Partitioning Harary graphs into connected subgraphs containing prescribed vertices

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Abstract

A graph G is *arbitrarily partitionable* (AP for short) if for every partition $(\tau_1, ..., \tau_p)$ of $|V(G)|$ there exists a partition $(V_1, ..., V_p)$ of $V(G)$ such that each V_i induces a connected subgraph of G with order τ_i. If, additionally, each of k of these subgraphs contains an arbitrary vertex of G prescribed beforehand, then G is *arbitrarily partitionable under k prescriptions* (AP+k for short). It is known that AP+k-graphs on n vertices are $(k+1)$-connected, and have thus at least $\lceil \frac{n(k+1)}{2} \rceil$ edges. We show that there exist AP+k-graphs on n vertices and $\lceil \frac{n(k+1)}{2} \rceil$ edges for every $k \geq 1$ and $n \geq k$.

Keywords: arbitrarily partitionable graph, partition under prescriptions, Harary graph

1 Introduction

We denote by $V(G)$ and $E(G)$ the sets of vertices and edges, respectively, of a graph G. The *order* (resp. *size*) of G is the cardinality of the set $V(G)$ (resp. $E(G)$).

Let G be a connected graph of order n. A sequence $\tau = (\tau_1, ..., \tau_p)$ of positive integers is *admissible for G* if it performs a partition of n, that is if $\sum_{i=1}^{p} \tau_i = n$. If, additionally, we can partition $V(G)$ into p parts $(V_1, ..., V_p)$ such that each V_i induces a connected subgraph of G with order τ_i, then τ is *realizable in G*, and the partition $(V_1, ..., V_p)$ is a *realization of τ in G*. If every admissible sequence for G is also realizable in G, then G is *arbitrarily partitionable* (AP for short).

The study of AP-graphs was motivated by the following computer science problem [1]. Suppose that we have a network of n computing resources we want to share between p users, where the i^{th} user needs exactly τ_i resources. For the sake of performance, we also suppose that the sharing must not be done arbitrarily but in such a way that the following two conditions are fulfilled:

1. each resource is attributed to exactly one user,
2. two resources attributed to a user must be able to communicate within his subnetwork.

Clearly, the problem of satisfying our users with this specific demand is equivalent to the problem of finding a realization of the sequence $(\tau_1, ..., \tau_p)$ in the graph modelling our network. Of course, the more different resource demands we can satisfy in a network, the more interesting

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it is. Hence, we want our network to have a topology which allows us to satisfy every user needs, that is an AP-graph topology.

The notion of AP-graph was mostly investigated in the context of some simple classes of graphs, like trees [1, 2]. Recall that a graph is *traceable* if it admits a Hamiltonian path. Since every path is obviously AP and the property of being AP is closed under edge addition, we get that traceable graphs are AP. Hence, the property of being AP can be considered as a generalization of the property of being Hamiltonian. The relationship between these two properties is illustrated in [7], where an Ore-type condition for a graph to be AP is exhibited.

We are here interested in a stronger version of this problem. Suppose now that our resources are not equivalent and that, for some reasons, k of our p users, for some $k \leq p$, are each allowed to request one specific resource to belong to their subnetwork. For instance, a resource could be needed by a user for it is the only one having a specific computing skill or it is conveniently located within the whole network. To model this requirement, we strengthen the definition of AP-graphs as follows. A *k-prescription of G* is a subset $P = (v_1, ..., v_k)$ of k distinct vertices of G. We say that a sequence $(\tau_1, ..., \tau_p)$ with $p \geq k$ elements is *realizable in G under P* if there exists a realization $(V_1, ..., V_p)$ of τ in G such that for every $i \in [1, k]$, the vertex v_i belongs to V_i. Notice that we have adopted the convention that the sizes associated to the prescribed vertices are located at the beginning of the sequence. The graph G is *arbitrarily partitionable under k prescriptions* (AP+k for short) if every sequence admissible for G consisting of at least k elements is realizable in G under every k-prescription. According to these definitions, an optimal network for the augmented problem above is a network whose topology allows our k special users to each choose one particular resource to belong to his subnetwork no matter what the other users needs are. Such a network should therefore have an AP+k-graph topology.

For any $k \geq 1$, the set of complete graphs on at least k vertices is a trivial class of AP+k-graphs. But these graphs are not very interesting regarding the problem above because of their extreme number of edges. Thus, we here focus on *optimal AP+k-graphs*, that is on AP+k-graphs with the least possible number of edges.

This paper is organized as follows. In Section 2, we introduce some notations and preliminary results which are useful to prove, in Sections 3 and 4, respectively, some results on the partitioning of powers of paths and cycles under many prescriptions. In Section 5, these results are then used to show that, for every $k \neq 2$, every $(k+1)$-connected Harary graph is an optimal AP+k-graph. We finally show, in Section 6, that 3-connected Harary graphs are not necessarily AP+2, and provide another class of optimal AP+2-graphs instead.

2 Definitions, notation, and preliminary results

A subgraph H of a graph G is a *spanning subgraph* of G if $V(H) = V(G)$. We will also say that G is *spanned by H*. Given a graph G and an integer $k \geq 1$, the *k^{th} power of G*, denoted by G^k, is the graph with the same vertex set as G, two of its vertices being adjacent if they are at distance at most k in G. Let P_n (resp. C_n) denote the path (resp. the cycle) on n vertices, and $\{v_0, ..., v_{n-1}\}$ its set of vertices. For convenience, the vertices v_0 and v_{n-1} of P_n are called the *first vertex* and the *last vertex* of P_n, respectively. We use the same terminology to deal with the vertices of P_n^k (resp. C_n^k) according to its natural spanning path P_n (resp. spanning cycle C_n).

Let $k \geq 1$ and $n \geq k$ be any two integers. The k-connected Harary graph on n vertices, denoted by $H_{k,n}$, has vertex set $\{v_0, ..., v_{n-1}\}$ and the following edges:

- if $k = 2r$ is even, then two vertices v_i and v_j are linked if and only if $i - r \leq j \leq i + r$;
- if $k = 2r + 1$ is odd and n is even, then $H_{k,n}$ is obtained by joining v_i and $v_{i+\frac{n}{2}}$ in $H_{2r,n}$ for every $i \in [0, \frac{n}{2} - 1]$;

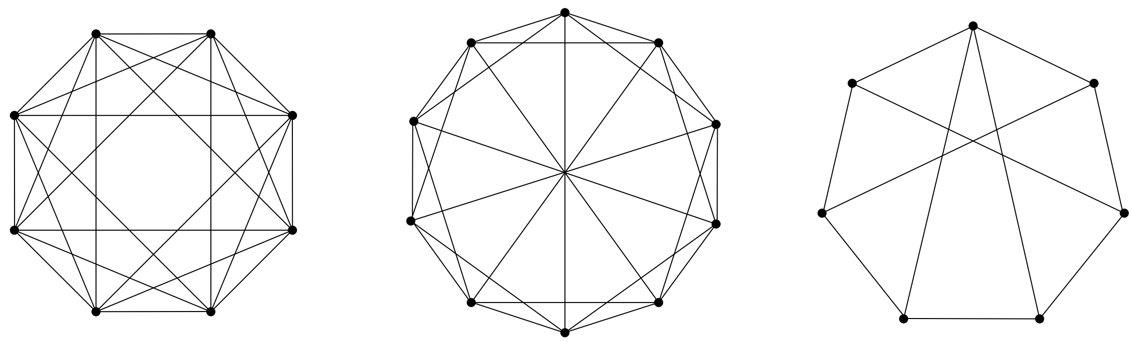

Figure 1: The Harary graphs $H_{6,8}$, $H_{5,10}$ and $H_{3,7}$

- if $k = 2r + 1$ and n are odd, then $H_{k,n}$ is obtained from $H_{2r,n}$ by first linking v_0 to both $v_{\lfloor \frac{n}{2} \rfloor}$ and $v_{\lceil \frac{n}{2} \rceil}$, and then each vertex v_i to $v_{i+\lceil \frac{n}{2} \rceil}$ for every $i \in [1, \lfloor \frac{n}{2} \rfloor - 1]$;

where the subscripts are taken modulo n. Three examples of Harary graphs are given in Figure 1. When k is odd, the neighbours of a vertex v of $H_{k,n}$ which are at distance strictly more than k from v in its spanning C_n are called the *antipodal neighbours of* v. Clearly, v has at most two antipodal neighbours in $H_{k,n}$. In particular, v has exactly two antipodal neighbours if and only if $i = 0$, and k and n are both odd. A *diagonal edge of* $H_{k,n}$ is an edge linking two antipodal neighbours of $H_{k,n}$.

