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Partitioning Harary graphs into connected subgraphs containing prescribed vertices

Olivier Baudon Julien Bensmail Éric Sopena*

Univ. Bordeaux, LaBRI, UMR 5800, F-33400 Talence, France
CNRS, LaBRI, UMR 5800, F-33400 Talence, France

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A graph G is *arbitrarily partitionable* (AP for short) if for every partition $(n_1, n_2, \dots, n_p)$ of $|V(G)|$ there exists a partition $(V_1, V_2, \dots, V_p)$ of $V(G)$ such that each V_i induces a connected subgraph of G with order n_i. If, additionally, k of these subgraphs ($k \leq p$) each contains an arbitrary vertex of G prescribed beforehand, then G is *arbitrarily partitionable under k prescriptions* (AP+k for short). Every AP+k graph on n vertices is $(k+1)$-connected, and thus has at least $\lceil \frac{n(k+1)}{2} \rceil$ edges. We show that there exist AP+k graphs on n vertices and $\lceil \frac{n(k+1)}{2} \rceil$ edges for every $k \geq 1$ and $n \geq k$.

Keywords: arbitrarily partitionable graph, partition under prescriptions, Harary graph

1 Introduction

We denote by $V(G)$ and $E(G)$ the sets of vertices and edges, respectively, of a graph G. The *order* (resp. *size*) of G is the cardinality of the set $V(G)$ (resp. $E(G)$). If X is a subset of $V(G)$, then $G[X]$ denotes the subgraph of G induced by X.

In the late 1970s, the following well-known result was proved.

Theorem 1 (Győri [5] and Lovász [7], independently)**.** *If G is a k-connected graph, then, given a sequence $(v_1, v_2, \dots, v_k)$ of k distinct vertices of G and a sequence $(n_1, n_2, \dots, n_k)$ of k positive integers adding up to $|V(G)|$, there exists a partition $(V_1, V_2, \dots, V_k)$ of $V(G)$ such that $v_i \in V_i$, the subgraph $G[V_i]$ is connected, and $|V_i| = n_i$ for every $i \in \{1, 2, \dots, k\}$.*

In this paper, we consider a more general partition problem resulting from the combination of the notion of *arbitrarily partitionable* graphs [1] with the constraint of prescribing a set of vertices from Theorem 1. Let G be a connected graph of order n. A sequence $\tau = (n_1, n_2, \dots, n_p)$ of positive integers is *admissible* for G if it performs a partition of n, that is if $\sum_{i=1}^{p} n_i = n$. If, additionally, we can partition $V(G)$ into p parts $(V_1, V_2, \dots, V_p)$ such that each V_i induces a connected subgraph of G with order n_i, then τ is *realizable* in G, the partition $(V_1, V_2, \dots, V_p)$ being a *realization* of τ in G. If every admissible sequence

*Corresponding author, sopena@labri.fr

for G is also realizable in G, then G is *arbitrarily partitionable* (AP for short). The interested reader is referred to [1, 2, 6, 8] for a review of some results on AP graphs.

Now suppose that we still want to partition G into an arbitrary number, say p, of connected subgraphs $G_1, G_2, \ldots, G_p$ of prescribed orders, but in such a way that for each $i \in \{1, 2, \ldots, k\}$ with fixed $k \in \{1, 2, \ldots, p\}$, the subgraph G_i contains a vertex v_i of G arbitrarily chosen beforehand. To model this additional requirement, the definition of AP graphs can be strenghtened as follows [3]. A *k-prescription* of G is a k-tuple $P = (v_1, v_2, \ldots, v_k)$ of k distinct vertices of G. We say that a sequence $\tau = (n_1, n_2, \ldots, n_p)$ with $p \geq k$ elements is *realizable* in G under P if there exists a realization $(V_1, V_2, \ldots, V_p)$ of τ in G such that the vertex v_i belongs to V_i for every $i \in \{1, 2, \ldots, k\}$. Notice that we have adopted the convention that the elements of τ associated with the prescribed vertices are the first elements of τ. We say that G is *(p, k)-partitionable* if every sequence admissible for G consisting of exactly p elements is realizable in G under every k-prescription. Finally, the graph G is *arbitrarily partitionable under k prescriptions* (AP+k for short) if G is (p, k)-partitionable for every $p \in \{k, k+1, \ldots, n\}$.

