12453, 13452, 14352, 15342, 23451, 24351, 25341, 34251, 35241, 45231\}.$$

Definition 29 *We denote by* FQSym *the Hopf algebra with basis* $(\mathbb{F}_\tau)_{\tau\in\mathfrak{S}}$ *and whose product and coproduct are defined by* $\mathbb{F}_\tau \cdot \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau \,\bar{\sqcup\!\sqcup}\, \tau'} \mathbb{F}_\sigma$ *and* $\triangle \mathbb{F}_\sigma = \sum_{\sigma \in \tau \star \tau'} \mathbb{F}_\tau \otimes \mathbb{F}_{\tau'}$.

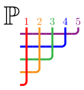
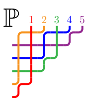
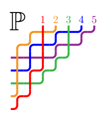
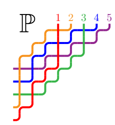
Proposition 30 *A product of weak order intervals in* FQSym *is a weak order interval: for any two weak order intervals* $[\mu, \omega]$ *and* $[\mu', \omega']$*, we have* $\Big(\sum_{\mu \le \tau \le \omega} \mathbb{F}_\tau\Big) \cdot \Big(\sum_{\mu' \le \tau' \le \omega'} \mathbb{F}_{\tau'}\Big) = \sum_{\mu\bar{\mu}' \le \sigma \le \bar{\omega}'\omega} \mathbb{F}_\sigma$.

3.2 *Subalgebra of* FQSym

We denote by Twist^k the vector subspace of FQSym generated by the elements

$$\mathbb{P}_{\mathrm{T}} := \sum_{\substack{\tau \in \mathfrak{S} \\ \mathsf{ins}^k(\tau) = \mathrm{T}}} \mathbb{F}_\tau = \sum_{\mathcal{L}(\mathrm{T}^\#)} \mathbb{F}_\tau,$$

for all acyclic k-twists T. For example, for the $(k, 5)$-twists of Figure 3, we have

$\mathbb{P}$  $= \sum_{\tau \in \mathfrak{S}_5} \mathbb{F}_\tau$ $\quad$ $\mathbb{P}$  $= \mathbb{F}_{13542} + \mathbb{F}_{15342} + \mathbb{F}_{31542} + \mathbb{F}_{51342} + \mathbb{F}_{35142} + \mathbb{F}_{53142} + \mathbb{F}_{35412} + \mathbb{F}_{53412}$ $\quad$ $\mathbb{P}$  $= \mathbb{F}_{31542} + \mathbb{F}_{35142}$ $\quad$ $\mathbb{P}$  $= \mathbb{F}_{31542}.$

Theorem 31 Twist^k *is a Hopf subalgebra of* FQSym*, that we call* k-twist Hopf algebra*.*

Example 32
- *For $k = 0$, the bijection which sends the unique $(0, n)$-twist to $X^n/n!$ defines an isomorphism from the 0-twist algebra* Twist^0 *to the polynomial ring* $\mathbb{K}[X]$*.*
- *For $k = 1$, the bijection of Remark 3 defines an isomorphism from the 1-twist algebra* Twist^1 *to M. Ronco and J.-L. Loday's Hopf algebra* PBT *on planar binary trees [LR98, HNT05].*

We now aim at understanding the product and the coproduct in Twist^k directly on k-twists. Although they are not always as satisfactory, our descriptions naturally extend classical results on the binary tree Hopf algebra PBT described in [LR98, AS06, HNT05].

Product. To describe the product in Twist^k, we need the following notation. For a (k, n)-twist T and a (k, n')-twist T', we denote by $\mathrm{T}\backslash\mathrm{T}'$ the $(k, n + n')$-twist obtained by inserting T in the first rows and columns of T' and by T/T' the $(k, n+n')$-twist obtained by inserting T' in the last rows and columns of T.