If G is a graph with a natural ordering of its vertices (like powers of paths and cycles, or Harary graphs), then, for any vertex v of G, we denote by v^+ (resp. v^-) the neighbour of v following v (resp. preceding v) in this ordering.

One of the first important property of AP-graphs is the following one.

Proposition 1 *If a graph G admits a spanning AP-subgraph (resp. a spanning AP+k-subgraph for some $k \geq 1$), then G is AP (resp. AP+k).*

Györi and Lovàsz independently considered the problem of determining whether a sequence τ can be realized in a given graph G under a given prescription P in the special case where G is k-connected, τ has exactly k elements, and P is a k-prescription. This led to the following well-known theorem.

Theorem 2 (Györi [4] and Lovàsz [6], independently) *If G is a k-connected graph, then every sequence consisting of exactly k elements is realizable in G under every k-prescription.*

Hence, throughout this paper, we only consider partitioning of k-connected graphs under a k-prescription into strictly more than k subgraphs. We now point out the following property of AP+k-graphs, from which we deduce the size of an optimal AP+k-graph.

Observation 3 *Every AP+k-graph is $(k+1)$-connected. Therefore, an optimal AP+k-graph has $\lceil \frac{n(k+1)}{2} \rceil$ edges.*

Indeed, if there exist p vertices $v_1, ..., v_p$, where $p \leq k$, such that $G - \{v_1, ..., v_p\}$ is disconnected into q connected components on $n_1, ..., n_q$ vertices, respectively, with $n_1 \geq ... \geq n_q$, then the sequence $(1, ..., 1, n_1 + 1, (\sum_{i=2}^{q} n_i) - 1)$ cannot be realized in G under $(v_1, ..., v_p)$. Since an edge-minimal $(k+1)$-connected graph has size $\lceil \frac{n(k+1)}{2} \rceil$, this is the size of an optimal AP+k-graph.

Baudon *et al.* [3] proved that, for every $k \geq 1$, there exist non complete AP+k-graphs, namely the powers of paths and the powers of cycles:

Theorem 4 (Baudon et al. [3]) *The graph P_n^{k+1} is AP+k for every $k \geq 0$ and $n \geq k+1$. The graph C_n^k is AP+$(2k-1)$ for every $k \geq 1$ and $n \geq 2k$.*

Since the number of edges of P_n^{k+1} is $(k+1)(n-(k+1))+\sum_{i=1}^{k} i$, this graph has more edges than an optimal AP+k-graph having the same number of vertices. However, C_n^k is $2k$-regular and hence is an edge-minimal $2k$-connected graph. According to Observation 3, it follows that the set of k^{th} powers of cycles is a set of optimal AP+$(2k-1)$-graphs.

3 Partitioning P_n^k under strictly more than $k-1$ prescriptions

In this section, we point out, in terms of prescribed vertices location, some situations in which it is possible to partition the graph P_n^k under strictly more than $k-1$ prescriptions.

Lemma 5 ([3]) *Let $P=(v_{i_1},...,v_{i_k})$ be a k-prescription of P_n^k with $k\geq 1$, $n\geq k$ and $0\leq i_1<...<i_k\leq n-1$. If $i_1=0$ or $i_k=n-1$, then every partition $\tau=(\tau_1,...,\tau_p)$ of n with $p\geq k$ elements is realizable in P_n^k under P.*

Lemma 6 *Let $P=(v_{i_1},...,v_{i_{k+1}})$ be a $(k+1)$-prescription of $G=P_n^k$ with $k\geq 1$, $n\geq k$ and $0\leq i_1<...<i_{k+1}\leq n-1$. If $i_1=0$ and $i_{k+1}=n-1$, then every partition $\tau=(\tau_1,...,\tau_p)$ of n with $p\geq k+1$ elements is realizable in G under P.*

Proof. We prove this claim by induction on k. For $k=1$, the result is obvious; thus, we now suppose that $k\geq 2$ and that the claim holds for every k' such that $k'<k$. If $\tau_1\leq i_2$, then a correct realization of τ in G under P is $(V_1,...,V_p)$ where $V_1=\{v_0,...,v_{\tau_1-1}\}$ and $(V_2,...,V_p)$ is a realization of $(\tau_2,...,\tau_p)$ in $G-V_1$ under the prescription $(v_{i_2},...,v_{i_{k+1}})$. This realization necessarily exists according to Lemma 5 since $v_{i_{k+1}}$ is the last vertex of $G-V_1$.

Suppose now that $\tau_1>i_2$. Observe that $[0,k-1]-(\bigcup_{j=2}^{k} i_j \mod k)$ is not empty, and let us denote by r one of these values. The subset V_1 of the realization is constructed as follows. It first contains all the vertices between v_0 and v_{i_2-1}, that is $V_1\supseteq\{v_0,...,v_{i_2-1}\}$. We then add the vertex v_j to V_1, where $j\in[i_2+1,i_2+k-1]$ is such that $j\equiv r \mod k$. Finally, as long as $|V_1|<\tau_1$ and we do not reach v_{n-1}, we repeatedly add to V_1 the vertex at distance k on the right from the last one added to V_1 (v_{j+k}, v_{j+2k}, etc.). According to our choice of r, these vertices are not prescribed ones and, at any moment of this procedure, the subgraph $G-V_1$ is spanned by the $(k-1)^{th}$ power of a path and the subgraph $G[V_1]$ is connected.

Thus, on the one hand, if $V_1=\tau_1$ holds after the procedure, then $(V_1,...,V_p)$ is a correct realization of τ under P, where $(V_2,...,V_p)$ is a realization of $(\tau_2,...,\tau_p)$ in $G-V_1$ under the prescription $(v_{i_2},...,v_{i_{k+1}})$ which necessarily exists by the induction hypothesis since v_{i_2} and $v_{i_{k+1}}$ are the first and last vertices of $G-V_1$.

On the other hand, if $|V_1|<\tau_1$ holds once the procedure is achieved, then each vertex from $V(G)-V_1$ has a neighbour in V_1. Hence, we can obtain a correct realization $(V_1\cup V_1',V_2,...,V_p)$ of τ in G under P, where $(V_2,...,V_p,V_1')$ is a realization of $(\tau_2,...,\tau_p,\tau_1-|V_1|)$ in $G-V_1$ under the prescription $(v_{i_2},...,v_{i_{k+1}})$. Once again, such a realization necessarily exists in this subgraph according to the induction hypothesis. ∎

Lemma 7 *Let $P=(v_{i_1},...,v_{i_k})$ be a k-prescription of $G=P_n^k$ with $k\geq 1$, $n\geq k$ and $0\leq i_1<...<i_k\leq n-1$. If $i_k\neq i_1+k-1$, then every partition $\tau=(\tau_1,...,\tau_p)$ of n with $p\geq k$ elements is realizable in P_n^k under P.*

Proof. If $x=\sum_{j=k+1}^{p}\tau_j\leq i_1$, then a correct realization of τ in G under P is $(V_1,...,V_p)$ where $(V_{k+1},...,V_p)$ is an arbitrary realization of $(\tau_{k+1},...,\tau_p)$ in the traceable subgraph $G[\{v_0,...,v_{x-1}\}]$, and $(V_1,...,V_k)$ is a realization of $(\tau_1,...,\tau_k)$ in $G-\{v_0,...,v_{x-1}\}$ under P obtained using Theorem 2.