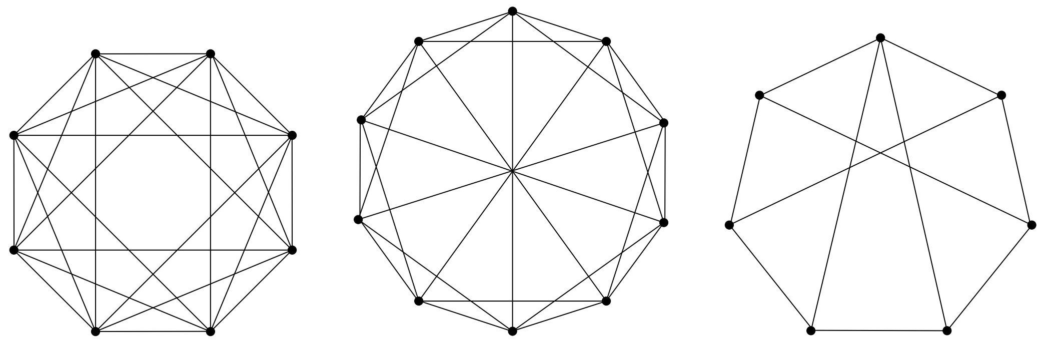
According to these definitions, an AP+0 graph is an AP graph. Stated differently, Theorem 1 asserts that every k-connected graph is (k, k)-partitionable. In the same flavour, note that every k-connected graph with $k \geq 2$ is trivially $(k, k-1)$-partitionable. Hence, when dealing with a k-connected graph, we only consider sequences with strictly more than k elements throughout this paper. It also has to be known that deciding whether a sequence is realizable in a graph under a prescription is NP-complete in general, even when the sequence or the prescription has a fixed number of elements [4].

Only a few classes of AP+k graphs are known. For every $k \geq 1$, the set of complete graphs on at least k vertices is a trivial class of AP+k graphs, these graphs having the largest possible size. Regarding graphs with less edges, it was proved in [3] that k^{th} powers of paths (resp. cycles) are AP+$(k-1)$ (resp. AP+$(2k-1)$) for every $k \geq 1$, these results being tight (*i.e.* we cannot always partition these graphs when more prescriptions are requested).

In this work, we investigate the least possible size of an AP+k graph. In this scope, we focus on *optimal* AP+k graphs, *i.e.* on AP+k graphs with the least possible number of edges regarding their order and connectivity. This is done by studying the family of well-known *Harary graphs*. After having introduced some notation and preliminary results in Section 2, we prove some more results regarding the partition of powers of paths or cycles in Section 3. These results are then used to show, in Section 4, that every $(k+1)$-connected Harary graph is an optimal AP+k graph for every $k \neq 2$. We finally deal with 3-connected Harary graphs in Section 5. In particular, we show that these graphs are not necessarily AP+2. We however provide another class of optimal AP+2 graphs instead. All these results imply that, for every $k \geq 1$ and $n \geq k$, every optimal AP+k graph with order n has size $\lceil \frac{n(k+1)}{2} \rceil$.

2 Definitions, notation, and preliminary results

A subgraph H of a graph G is a *spanning subgraph* of G if $V(H) = V(G)$. We also say that G is *spanned by* H. Given an integer $k \geq 1$, the k^{th} *power* of G, denoted by G^k, is the graph with the same vertex set as G, two vertices of G^k being adjacent if they are at distance at most k in G. We denote by P_n (resp. C_n) the path (resp. cycle) on n vertices. The vertices of P_n or C_n are consecutively denoted by $v_0, v_1, \ldots, v_{n-1}$. Regarding P_n, the vertices v_0 and v_{n-1} are its *first* and *last* vertices, respectively, sometimes called its *endvertices*. We use the same terminology to deal with the vertices of P_n^k (resp. C_n^k) according to its natural spanning P_n (resp. C_n).



(a) $H_{6,8}$. (b) $H_{5,10}$. (c) $H_{3,7}$.

Fig. 1: Three examples of Harary graphs.