Proposition 33 *For any acyclic k-twists $\mathrm{T} \in \mathcal{AT}^k(n)$ and $\mathrm{T}' \in \mathcal{AT}^k(n')$, we have $\mathbb{P}_{\mathrm{T}} \cdot \mathbb{P}_{\mathrm{T}'} = \sum_{\mathrm{S}} \mathbb{P}_{\mathrm{S}}$, where S runs over the interval between $\mathrm{T}\backslash\mathrm{T}'$ and T/T' in the $(k, n + n')$-twist lattice.*

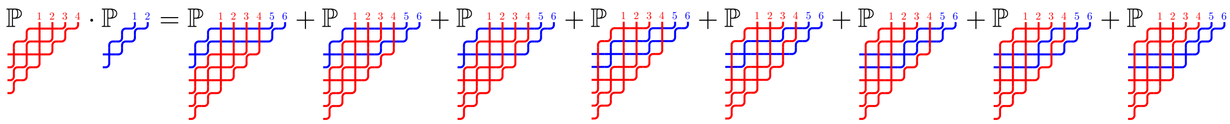


Fig. 7: An example of product in the 2-twist algebra Twist^2.

Coproduct. Our description of the coproduct in Twist^k is unfortunately not as simple as the coproduct in PBT. We need the following definition. A *cut* in a k-twist S is a set γ of edges of the contact graph $\mathrm{S}^\#$ such that any path in $\mathrm{S}^\#$ from a leaf to the root contains precisely one edge of γ. We then denote by $A^\#(\mathrm{S}, \gamma)$ (resp. $B^\#(\mathrm{S}, \gamma)$) the restriction of the contact graph $\mathrm{S}^\#$ to the nodes above (resp. below) γ. Moreover, $A^\#(\mathrm{S}, \gamma)$ is the contact graph of the k-twist $A(\mathrm{S}, \gamma)$ obtained from S by deleting all pipes below γ in $\mathrm{S}^\#$. Nevertheless, note that $B^\#(\mathrm{S}, \gamma)$ is not *a priori* the contact graph of a k-twist.

Proposition 34 *For any acyclic k-twist $\mathrm{S} \in \mathcal{AT}^k(m)$, we have $\triangle\mathbb{P}_{\mathrm{S}} = \sum_\gamma \big(\sum_\tau \mathbb{P}_{\mathsf{ins}^k(\tau)}\big) \otimes \mathbb{P}_{A(\mathrm{S},\gamma)}$, where γ runs over all cuts of S and τ runs over a set of representatives of the k-twist congruence classes of the linear extensions of $B^\#(\mathrm{S}, \gamma)$.*

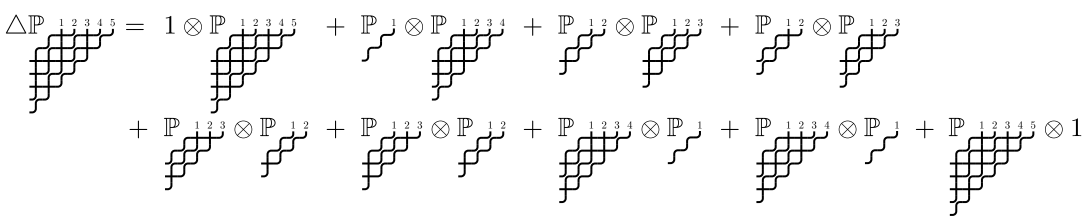


Fig. 8: An example of coproduct in the 2-twist algebra Twist^2.

Matriochka algebras. Consider the k-recoil algebra Rec^k defined in [NRT11] as the Hopf subalgebra of FQSym generated by the elements $\mathbb{X}_{\mathrm{O}} := \sum_{\mathsf{rec}^k(\tau)=\mathrm{O}} \mathbb{F}_\tau$, for all acyclic orientations O of the graph $\mathrm{G}^k(n)$ for all $n \in \mathbb{N}$. The commutative diagram of Proposition 24 ensures that $\mathbb{X}_{\mathrm{O}} = \sum_{\mathsf{can}^k(\mathrm{T})=\mathrm{O}} \mathbb{P}_{\mathrm{T}}$, and thus that Rec^k is a Hopf subalgebra of Twist^k.