Suppose now that $x>i_1$. On the one hand, if $\tau_1>i_1$, then a correct realization of τ in G under P is $(V_1'\cup V_1'',V_2,...,V_p)$, where $V_1'=\{v_0,...,v_{i_1-1}\}$ and $(V_1'',V_2,...,V_p)$ is a realization of $(\tau_1-i_1,\tau_2,...,\tau_p)$ in $G-V_1'$ under P obtained via Lemma 5. On the other hand, if $\tau_1\leq i_1$, then let V_1 be a subset of $\{v_0,...,v_{i_1}\}$ obtained as follows. First, we let $V_1=\{v_{i_1}\}$ and we then repeatedly add to V_1 the vertex located at distance 2 on the left of the last vertex added to V_1 as long as $|V_1| < \tau_1$ and v_0 is not reached. If there is no vertex at distance 2 on the left of the last vertex added to V_1, then we add to V_1 every remaining vertex from $\{v_0, ..., v_{i_1-1}\} - V_1$ from left to right until V_1 has size τ_1. Let $X = \{v_0, ..., v_{i_1-1}\} - V_1$. Notice that, at the end of the previous procedure, $G[V_1]$ is connected, $G[X]$ is traceable, and $v_{i_1-1} \in X$. Now, if there exists $y \in [k+1, p]$ such that $\sum_{j=k+1}^{y} \tau_j = |X|$, then a correct realization of τ in G under P is $(V_1, ..., V_p)$ where $(V_{k+1}, ..., V_y)$ is a realization of $(\tau_{k+1}, ..., \tau_y)$ in $G[X]$ and $(V_2, ..., V_k, V_{y+1}, ..., V_p)$ is a realization of $(\tau_2, ..., \tau_k, \tau_{y+1}, ..., \tau_p)$ in $G - \{v_1, ..., v_{i_1}\}$ under $\{v_{i_2}, ..., v_{i_k}\}$ obtained using Lemma 5.

Theorem 2. On the contrary, if for some y we have $\sum_{j=k+1}^{y-1} \tau_j < |X|$ and $\sum_{j=k+1}^{y} \tau_j > |X|$, then let $\tau'_y = |X| - \sum_{j=k+1}^{y-1} \tau_j$, $\tau''_y = \tau_y - \tau'_y$, and $v_\alpha \notin P$ be the nearest neighbour of v_{i_1-1} located on the right of v_{i_1}. Such a vertex necessarily exists since otherwise this would imply that our k prescribed vertices are located consecutively along G. Moreover, v_α is the first vertex of $G - \{v_0, ..., v_{i_1}\}$ if $v_{i_1}^+ \neq v_{i_2}$. We then obtain a desired realization $(V_1, ..., V_{y-1}, V'_y \cup V''_y, V_{y+1}, ..., V_p)$ of τ in G under P, where $(V'_y, V_{k+1}, ..., V_{y-1})$ is a realization of $(\tau'_y, \tau_{k+1}, ..., \tau_{y-1})$ in $G[X]$ under (v_{i_1-1}), and $(V_2, ..., V_k, V''_y, V_{y+1}, ..., V_p)$ is a realization of $(\tau_2, ..., \tau_k, \tau''_y, \tau_{y+1}, ..., \tau_p)$ in $G[\{v_{i_1+1}, ..., v_{n-1}\}]$ under $(v_{i_2}, ..., v_{i_k}, v_\alpha)$. Those two realizations exist according to Lemma 5. ∎

4 Partitioning C_n^k under $2k$ prescriptions

We now prove, for any $k \geq 1$ and $n \geq 2k$, that C_n^k can be partitioned under $2k$ prescriptions when the prescribed vertices are "conveniently" located in C_n^k. Since the Harary graph $H_{2k+1,n}$ is spanned by C_n^k, these results imply that, in some situations, we can deduce a partitioning of $H_{2k+1,n}$ under $2k$ prescriptions using a spanning subgraph argument.

We first introduce some notations. Suppose that $G = C_n^k$ for some $k \geq 1$ and $n \geq 2k$, and that we want to realize the sequence $\tau = (\tau_0, ..., \tau_{p-1})$ in G under $P = (v_{i_0}, ..., v_{i_{2k-1}})$, with $p \geq 2k$ and $0 \leq i_0 < ... < i_{2k-1} \leq n-1$. For every $j \in [0, 2k-1]$, we denote by d_j the value $i_j - i_{j-1} - 1$. In particular, we have $n = 2k + \sum_{j=0}^{2k-1} d_j$. Given two distinct integers $x, y \in [0, 2k-1]$ such that $x < y$, the *garden of the prescribed vertices* $v_{i_x}, ..., v_{i_y}$ *in* G is defined as $G_{x,y} = \{v_{i_{x-1}}^+, ..., v_{i_{y+1}}^-\}$ and is the maximum subset of consecutive vertices of G containing no other prescribed vertices than $v_{i_x}, ..., v_{i_y}$. In particular, $|G_{x,y}| = (y - x + 1) + \sum_{j=x}^{y+1} d_j$. We say that the prescribed vertices $v_{i_x}, ..., v_{i_y}$ are *saturated for* τ *in their garden* if $\sum_{j=x}^{y} \tau_j > |G_{x,y}|$, that is if the parts associated to the prescribed vertices $v_{i_x}, ..., v_{i_y}$ of a realization of τ in G under P must contain some vertices of $V(G) - G_{x,y}$.

Lemma 8 *Let $P = (v_{i_0}, ..., v_{i_{2k-1}})$ be a $2k$-prescription of $G = C_n^k$ with $k \geq 1$, $n \geq 2k$ and $0 \leq i_1 < ... < i_{2k-1} \leq n-1$, and $\tau = (\tau_0, ..., \tau_{p-1})$ be an admissible sequence for G with $p \geq 2k$. For any $x \in [0, 2k-1]$, if the prescribed vertices $v_{i_x}, ..., v_{i_{x+k-1}}$ are saturated for τ in their garden, then neither the prescribed vertices $v_{i_{x+1}}, ..., v_{i_{x+k}}$ nor $v_{i_{x+k+1}}, ..., v_{i_x}$ are saturated in their respective garden.*

Proof. Let us consider, without loss of generality, that $x = 0$. If the prescribed vertices $v_{i_0}, ..., v_{i_{k-1}}$ are saturated for τ in $G_{0,k-1}$, then we have:

$$\sum_{j=0}^{k-1} \tau_j > k + \sum_{j=0}^{k} d_j,$$

and

$$\sum_{j=k}^{2k-1} \tau_j < k + \sum_{j=k+1}^{2k-1} d_j.$$

If the prescribed vertices $v_{i_1}, ..., v_{i_k}$ are also saturated for τ in $G_{1,k}$, then

$$\sum_{j=1}^{k} \tau_j > k + \sum_{j=1}^{k+1} d_j,$$

and

$$\tau_0 + \sum_{j=k+1}^{2k-1} \tau_j < k + d_0 + \sum_{j=k+2}^{2k-1} d_j,$$

which implies that both $n_1 - n_{k+1} > d_1 - d_{k+2}$ and $n_1 - n_{k+1} < d_1 - d_{k+2}$ hold, a contradiction.

If we now suppose that $v_{i_{k+1}}, ..., v_0$ are saturated for τ in $G_{k+1,0}$, then

$$\sum_{j=1}^{k} \tau_j < k + \sum_{j=2}^{k} d_j,$$

and

$$\tau_0 + \sum_{j=k+1}^{2k-1} \tau_j > k + d_0 + d_1 + \sum_{j=k+1}^{2k-1} d_j,$$

and we thus get that both

$$(\sum_{j=1}^{k-1} \tau_j) - (\sum_{j=k+1}^{2k-1} \tau_j) > (\sum_{j=2}^{k} d_j) - (\sum_{j=k-1}^{2k-1} d_j)$$

and

$$(\sum_{j=1}^{k-1} \tau_j) - (\sum_{j=k+1}^{2k-1} \tau_j) < (\sum_{j=2}^{k} d_j) - (\sum_{j=k-1}^{2k-1} d_j)$$

hold, again a contradiction. ∎

In what follows, a *maximal block B of P in C_n^k* is a maximal subset $B \subseteq P$ of consecutive vertices of C_n^k. In particular, if $\{v_{i_q}, ..., v_{i_{q+\alpha}}\}$ is a maximal block of P in C_n^k, then neither $v_{i_q}^-$ nor $v_{i_{q+\alpha}}^+$ are prescribed vertices.