Let $k \geq 1$ and $n \geq k$ be two integers. The k-connected Harary graph on n vertices, denoted by $H_{k,n}$, has a vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ and the following edges:

- if $k = 2r$ is even, then two vertices v_i and v_j are linked if and only if $i - r \leq j \leq i + r$;
- if $k = 2r + 1$ is odd and n is even, then $H_{k,n}$ is obtained by joining v_i and $v_{i+\frac{n}{2}}$ in $H_{2r,n}$ for every $i \in \{0, 1, \ldots, \frac{n}{2} - 1\}$;
- if $k = 2r + 1$ and n are odd, then $H_{k,n}$ is obtained from $H_{2r,n}$ by first linking v_0 to both $v_{\lfloor \frac{n}{2} \rfloor}$ and $v_{\lceil \frac{n}{2} \rceil}$, and then each vertex v_i to $v_{i+\lceil \frac{n}{2} \rceil}$ for every $i \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$;

where the subscripts are taken modulo n. Three examples of Harary graphs are given in Figure 1. When k is odd, the neighbours of a vertex v_i of $H_{k,n}$ which are at distance strictly more than r from v_i in the underlying C_n (there are at most two of them) are called the *antipodal neighbours* of v_i. In particular, the vertex v_i has two antipodal neighbours if and only if $i = 0$, and k and n are both odd. A *diagonal edge* of $H_{k,n}$ is an edge linking two vertices each of which is an antipodal neighbour of the other one.

If G is a graph with a natural ordering of its vertices (like powers of paths and cycles, or Harary graphs), then, for every vertex v of G, we denote by v^+ (resp. v^-) the neighbour of v succeeding (resp. preceding) v in this ordering. Every power of path P_n^k with underlying path $P_n = v_0 v_1 \ldots v_{n-1}$ is considered to be depicted in a "usual" way, *i.e.* from its leftmost vertex v_0 to its rightmost vertex v_{n-1}. By uGv we refer to the graph $G[\{u, u^+, (u^+)^+, \ldots, v^-, v\}]$ for every two vertices u and v of G. Assuming P is a prescription of G, a *prescribed block* B of P in G is a set $\{v_{i_{j_1}}, v_{i_{j_2}}, \ldots, v_{i_{j_\ell}}\}$ of consecutive prescribed vertices, *i.e.* $v_{i_{j_2}} = v_{i_{j_1}}^+, v_{i_{j_3}} = v_{i_{j_2}}^+, \ldots, v_{i_{j_\ell}} = v_{i_{j_{\ell-1}}}^+$. We say that B is *maximal* if neither $v_{i_{j_1}}^-$ nor $v_{i_{j_\ell}}^+$ are prescribed vertices.

One important property of AP graphs is the following.

Observation 2. *If a graph G admits a spanning AP (resp. AP+k) subgraph (resp. for some $k \geq 1$), then G is AP (resp. AP+k).*

Recall that a graph is *traceable* if it admits a Hamiltonian path. Since every path is AP, Observation 2 implies the following result.

Corollary 3. *Every traceable graph is AP.*

We now point out the following property of AP+k graphs, from which we deduce a bound on the size of an optimal AP+k graph.

Observation 4. *Let $k \geq 1$. Every AP+k graph is $(k+1)$-connected. Therefore, an optimal AP+k graph on n vertices has at least $\lceil \frac{n(k+1)}{2} \rceil$ edges.*

Proof: Assume G is a graph with order n. If there exist k vertices $v_1, v_2, \dots, v_k$ such that $G - \{v_1, v_2, \dots, v_k\}$ is not connected, then the sequence $(1, 1, \dots, 1, n-k)$ with the value 1 appearing k times cannot be realized in G under $(v_1, v_2, \dots, v_k)$. Therefore, a necessary condition for G to be AP+k is to be $(k+1)$-connected. The lower bound then follows. □

As mentioned by Corollary 3, paths are AP+0, while it is easy to check that cycles are AP+1. Baudon *et al.* generalized these observations to powers of paths and cycles [3].