Remark 35 *Further properties of Twist^k can be found in [Pil15]. In particular, we describe combinatorially in terms of (k, n)-twists the product and coproduct operations on the dual Hopf algebra Twist^{k*} of Twist^k, we study some multiplicative bases of Twist^k and their indecomposable elements, and we observe a connection to integer point transform in certain cones related to k-twists.*

3.3 k-twistiform algebras

In this section, we extend the notion of dendriform algebras [Lod01, Chap. 5] to k-twistiform algebras. In a dendriform algebra, the product $\cdot$ is decomposed into two partial products $\prec$ and $\succ$ satisfying:

$$\mathsf{x} \prec (\mathsf{y} \cdot \mathsf{z}) = (\mathsf{x} \prec \mathsf{y}) \prec \mathsf{z}, \qquad \mathsf{x} \succ (\mathsf{y} \prec \mathsf{z}) = (\mathsf{x} \succ \mathsf{y}) \prec \mathsf{z}, \quad \text{and} \quad \mathsf{x} \succ (\mathsf{y} \succ \mathsf{z}) = (\mathsf{x} \cdot \mathsf{y}) \succ \mathsf{z}.$$

In our context, we will still decompose the product of FQSym (and of Twist^k) into partial products, but we will use 2^k partial products satisfying 3^k relations. We need to fix some natural notations on words. We denote by $|W|$ the length of a word W. For a subset P of positions in W and a subset L of letters of W, we denote by W_P the subword of W consisting only of the letters at positions in P and by W^L the subword of W consisting only of the letters which belong to L.

Definition 36 *A k-twistiform algebra is a vector space Alg endowed with a collection $\mathfrak{B} := \{\prec, \bowtie, \succ\}^k$ of 3^k bilinear operations which satisfy the following $k3^{k-1} + 3^k$ relations:*

Split relations *For any $\mathfrak{b}, \mathfrak{b}' \in \{\prec, \bowtie, \succ\}^*$ with $|\mathfrak{b}| + |\mathfrak{b}'| = k-1$, the operation $\mathfrak{b}{\bowtie}\mathfrak{b}' \in \mathfrak{B}$ decomposes into the operations $\mathfrak{b}{\prec}\mathfrak{b} \in \mathfrak{B}$ and $\mathfrak{b}{\succ}\mathfrak{b} \in \mathfrak{B}$:* $\mathsf{x}\ \mathfrak{b}{\bowtie}\mathfrak{b}'\ \mathsf{y} = \mathsf{x}\ \mathfrak{b}{\prec}\mathfrak{b}'\ \mathsf{y} + \mathsf{x}\ \mathfrak{b}{\succ}\mathfrak{b}'\ \mathsf{y}$ *for all* $\mathsf{x}, \mathsf{y} \in \mathsf{Alg}$.

Associativity relations *For any $W \in \{x, y, z\}^k$, the operations $\mathfrak{b}_W, \mathfrak{b}'_W, \mathfrak{b}''_W, \mathfrak{b}'''_W \in \mathfrak{B}$ defined by*

$$(\mathfrak{b}_W)_p := \begin{cases} \prec & \text{if } W_p = x \\ \succ & \text{if } W_p \in \{y, z\} \end{cases} \qquad (\mathfrak{b}'_W)_p := \begin{cases} \prec & \text{if } |W^{\{y,z\}}| \ge p \text{ and } (W^{\{y,z\}})_p = y \\ \succ & \text{if } |W^{\{y,z\}}| \ge p \text{ and } (W^{\{y,z\}})_p = z \\ \bowtie & \text{otherwise} \end{cases}$$

$$(\mathfrak{b}''_W)_p := \begin{cases} \prec & \text{if } |W^{\{x,y\}}| \ge p \text{ and } (W^{\{x,y\}})_p = x \\ \succ & \text{if } |W^{\{x,y\}}| \ge p \text{ and } (W^{\{y,z\}})_p = y \\ \bowtie & \text{otherwise} \end{cases} \qquad (\mathfrak{b}'''_W)_p := \begin{cases} \prec & \text{if } W_p \in \{x, y\} \\ \succ & \text{if } W_p = z \end{cases}$$

satisfy the associativity relation $\quad \mathsf{x}\ \mathfrak{b}_W\ (\mathsf{y}\ \mathfrak{b}'_W\ \mathsf{z}) = (\mathsf{x}\ \mathfrak{b}''_W\ \mathsf{y})\ \mathfrak{b}'''_W\ \mathsf{z}$ *for all* $\mathsf{x}, \mathsf{y}, \mathsf{z} \in \mathsf{Alg}$.