Lemma 9 *Let $P = (v_{i_0}, ..., v_{i_{2k-1}})$ be a $2k$-prescription of $G = C_n^k$ with $k \geq 1$, $n \geq 2k$ and $0 \leq i_1 < ... < i_k \leq n-1$. If there exists exactly one maximal block B of P in G of size at least k, then every partition $\tau = (\tau_0, ..., \tau_{p-1})$ of n with $p \geq 2k$ elements is realizable in G under P.*

Proof. Let us suppose, without loss of generality, that $v_{i_0}, ..., v_{i_{k-1}}$ belong to B. We distinguish two subcases depending on whether $v_{i_0}, ..., v_{i_{k-1}}$ are saturated for τ in their garden or not.

Case 1: The prescribed vertices $v_{i_0}, ..., v_{i_{k-1}}$ are not saturated for τ in $G_{0,k-1}$.

Let X be a subset of $\sum_{j=0}^{k-1} \tau_j$ consecutive vertices of $G_{0,k-1}$ such that $X \cap P = \{v_{i_0}, ..., v_{i_{k-1}}\}$. According to Theorem 2, there exists a realization $(V_0, ..., V_{k-1})$ of $(\tau_0, ..., \tau_{k-1})$ in $G[X]$ under the prescription $(v_{i_0}, ..., v_{i_{k-1}})$ since this subgraph is k-connected. Now observe that a realization $(V_k, ..., V_{p-1})$ of $(\tau_k, ..., \tau_{p-1})$ in $G - X$ under the prescription $(v_{i_k}, ..., v_{i_{2k-1}})$ necessarily exists since $G - X$ is isomorphic to the k^{th} power of a path and we supposed that the prescribed vertices perform only one maximal block of size at least k in G. This implies that either v_{i_k} is the first vertex of $G - X$, $v_{i_{2k-1}}$ is the last vertex of $G - X$, or the prescribed vertices $v_{i_k}, ..., v_{i_{2k-1}}$ are not consecutive in $G - X$. In the first two cases, the realization is obtained thanks to Lemma 5, and it follows from Lemma 7 in the third case. Finally, observe that $(V_0, ..., V_{p-1})$ is a correct realization of τ in G under P.

Case 2: The prescribed vertices $v_{i_0}, ..., v_{i_{k-1}}$ are saturated for τ in $G_{0,k-1}$.

By Lemma 8, we know that neither $v_{i_1},...,v_{i_k}$ nor $v_{i_{k+1}},...,v_{i_0}$ are saturated for τ in $G_{1,k}$ and $G_{k+1,0}$, respectively. Hence, only two cases may occur.

Case 2.a. If there exists a subset X' of $\sum_{j=1}^{k}\tau_j$ consecutive vertices of $G_{1,k}$ such that $P\cap X'=\{v_{i_1},...,v_{i_k}\}$, then, since v_{i_0} is the last vertex of $G-X'$, all the conditions are met to apply the procedure used in *Case 1* with $v_{i_1},...,v_{i_k}$ and X' to get the desired realization. A similar realization can be obtained analogously if the assumption above holds for $v_{i_{k+1}},...,v_{i_0}$ and their garden $G_{k+1,0}$.

Case 2.b. Suppose now that both $\sum_{j=1}^{k}\tau_j<k+\sum_{j=2}^{k}d_j$ and $\tau_0+\sum_{j=k+1}^{2k-1}\tau_j<k+d_0+\sum_{j=k+2}^{2k-1}d_j$ hold. Let $X=\{v_{i_1},...,v_{i_k}\}$. If there exists a $x\in[2k,p-1]$ such that $(\sum_{j=1}^{k}\tau_j)+(\sum_{j=2k}^{x}\tau_j)=|X|$ then we can partition G into two graphs $G[X]$ and $G-X$ and obtain a whole realization of τ in G under P by considering realizations of $(\tau_1,...,\tau_k,\tau_{2k},...,\tau_x)$ in $G[X]$ under $(v_{i_1},...,v_{i_k})$ and of $(\tau_0,\tau_{k+1},...,\tau_{2k-1},\tau_{x+1},...,\tau_{p-1})$ in $G-X$ under $(v_{i_0},v_{i_{k+1}},...,v_{i_{2k-1}})$ which necessarily exist since the conditions of Lemma 5 are satisfied in both cases (v_{i_0} is the last vertex of $G-X$ and v_{i_1} is the first vertex of $G[X]$). Finally, $(V_0,...,V_{p-1})$ is a realization of τ in G under P.

If such a x does not exist, then let $x\in[2k,p-1]$ be such that $(\sum_{j=1}^{k}\tau_j)+(\sum_{j=2k}^{x-1}\tau_j)<|X|$ and $(\sum_{j=1}^{k}\tau_j)+(\sum_{j=2k}^{x}\tau_j)>|X|$, and let $\tau'_x=|X|-[(\sum_{j=1}^{k}\tau_j)+(\sum_{j=2k}^{x-1}\tau_j)]$ and $\tau''_x=\tau_x-\tau'_x$. Since we supposed that there is only one maximal block of P in G with size at least k and v_{i_0} and v_{i_1} are adjacent according to the natural ordering of $V(G)$, there exist $\alpha\in[i_1,i_k]$ and $\beta\in[i_k+1,i_k+k-1]$ such that v_α and v_β are adjacent and are not prescribed vertices (otherwise there would exist a second maximal block of P in G with size at least k since $v_{i_1},...,v_{i_k}$ are not saturated for τ in their garden). In particular, let us choose v_β in such a way that the distance between v_α and v_β is the smallest possible according to the natural ordering of $V(G)$. We get that either v_β or $v_{i_{k+1}}$ is the first vertex of $G-X$.

Finally, we obtain the desired realization of τ in G under P as follows. First, let $(V_1,...,V_k,V'_x,V_{2k},...,V_{x-1})$ be a realization of $(\tau_1,...,\tau_k,\tau'_x,\tau_{2k},...,\tau_{x-1})$ in $G[X]$ under $(v_{i_1},...,v_{i_k},v_\alpha)$. Then, let $(V_0,V_{k+1},...,V_{2k-1},V''_x,V_{x+1},...,V_{p-1})$ be a realization of $(\tau_0,\tau_{k+1},...,\tau_{2k-1},\tau''_x,\tau_{x+1},...,\tau_{p-1})$ in $G-X$ under $(v_{i_0},v_{i_{k+1}},...,v_{i_{2k-1}},v_\beta)$. These two realizations necessarily exist according to Lemma 6. Moreover, the subgraph $G[V'_x\cup V''_x]$ is connected since $v_\alpha v_\beta\in E(G)$; therefore, we eventually get that $(V_0,...,V_{x-1},V'_x\cup V''_x,V_{x+1},...,V_{p-1})$ is a correct realization of τ in G under P. ■

Lemma 10 *Let $P=(v_{i_0},...,v_{i_{2k-1}})$ be a $2k$-prescription of $G=C_n^k$ with $k\geq 1$, $n\geq 2k$ and $0\leq i_1<...<i_k\leq n-1$. If there does not exist a maximal block of P in G of size at least k, then every partition $\tau=(\tau_0,...,\tau_{p-1})$ of n with $p\geq 2k$ elements is realizable in G under P.*

Proof. We can suppose, without loss of generality, that neither $v_{i_0},...,v_{i_{k-1}}$ nor $v_{i_k},...,v_{i_{2k-1}}$ are saturated for τ in $G_{0,k-1}$ and $G_{k,2k-1}$, respectively. Indeed, if one of these conditions holds, then this cannot be simultaneously the case for $v_{i_1},...,v_{i_k}$ nor $v_{i_{k+1}},...,v_{i_{2k-1}},v_{i_0}$ in their respective garden according to Lemma 8; we could thus easily relabel our prescribed vertices and part sizes so that the above assertion holds.

We distinguish the following two subcases.

Case 1. If we can find a subset X of $\sum_{j=0}^{k-1}\tau_j$ consecutive vertices of $G_{0,k-1}$ such that $P\cap X=\{v_{i_0},...,v_{i_{k-1}}\}$, then we can obtain a correct realization of τ in G under P in a same way as what we did in *Case 2.a* of Lemma 9. A similar argument holds if there exists a satisfying subset X included in the garden of $v_{i_k},...,v_{i_{2k-1}}$.

Case 2. The last case which may happen is the one where both $\sum_{j=0}^{k-1}\tau_j<k+\sum_{j=1}^{k-1}d_j$ and $\sum_{j=k}^{2k-1}\tau_j<k+\sum_{j=k+1}^{2k-1}d_j$ hold, and neither v_{i_0} and $v_{i_{2k-1}}$ nor $v_{i_{k-1}}$ and v_{i_k} are consecutive vertices of G (otherwise, we could use the arguments of *Case 2.b* of Lemma 9 to deduce a realization of τ in G under P). We can suppose that there does not exist $x\in[2k,p-1]$ such that $\sum_{j=0}^{k-1}\tau_j+\sum_{j=2k}^{x}\tau_j=k+\sum_{j=1}^{k-1}d_j$ or $\sum_{j=k}^{2k-1}\tau_j+\sum_{j=2k}^{x}\tau_j=k+\sum_{j=k+1}^{2k-1}d_j$ for otherwise

we could find a realization of τ in G under P using Lemma 7 since there is no maximal block of P in G with size at least k. We explain below how to deduce a realization of τ in G under P.

If $\tau_0 + \tau_{2k-1} \geq d_0 + 2$, then we can obtain a correct realization of τ in G under P as follows. Let $V'_0 = \{v_{i_0-1}, v_{i_0-2}, ..., v_\alpha\}$ and $V'_{2k-1} = \{v_{i_{2k-1}+1}, v_{i_{2k-1}+2}, ..., v_\alpha^-\}$ where $|V'_0| < \tau_0$, $|V'_{2k-1}| < \tau_{2k-1}$, and $\alpha \in [i_{2k-1}+1, i_0-1]$. Then, consider a realization $(V''_0, V_1, ..., V_{2k-2}, V''_{2k-1}, V_{2k}, ..., V_{p-1})$ of the sequence $(\tau_0 - |V'_0|, \tau_1, ..., \tau_{2k-2}, \tau_{2k-1} - |V'_{2k-1}|, \tau_{2k}, ..., \tau_{p-1})$ in the subgraph $G[\{v_{i_0}, ..., v_{i_{2k-1}}\}]$ under P obtained in the same way as what we did for *Case 2.b* (that is by doing as if v_{i_0} and $v_{i_{2k-1}}$ were consecutive vertices of G). Such a realization necessarily exists since we supposed that there is no maximal block of P in G with size at least k. Finally, observe that the realization $(V'_0 \cup V''_0, V_1, ..., V_{2k-2}, V'_{2k-1} \cup V''_{2k-1}, V_{2k}, ..., V_{p-1})$ is a correct realization of τ in G under P.

Now, if $\tau_0 + \tau_{2k-1} < d_0 + 2$, then $\tau_0 - 1 \geq \lfloor \frac{d_0(k-1)}{k} \rfloor$ and $\tau_{2k-1} - 1 \geq \lfloor \frac{d_0(k-1)}{k} \rfloor$ cannot hold simultaneously since otherwise we would get a contradiction. Let $Z = \{v^+_{i_{2k-1}}, ..., v^-_{i_0}\}$ and suppose that $\tau_0 - 1 < \lfloor \frac{d_0(k-1)}{k} \rfloor$; thus, there exists a subset V_0 of $Z \cup \{v_{i_0}\}$ such that $G[V_0]$ is connected on τ_0 vertices, V_0 contains the vertex v_{i_0} but does not contain neither $v^+_{i_{2k-1}}$ nor $v^-_{i_0}$, and the subgraph $G[Z - V_0]$ is traceable.

Once again, we can suppose that there does not exist $x \in [2k, p-1]$ such that $\sum_{j=1}^{k-1} \tau_j + \sum_{j=2k}^{x} \tau_j = |G_{1,k-1}|$ since otherwise we could easily deduce a realization of τ in G under P using Lemmas 5 and 6 (recall that $v_{i_k}, ..., v_{i_{2k-1}}$ are not saturated for τ in their garden). Thus, let $x \in [2k, p-1]$ be such that $\sum_{j=1}^{k-1} \tau_j + \sum_{j=2k}^{x-1} \tau_j < |G_{1,k-1}|$ and $\sum_{j=1}^{k-1} \tau_j + \sum_{j=2k}^{x} \tau_j > |G_{1,k-1}|$, and let $\tau'_x = |G_{1,k-1}| - \sum_{j=1}^{k-1} \tau_j - \sum_{j=2k}^{x-1} \tau_j$ and $\tau''_x = \tau_x - \tau'_x$. For the sake of the proof, let us also denote τ_p instead of τ''_x. Since $G_{1,k-1}$ only includes $k-1$ prescribed vertices, then we can obtain a realization $(V_1, ..., V_{k-1}, V'_x, V_{2k}, ..., V_{x-1})$ of $(\tau_1, ..., \tau_{k-1}, \tau'_x, \tau_{2k}, ..., \tau_{x-1})$ in $G[G_{1,k-1}]$ under $(v_{i_1}, ..., v_{i_{k-1}}, v_\alpha)$ where v_α is a non-prescribed vertex of $G_{1,k-1}$ neighbouring $v^-_{i_0}$. Such a vertex necessarily exists since there are no maximal block of P of size at least k in G; in particular, choose $v_\alpha = v^+_{i_0}$ if $v_{i_1} \neq v^+_{i_0}$. The previous realization exists by Lemma 5.

For the same reasons as above, let us suppose that there does not exist $y \in [x+1, p]$ such that $\sum_{j=k}^{2k-1} \tau_j + \sum_{j=x+1}^{y} \tau_j = k + \sum_{j=k+1}^{2k-1} d_j$. Then denote by $y \in [x+1, p]$ the least integer such that $\sum_{j=k}^{2k-1} \tau_j + \sum_{j=x+1}^{y-1} \tau_j < k + \sum_{j=k+1}^{2k-1} d_j$ and $\sum_{j=k}^{2k-1} \tau_j + \sum_{j=x+1}^{y} \tau_j > k + \sum_{j=k+1}^{2k-1} d_j$. Next, let us split τ_y into two elements τ'_y and τ''_y as above, and use the arguments above once again to obtain a realization $(V_k, ..., V_{2k-1}, V'_y, V_{x+1}, ..., V_{y-1})$ of $(\tau_k, ..., \tau_{2k-1}, \tau'_y, \tau_{x+1}, ..., \tau_{y-1})$ in $G[\{v_{i_k}, ..., v_{i_{2k-1}}\}]$ under $(v_{i_k}, ..., v_{i_{2k-1}}, v_\beta)$ where v_β is a non-prescribed vertex of $\{v_{i_k}, ..., v_{i_{2k-1}}\}$ neighbouring $v^+_{i_{2k-1}}$. This realization exists by Lemma 6 since v_{i_k} and $v_{i_{2k-1}}$ are the first and last vertices of $G[\{v_{i_k}, ..., v_{i_{2k-1}}\}]$, respectively.

On the one hand, if $y = p$, then it means that the element τ_x was split into three elements τ'_x, τ'_y and τ''_y. In this case, the realization $(V_0, ..., V_{x-1}, V'_x \cup V'_y \cup Z, V_{y+1}, ..., V_{p-1})$ is a correct realization of τ in G under P. On the other hand, if $y < p$, then, since $G[Z - V_0]$ is spanned by a path whose first and last vertices are $v^+_{i_{2k-1}}$ and $v^-_{i_0}$, respectively, by Lemma 6 there exists a realization $(V''_x, V''_y, V_{y+1}, ..., V_{p-1})$ of $(\tau''_x, \tau''_y, \tau_{y+1}, ..., \tau_{p-1})$ in $G[Z - V_0]$ under $(v^-_{i_0}, v^+_{i_{2k-1}})$. By construction, $G[V'_x \cup V''_x]$ and $G[V'_y \cup V''_y]$ both induce a connected subgraph of G. It follows that $(V_0, ..., V_{x-1}, V'_x \cup V''_x, V_{x+1}, ..., V_{y-1}, V'_y \cup V''_y, V_{y+1}, ..., V_{p-1})$ is a realization of τ in G under P. ∎

5 Partitioning Harary graphs under prescriptions

Harary graphs are known to be graphs having the smallest possible size regarding their connectivity [5]. They trivially have the property of being AP since they are Hamiltonian; thus, one could wonder about the maximum number of prescribed vertices we can allow while partitioning them. In particular, can we always prescribe k vertices when partitioning a $(k+1)$-connected Harary graph? If this is the case, then such a graph will also be an optimal AP+k-graph.