Example 37 *1-twistiform algebras are precisely dendriform algebras. 2-twistiform algebras are vector spaces endowed with 9 operations $\prec\prec$, $\prec\bowtie$, $\prec\succ$, $\bowtie\prec$, $\bowtie\bowtie$, $\bowtie\succ$, $\succ\prec$, $\succ\bowtie$, $\succ\succ$ such that:*

$$\begin{array}{lll}
\mathsf{x} \bowtie\prec \mathsf{y} = \mathsf{x} \prec\prec \mathsf{y} + \mathsf{x} \succ\prec \mathsf{y}, & \mathsf{x} \bowtie\bowtie \mathsf{y} = \mathsf{x} \prec\bowtie \mathsf{y} + \mathsf{x} \succ\bowtie \mathsf{y}, & \mathsf{x} \bowtie\succ \mathsf{y} = \mathsf{x} \prec\succ \mathsf{y} + \mathsf{x} \succ\succ \mathsf{y}, \\
\mathsf{x} \prec\bowtie \mathsf{y} = \mathsf{x} \prec\prec \mathsf{y} + \mathsf{x} \prec\succ \mathsf{y}, & \mathsf{x} \bowtie\bowtie \mathsf{y} = \mathsf{x} \bowtie\prec \mathsf{y} + \mathsf{x} \bowtie\succ \mathsf{y}, & \mathsf{x} \succ\bowtie \mathsf{y} = \mathsf{x} \succ\prec \mathsf{y} + \mathsf{x} \succ\succ \mathsf{y}, \\
\mathsf{x} \prec\prec (\mathsf{y} \bowtie\bowtie \mathsf{z}) = (\mathsf{x} \prec\prec \mathsf{y}) \prec\prec \mathsf{z}, & \mathsf{x} \prec\succ (\mathsf{y} \prec\bowtie \mathsf{z}) = (\mathsf{x} \prec\succ \mathsf{y}) \prec\prec \mathsf{z}, & \mathsf{x} \prec\succ (\mathsf{y} \succ\bowtie \mathsf{z}) = (\mathsf{x} \prec\bowtie \mathsf{y}) \prec\succ \mathsf{z}, \\
\mathsf{x} \succ\prec (\mathsf{y} \prec\bowtie \mathsf{z}) = (\mathsf{x} \succ\prec \mathsf{y}) \prec\prec \mathsf{z}, & \mathsf{x} \succ\succ (\mathsf{y} \prec\prec \mathsf{z}) = (\mathsf{x} \succ\succ \mathsf{y}) \prec\prec \mathsf{z}, & \mathsf{x} \succ\succ (\mathsf{y} \prec\succ \mathsf{z}) = (\mathsf{x} \succ\bowtie \mathsf{y}) \prec\succ \mathsf{z}, \\
\mathsf{x} \succ\prec (\mathsf{y} \succ\bowtie \mathsf{z}) = (\mathsf{x} \prec\bowtie \mathsf{y}) \succ\prec \mathsf{z}, & \mathsf{x} \succ\succ (\mathsf{y} \succ\prec \mathsf{z}) = (\mathsf{x} \succ\bowtie \mathsf{y}) \succ\prec \mathsf{z}, & \mathsf{x} \succ\succ (\mathsf{y} \succ\succ \mathsf{z}) = (\mathsf{x} \bowtie\bowtie \mathsf{y}) \succ\succ \mathsf{z}.
\end{array}$$

We define a k-twistiform structure on FQSym as follows. For an operation $\mathfrak{b} \in \mathfrak{B}$ and two permutations $\tau \in \mathfrak{S}_n$ and $\tau' \in \mathfrak{S}_{n'}$, we define $\tau\, \mathfrak{b}\, \tau'$ to be the set of permutations $\sigma \in \tau \,\bar{\sqcup}\, \tau'$ such that for all $i \in [k]$, we have $\sigma_{n+n'+1-i} \le n$ if $\mathfrak{b}_i = \prec$ while $\sigma_{n+n'+1-i} > n$ if $\mathfrak{b}_i = \succ$. Finally, we define the operations $\mathfrak{B}$ on the Hopf algebra FQSym itself by $\mathbb{F}_\tau\, \mathfrak{b}\, \mathbb{F}_{\tau'} = \sum_{\sigma \in \tau\, \mathfrak{b}\, \tau'} \mathbb{F}_\sigma$.