We investigate, in this section, the question for the three different constructions of Harary graphs, according to the parities of both n and k.

5.1 Construction 1: k is even

For every $k \geq 2$ and $n \geq 2k$ such that k is even, $H_{k,n}$ is isomorphic to $C_n^{\frac{k}{2}}$ which is AP+$(k-1)$ according to Theorem 4. Thus, the following result derives directly.

Corollary 11 *For every even $k \geq 2$ and $n \geq 2k$, the Harary graph $H_{k,n}$ is AP+$(k-1)$.*

5.2 Construction 2: k is odd and n is even

Let $k \geq 2$ and $n \geq 2$ be two integers such that k is odd and n is even. By construction, $H_{2k+1,n}$ is spanned by $H_{2k,n}$ and thus is AP+$(2k-1)$ according to Corollary 11. Although this number of prescribed vertices is good since $H_{2k+1,n}$ is $(2k+1)$-connected, the connectivity of this graph suggests that we could up to one more prescribed vertex while partitioning it (see Observation 3).

Before proving that $H_{2k+1,n}$ is actually AP+$2k$, we first introduce the following lemma which deals with the traceability of a graph composed by two linked squares of paths.

Lemma 12 *If G is a graph such that $V(G) = V_1 \cup V_2$, the subgraphs $G[V_1]$ and $G[V_2]$ are both spanned by the square of a path, and there exists an edge joining one vertex of V_1 and one of V_2, then G is traceable.*

Proof. Let $v_1, ..., v_{n_1}$ and $u_1, ..., u_{n_2}$ denote the vertices of $G[V_1]$ and $G[V_2]$, from left to right, and v_α and u_β be two vertices of G such that $v_\alpha u_\beta \in E(G)$. Consider the following subpaths of G:

- $P = v_1 v_2 ... v_{\alpha-1}$;
- $Q = \begin{cases} v_{\alpha+1} v_{\alpha+3} ... v_{n_1-1} v_{n_1} v_{n_1-2} v_{n_1-4} ... v_{\alpha+2} & \text{if } n_1 - \alpha \text{ is even,} \\ v_{\alpha+1} v_{\alpha+3} ... v_{n_1} v_{n_1-1} v_{n_1-3} ... v_{\alpha+2} & \text{otherwise;} \end{cases}$
- $R = \begin{cases} u_{\beta+2} u_{\beta+4} ... u_{n_2} u_{n_2-1} u_{n_2-3} ... u_{\beta+1} & \text{if } n_2 - \beta \text{ is even,} \\ u_{\beta+2} u_{\beta+4} ... u_{n_2-1} u_{n_2} u_{n_2-2} u_{n_2-4} ... u_{\beta+1} & \text{otherwise;} \end{cases}$
- $S = u_{\beta-1} u_{\beta-2} ... u_1$.

It is then easy to check that $PQv_\alpha u_\beta RS$ is a Hamiltonian path of G. ∎

We now prove the main result of this section.

Theorem 13 *For every odd $k \geq 2$ and even $n \geq 2k+1$, the Harary graph $H_{2k+1,n}$ is AP+$2k$.*

Proof. Let $G = H_{2k+1,n}$ be the $(2k+1)$-connected Harary graph on n vertices. We prove that every partition $\tau = (\tau_0, ..., \tau_{p-1})$ of n with $p \geq 2k+1$ elements is realizable in G under every prescription $P = (v_{i_0}, ..., v_{i_{2k-1}})$ with $0 \leq i_0 < ... < i_{2k-1} \leq n-1$. We distinguish three cases depending on whether there exists a realization of τ in C_n^k under P or not. Since C_n^k is a spanning subgraph of G, such a realization would also be a realization of τ in G under P.

Case 1. First observe that if there exists exactly one maximal block of P in G with size at least k, then we can obtain a correct realization of τ in G under P by considering a realization of τ in C_n^k under P. Such a realization necessarily exists according to Lemma 9.

Case 2. Similarly, if there does not exist a maximal block of P in G with size at least k, then we can use Lemma 10 to deduce a realization of τ in C_n^k under P. This realization is also a realization of τ in G under P.

Case 3. The last case we have to consider is the one where there exist two maximal blocks B_1 and B_2 of P in G with size exactly k. In this case, the prescribed vertices disconnect the graph into two components linked by some diagonal edges. Indeed, without loss of generality, let us suppose that $B_1 = \{v_{i_0}, ..., v_{i_{k-1}}\}$ and $B_2 = \{v_{i_k}, ..., v_{i_{2k-1}}\}$; necessarily, the antipodal neighbours of $v_{i_0}^-$ and $v_{i_{k-1}}^+$ do not both belong to P since otherwise there would exist a maximal block of P in G with size at least $k+2$. Let us denote by v_α and v_β two antipodal neighbours of G such that $v_\alpha, v_\beta \notin B_1 \cup B_2$, $\alpha \in [i_{k-1}+1, i_k-1]$ and $\beta \in [i_{2k-1}+1, i_0-1]$. Besides, let $a_1 = \alpha - i_{k-1} - 1$, $a_2 = i_k - \alpha - 1$, $a_3 = i_0 - \beta - 1$ and $a_4 = \beta - i_{2k-1} - 1$ denote the number of consecutive vertices between B_1, B_2 and the two vertices v_α and v_β according to the natural ordering of $V(G)$.

If $\sum_{j=0}^{k-1} \tau_j \leq a1 + a3 + k$ and $\sum_{j=k}^{2k-1} \tau_j \leq a_2 + a_4 + k$, then we can find two subsets X and Y of consecutive vertices of G such that $|X| = \sum_{j=0}^{k-1} \tau_j$, $|Y| = \sum_{j=k}^{2k-1} \tau_j$, $\{v_{i_0}, ..., v_{i_{k-1}}\} \subseteq X$, $\{v_{i_k}, ..., v_{i_{2k-1}}\} \subseteq Y$, and $v_\alpha, v_\beta \notin X \cup Y$. Since $G[X]$ and $G[Y]$ are both isomorphic to the k^{th} power of a path, then using Theorem 2 we can deduce two realizations $(V_0, ..., V_{k-1})$ and $(V_k, ..., V_{2k-1})$ of $(\tau_0, ..., \tau_{k-1})$ and $(\tau_k, ..., \tau_{2k-1})$ in $G[X]$ and $G[Y]$ under the prescriptions $(v_{i_0}, ..., v_{i_k-1})$ and $(v_{i_k}, ..., v_{i_{2k-1}})$. Moreover, since $k \geq 2$, $G - (X, Y)$ is traceable according to Lemma 12, there exists a realization $(V_{2k}, ..., V_{p-1})$ of $(\tau_{2k}, ..., \tau_{p-1})$ in $G - (X, Y)$. Finally, $(V_0, ..., V_{p-1})$ is a correct realization of τ in G under P.

Let us now suppose that, without loss of generality, $\sum_{j=0}^{k-1} \tau_j > a_1 + a_3 + k$ and $\sum_{j=k}^{2k-1} \tau_j < a_2 + a_4 + k$. If $\sum_{j=0}^{2k-1} \tau_j \geq a_1 + a_2 + 2k + 1$, then we can find two subsets of consecutive vertices $X, Y \subseteq V(G)$ such that, for some $\gamma \in [i_{k-1}+1, i_k-1]$, $v_\gamma \in X$, $v_\gamma^+ \in Y$, $\{v_{i_0}, ..., v_{i_{k-1}}\} \subseteq X$, $\{v_{i_k}, ..., v_{i_{2k-1}}\} \subseteq Y$, $|X| = \sum_{j=0}^{k-1} \tau_j$ and $|Y| = \sum_{j=k}^{2k-1} \tau_j$. By Theorem 2, we know that there exists a realization $(V_0, ..., V_{k-1})$ of $(\tau_0, ..., \tau_{k-1})$ in $G[X]$ under the prescription $(v_{i_0}, ..., v_{i_{k-1}})$, and a realization $(V_k, ..., V_{2k-1})$ of $(\tau_k, ..., \tau_{2k-1})$ in $G[Y]$ under $(v_{i_k}, ..., v_{i_{2k-1}})$. Finally, $G-(X, Y)$ is isomorphic to the k^{th} power of a path, and thus there exists a realization $(V_{2k}, ..., V_p)$ of the remaining sequence $(\tau_{2k}, ..., \tau_{p-1})$ in it. We get that $(V_0, ..., V_{p-1})$ is a realization of τ in G under P.

If it is not possible to choose the subsets X and Y in such a way that they have two neighbouring consecutive vertices along the arc of size $a_1 + a_2 + 1$ of G, then we can necessarily find two such subsets along the arc of size $a_3 + a_4 + 1$ of G. Indeed, in such a situation we have $\sum_{j=0}^{k-1} \tau_j > a_1 + a_3 + k$ by hypothesis, but $\sum_{j=0}^{k-1} \tau_j < a_1 + a_2 + k + 1$. This implies that $a_2 \geq a_3$, and, since $a_1 + a_3 = a_2 + a_4$, that $a_1 \geq a_4$. Hence, we get $\sum_{j=0}^{k-1} \tau_j > a_4 + a3 + k$ and it is therefore possible to choose the two subsets X and Y along the arc of size $a_3 + a_4 + 1$ of G to deduce a realization of τ in G under P. ∎

5.3 Construction 3: k and n are odd

The Harary graph $G = H_{2k+1,n+1}$, where $k \geq 2$ and $n+1 \geq 2k+1$ are odd, is spanned by C_n^k. Thus, it follows, according to Lemmas 9 and 10, that there always exists a realization of τ in G under a $2k$-prescription which admits at most one maximal block in G with size at least k. Moreover, $H_{2k+1,n}$ and $H_{2k+1,n+1}$ differ by their diagonal edges, but the arguments we used to prove *Case 3* of Theorem 13 also hold when considering $H_{2k+1,n+1}$ instead of $H_{2k+1,n}$.

Thus, we can directly derive the following result.

Theorem 14 *For every odd $k \geq 2$ and odd $n \geq 2k+1$, the Harary graph $H_{2k+1,n}$ is AP+$2k$.*

6 On the existence of optimal AP+2-graphs

The proof of Theorems 13 and 14 relies on the fact that some particular subgraphs of a Harary graph $H_{k,n}$ necessarily satisfy the conditions of Lemma 12 when $k \neq 3$. Although, one can easily check that this lemma cannot be used when $H_{k,n}$ is not connected enough, in particular when

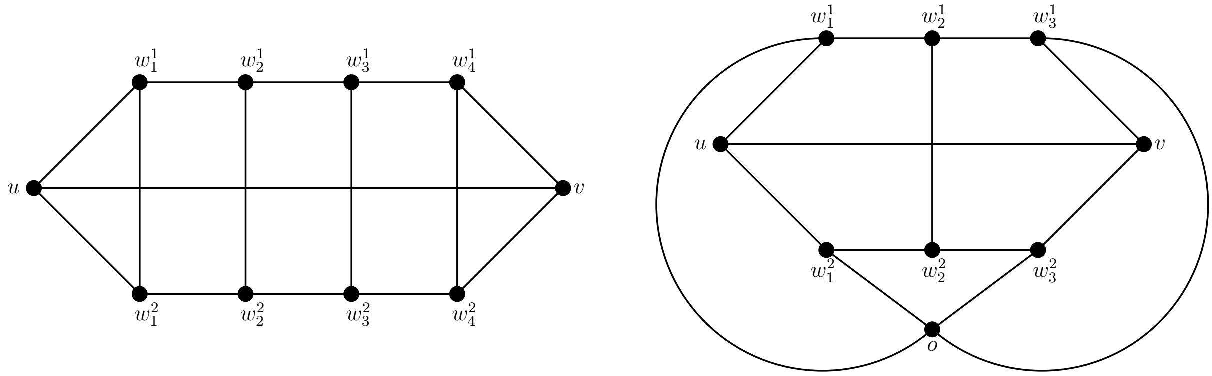

Figure 2: The graphs Pr_{10} and Pr_9

$k = 3$. Hence, our proof cannot be used directly to prove that a Harary graph $H_{3,n}$ is AP+2 for some n.

Besides, these graphs are not all AP+2 anyway:

Observation 15 *For every $n \equiv 2 \mod 4$, the Harary graph $H_{3,n}$ is not AP+2.*

Indeed, let $n \equiv 2 \mod 4$ and $G = H_{3,n}$. One can check that the subgraph $G - \{v_0, v_2\}$ does not admit a perfect matching. Therefore, the sequence $(1, 1, 2, ..., 2)$ is not realizable in G under (v_0, v_2).

In order to prove that there actually exist optimal AP+2-graphs, we introduce a new class of 3-connected graphs.

Definition 16 Let $n \geq 4$. The graph Pr_n is constructed as follows:

- If n is even, Pr_n is obtained from the cycle C_n, whose vertices are successively denoted by $u, w_1^1, ..., w_{\frac{n-2}{2}}^1, v, w_{\frac{n-2}{2}}^2, ..., w_1^2$, by adding it the edge uv and all edges $w_i^1 w_i^2$, for every $i \in [1, \frac{n-2}{2}]$.
- If n is odd, Pr_n is obtained by first removing the edges $w_1^1 w_1^2$ and $w_{\frac{n-3}{2}}^1 w_{\frac{n-3}{2}}^2$ from Pr_{n-1}, and then adding it a new vertex o linked to w_1^1, w_1^2, $w_{\frac{n-3}{2}}^1$, and $w_{\frac{n-3}{2}}^2$.

The two graphs Pr_9 and Pr_{10} are drawn in Figure 2.

For every $n \geq 4$, the graph Pr_n is 3-connected and has the least possible number of edges, that is $\lceil \frac{3n}{2} \rceil$. To now prove that such a graph is AP+2, we consider the following sufficient condition for a graph to be AP+2 which is easier to determine for Pr graphs. Recall that a graph G is *Hamiltonian-connected* if, for every two distinct vertices u and v of G, there exists a Hamiltonian path in G with endvertices u and v.

Lemma 17 *If a graph G is Hamiltonian-connected, then G is AP+2.*

This lemma obviously holds since, by Lemma 6, we can partition P_n under every 2-prescription (u, v) as long as u and v are the endvertices of P_n.

Before showing that $G = Pr_n$ is Hamiltonian-connected for every $n \geq 4$, we first introduce some notations to deal with the vertices of such a graph. Let $q = \frac{n-2}{2}$ if n is even (resp. $q = \frac{n-3}{2}$ if n is odd); given two distinct integers x and y in $[1, q]$ (resp. in $[2, q-1]$) such that $x < y$, we denote by $P_{x,y}^{\nearrow}(G)$ and $P_{x,y}^{\searrow}(G)$ the paths of G defined as follows.

$$P_{x,y}^{\nearrow}(G) = \begin{cases} w_x^2 w_x^1 \text{ if } x = y, \\ w_x^2 w_x^1 P_{x+1,y}^{\searrow}(G) \text{ otherwise.} \end{cases}$$

$$P_{x,y}^{\searrow}(G) = \begin{cases} w_x^1 w_x^2 \text{ if } x = y, \\ w_x^1 w_x^2 P_{x+1,y}^{\nearrow}(G) \text{ otherwise.} \end{cases}$$

s	t	P
u	v	$uP^{\nearrow}_{1,q}(G)v$
u	w_i^1	$uP^{\nearrow}_{1,i-1}(G)w_i^2P^{2,\rightarrow}_{i+1,q}(G)vP^{1,\leftarrow}_{q,i}(G)$ if $i-1$ is even $uP^{\searrow}_{1,i-1}(G)w_i^2P^{2,\rightarrow}_{i+1,q}(G)vP^{1,\leftarrow}_{q,i}(G)$ otherwise
w_i^1	w_j^1	$P^{1,\rightarrow}_{i,j-1}(G)P^{2,\leftarrow}_{j-1,i}(G)P^{\nwarrow}_{i-1,1}(G)uvP^{\nwarrow}_{q,j}(G)$ if $q-j$ is even $P^{1,\rightarrow}_{i,j-1}(G)P^{2,\leftarrow}_{j-1,i}(G)P^{\nwarrow}_{i-1,1}(G)uvP^{\swarrow}_{q,j}(G)$ otherwise
w_i^1	w_j^2	$P^{1,\rightarrow}_{i,j-1}(G)P^{2,\leftarrow}_{j-1,i}(G)P^{\nwarrow}_{i-1,1}(G)uvP^{\swarrow}_{q,j}(G)$ if $q-j$ is even $P^{1,\rightarrow}_{i,j-1}(G)P^{2,\leftarrow}_{j-1,i}(G)P^{\nwarrow}_{i-1,1}(G)uvP^{\nwarrow}_{q,j}(G)$ otherwise