Proposition 38 *The Hopf algebra FQSym, endowed with the operations $\mathfrak{B}$ described above, defines a k-twistiform algebra. The product of FQSym is then given by $\cdot = \bowtie^k$. The subalgebra Twist^k of FQSym is stable by the operations $\mathfrak{B}$ and therefore inherits a k-twistiform structure.*

This statement motivates the study of the k-twistiform operad. Several similar operads generalizing the dendriform and the diassociative operads are studied in [HP15].

References

[AS06] M. Aguiar and F. Sottile. Structure of the Loday-Ronco Hopf algebra of trees. *J. Algebra*, 295(2):473–511, 2006.

[BB93] N. Bergeron and S. Billey. RC-graphs and Schubert polynomials. *Experiment. Math.*, 2(4):257–269, 1993.

[HNT05] F. Hivert, J.-C. Novelli, and J.-Y. Thibon. The algebra of binary search trees. *Theoret. Comput. Sci.*, 339(1):129–165, 2005.

[HP15] F. Hivert and V. Pilaud. Multitwistiform algebras and operads. Work in progress, 2015.

[Jon05] J. Jonsson. Generalized triangulations and diagonal-free subsets of stack polyominoes. *J. Combin. Theory Ser. A*, 112(1):117–142, 2005.

[KM05] A. Knutson and E. Miller. Gröbner geometry of Schubert polynomials. *Ann. of Math. (2)*, 161(3):1245–1318, 2005.

[Lod01] J.-L. Loday. Dialgebras. In *Dialgebras and related operads*, volume 1763 of *Lecture Notes in Math.*, pages 7–66. Springer, Berlin, 2001.

[Lod04] J.-L. Loday. Realization of the Stasheff polytope. *Arch. Math. (Basel)*, 83(3):267–278, 2004.

[LR98] J.-L. Loday and M. Ronco. Hopf algebra of the planar binary trees. *Adv. Math.*, 139(2):293–309, 1998.

[MR95] C. Malvenuto and C. Reutenauer. Duality between quasi-symmetric functions and the Solomon descent algebra. *J. Algebra*, 177(3):967–982, 1995.

[NRT11] J.-C. Novelli, C. Reutenauer, and J.-Y. Thibon. Generalized descent patterns in permutations and associated Hopf algebras. *European J. Combin.*, 32(4):618–627, 2011.

[OEI10] The On-Line Encyclopedia of Integer Sequences. `http://oeis.org`, 2010.

[Pil15] V. Pilaud. Brick polytopes, lattice quotients, and Hopf algebras. `arXiv:1505.07665`, 2015.

[PP12] V. Pilaud and M. Pocchiola. Multitriangulations, pseudotriangulations and primitive sorting networks. *Discrete Comput. Geom.*, 48(1):142–191, 2012.

[PS09] V. Pilaud and F. Santos. Multitriangulations as complexes of star polygons. *Discrete Comput. Geom.*, 41(2):284–317, 2009.

[PS12] V. Pilaud and F. Santos. The brick polytope of a sorting network. *European J. Combin.*, 33(4):632–662, 2012.

[Rea05] N. Reading. Lattice congruences, fans and Hopf algebras. *J. Combin. Theory Ser. A*, 110(2):237–273, 2005.

[Sol76] L. Solomon. A Mackey formula in the group ring of a Coxeter group. *J. Algebra*, 41(2):255–264, 1976.

[Vie07] X. Viennot. Catalan tableaux and the asymmetric exclusion process. In *Proc. FPSAC'07*. 2007.