Table 1: Proof that Pr_n is Hamiltonian-connected for every $n \geq 4$ even

The paths $P^{\nwarrow}_{x,y}(G)$ and $P^{\swarrow}_{x,y}(G)$ of G are defined analogously from right to left when $x > y$. For every $\alpha \in \{1, 2\}$, we additionally define $P^{\alpha,\rightarrow}_{x,y}(G)$ for $x < y$ (resp. $P^{\alpha,\leftarrow}_{x,y}(G)$ for $x > y$) to be the path $w_x^\alpha w_{x+1}^\alpha ... w_y^\alpha$ of G (resp. the path $w_x^\alpha w_{x-1}^\alpha ... w_y^\alpha$ of G). For convenience, let us assume that $P^{\nearrow}_{x,y}(G) = \emptyset$, $P^{\searrow}_{x,y}(G) = \emptyset$ and $P^{\alpha,\rightarrow}_{x,y}(G) = \emptyset$ (resp. $P^{\nwarrow}_{x,y}(G) = \emptyset$, $P^{\swarrow}_{x,y}(G) = \emptyset$ and $P^{\alpha,\leftarrow}_{x,y}(G) = \emptyset$) whenever x or y do not belong to the interval above or when $x > y$ (resp. when $x < y$). Using these notations, we get, for instance, that $uP^{1,\rightarrow}_{1,4}(Pr_{10})vP^{2,\leftarrow}_{4,1}(Pr_{10})$ and $uP^{\nearrow}_{1,4}(Pr_{10})v$ are Hamiltonian paths of Pr_{10}.

We are now ready to prove that every Pr_n graph is Hamiltonian-connected, and thus AP+2 according to Lemma 17.

Proposition 18 *For every $n \geq 4$, the graph Pr_n is Hamiltonian-connected.*

Proof. Let $G = Pr_n$, and $q = \frac{n-2}{2}$ if n is even, $q = \frac{n-3}{2}$ otherwise. Table 1 (resp. Table 2) exhibits, given two distinct vertices s and t of G, a Hamiltonian path P of G whose endvertices are s and t when n is even (resp. n is odd). In Table 1 (resp. Table 2), it is assumed that $1 \leq i \leq q$ when j is not defined (resp. $1 < i < q$), and $0 \leq i < j \leq q$ otherwise (resp. $1 < i < j < q$). All the Hamiltonian paths which do not appear in these two tables can be deduced from other Hamiltonian paths using the symmetries of G. ∎

Corollary 19 *For every $n \geq 4$, the graph Pr_n is AP+2.*

7 Conclusions

We summarize Corollaries 11 and 19 and Theorems 13 and 14 in this concluding theorem.

Theorem 20 *For every $k \geq 1$ and $n \geq k$, there exist optimal AP+k-graphs on n vertices.*

According to Observation 3, we know that the number of optimal AP+k-graphs is upper bounded by the number of edge-minimal $(k + 1)$-connected graphs. However, it seems difficult to make an estimation on the number of optimal AP+k-graphs in general.

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s	t	P
o	u	$oP^{1,\rightarrow}_{1,q}(G)vP^{2,\leftarrow}_{q,1}(G)u$
o	w^1_1	$ow^1_qvw^2_qP^{\nwarrow}_{q-1,2}w^2_1uw^1_1$ if q is even $ow^2_qvw^1_qP^{\swarrow}_{q-1,2}w^2_1uw^1_1$ otherwise
o	w^1_i	$ow^1_1uw^2_1P^{\nearrow}_{2,i-1}(G)w^2_iP^{2,\rightarrow}_{i+1,q}(G)vP^{1,\leftarrow}_{q,i}(G)$ if i is even $ow^2_1uw^1_1P^{\searrow}_{2,i-1}(G)w^2_iP^{2,\rightarrow}_{i+1,q}(G)vP^{1,\leftarrow}_{q,i}(G)$ otherwise
u	v	$uP^{2,\rightarrow}_{1,q}(G)oP^{1,\rightarrow}_{1,q}(G)v$
u	w^1_1	$uvP^{2,\leftarrow}_{q,1}(G)oP^{1,\leftarrow}_{q,1}(G)$
u	w^1_q	$uvP^{2,\leftarrow}_{q,1}(G)oP^{1,\rightarrow}_{1,q}(G)$
u	w^1_i	$uP^{1,\rightarrow}_{1,i-1}(G)P^{2,\leftarrow}_{i-1,1}(G)ow^2_qvw^1_qP^{\swarrow}_{q-1,i}(G)$ if $q-i$ is even $uP^{1,\rightarrow}_{1,i-1}(G)P^{2,\leftarrow}_{i-1,1}(G)ow^1_qvw^2_qP^{\nwarrow}_{q-1,i}(G)$ otherwise
w^1_1	w^2_1	$w^1_1uvP^{1,\leftarrow}_{q,2}(G)P^{2,\rightarrow}_{2,q}(G)ow^2_1$
w^1_1	w^1_q	$P^{1,\rightarrow}_{1,q-1}(G)P^{1,\leftarrow}_{q-1,1}(G)uvw^2_qow^1_q$
w^1_1	w^2_q	$w^1_1ow^2_1uvP^{1,\leftarrow}_{q,2}(G)P^{2,\rightarrow}_{2,q}(G)$
w^1_i	w^1_j	$P^{1,\rightarrow}_{i,j-1}(G)P^{2,\leftarrow}_{j-1,i}(G)P^{\nwarrow}_{i-1,2}(G)w^2_1uw^1_1ow^2_qvw^1_qP^{\swarrow}_{q-1,j}(G)$ if i and $q-j$ are even $P^{1,\rightarrow}_{i,j-1}(G)P^{2,\leftarrow}_{j-1,i}(G)P^{\nwarrow}_{i-1,2}(G)w^1_1uw^2_1ow^2_qvw^1_qP^{\swarrow}_{q-1,j}(G)$ if i is odd and $q-j$ is even $P^{1,\rightarrow}_{i,j-1}(G)P^{2,\leftarrow}_{j-1,i}(G)P^{\nwarrow}_{i-1,2}(G)w^2_1uw^1_1ow^1_qvw^2_qP^{\nwarrow}_{q-1,j}(G)$ if i is even and $q-j$ is odd $P^{1,\rightarrow}_{i,j-1}(G)P^{2,\leftarrow}_{j-1,i}(G)P^{\nwarrow}_{i-1,2}(G)w^1_1uw^2_1ow^1_qvw^2_qP^{\nwarrow}_{q-1,j}(G)$ if i and $q-j$ are odd
w^1_i	w^2_j	$P^{1,\rightarrow}_{i,j-1}(G)P^{2,\leftarrow}_{j-1,i}(G)P^{\nwarrow}_{i-1,2}(G)w^2_1uw^1_1ow^1_qvw^2_qP^{\nwarrow}_{q-1,j}(G)$ if i and $q-j$ are even $P^{1,\rightarrow}_{i,j-1}(G)P^{2,\leftarrow}_{j-1,i}(G)P^{\nwarrow}_{i-1,2}(G)w^1_1uw^2_1ow^1_qvw^2_qP^{\nwarrow}_{q-1,j}(G)$ if i is odd and $q-j$ is even $P^{1,\rightarrow}_{i,j-1}(G)P^{2,\leftarrow}_{j-1,i}(G)P^{\nwarrow}_{i-1,2}(G)w^2_1uw^1_1ow^2_qvw^1_qP^{\swarrow}_{q-1,j}(G)$ if i is even and $q-j$ is odd $P^{1,\rightarrow}_{i,j-1}(G)P^{2,\leftarrow}_{j-1,i}(G)P^{\nwarrow}_{i-1,2}(G)w^1_1uw^2_1ow^2_qvw^1_qP^{\swarrow}_{q-1,j}(G)$ if i and $n-j$ are odd

Table 2: Proof that Pr_n is Hamiltonian-connected for every $n \geq 5$ odd

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