Theorem 5 ([3])**.** *The graph P_n^k is AP+$(k-1)$ for every $k \geq 1$ and $n \geq k$. The graph C_n^k is AP+$(2k-1)$ for every $k \geq 1$ and $n \geq 2k$.*

Provided that $n \geq 2k+2$, note that the size of P_n^{k+1} is $(k+1)(n-(k+1)) + \sum_{i=1}^{k} i$. Then, since $|E(P_n^{k+1})| > \lceil \frac{n(k+1)}{2} \rceil$, an optimal AP+$k$ graph on n vertices may have less edges than P_n^{k+1}. On the contrary, every graph C_n^k is $2k$-regular and hence is an edge-minimal $2k$-connected graph. According to Observation 4, it follows that the set of k^{th} powers of cycles is a set of optimal AP+$(2k-1)$ graphs for every $k \geq 1$.

3 Partitioning powers of paths and cycles under prescriptions

As pointed out in Theorem 5, recall that k^{th} powers of paths and cycles are AP+$(k-1)$ and AP+$(2k-1)$, respectively. This result is tight according to Observation 4, in the sense that we cannot always prescribe more vertices while partitioning these graphs. In this section, we exhibit situations under which these graphs can be partitioned under more prescriptions than indicated by their connectivity.

The following first result asserts that k^{th} powers of paths can be partitioned under k-prescriptions when either the first or the last vertex is prescribed.

Lemma 6 ([3])**.** *Let $P = (v_{i_1}, v_{i_2}, \dots, v_{i_k})$ be a k-prescription of P_n^k with $k \geq 1$, $n \geq k$ and $0 \leq i_1 < i_2 < \dots < i_k \leq n-1$. If $i_1 = 0$ or $i_k = n-1$, then every sequence $\tau = (n_1, n_2, \dots, n_p)$ admissible for P_n^k with $p \geq k$ elements is realizable in P_n^k under P.*

In the next result, we prove that k^{th} powers of paths are also partitionable under k-prescriptions when the prescribed vertices do not form a prescribed block with size k.

Lemma 7. *Let $P = (v_{i_1}, v_{i_2}, \dots, v_{i_k})$ be a k-prescription of P_n^k with $k \geq 1$, $n \geq k$ and $0 \leq i_1 < i_2 < \dots < i_k \leq n-1$. If the prescribed vertices do not form a prescribed block with size k, then every sequence $\tau = (n_1, n_2, \dots, n_p)$ admissible for P_n^k with $p \geq k$ elements is realizable in P_n^k under P.*

Proof: Let $G = P_n^k$ for given values of $k \geq 1$ and $n \geq k$. If $s = \sum_{j=k+1}^{p} n_j \leq i_1$, then a realization of τ in G under P is $(V_1, V_2, \dots, V_p)$ where $(V_{k+1}, V_{k+2}, \dots, V_p)$ is a realization of $(n_{k+1}, n_{k+2}, \dots, n_p)$ in the traceable graph $G[\{v_0, v_1, \dots, v_{s-1}\}]$, and $(V_1, V_2, \dots, V_k)$ is a realization of $(n_1, n_2, \dots, n_k)$ in $G - \{v_0, v_1, \dots, v_{s-1}\}$ under P which exists according to Theorem 1.

Suppose now that $s > i_1$. On the one hand, if $n_1 > i_1$, then a realization of τ in G under P is $(V_1' \cup V_1'', V_2, V_3, \ldots, V_p)$, where $V_1' = \{v_0, v_1, \ldots, v_{i_1-1}\}$ and $(V_1'', V_2, V_3, \ldots, V_p)$ is a realization of $(n_1 - i_1, n_2, \ldots, n_p)$ in $G - V_1'$ under P obtained via Lemma 6. On the other hand, if $n_1 \leq i_1$, then let V_1 be a subset of $\{v_0, v_1, \ldots, v_{i_1}\}$ obtained as follows. First, we set $V_1 = \{v_{i_1}\}$ and we then repeatedly add to V_1 the vertex located at distance 2 on the left of the last vertex added to V_1 as long as $|V_1| < n_1$ and v_0 is not reached. If there is no vertex at distance 2 on the left of the last vertex added to V_1 (but V_1 needs additional vertices), then we add to V_1 every remaining vertex from $\{v_0, v_1, \ldots, v_{i_1-1}\} - V_1$ from left to right until V_1 has size n_1. Let $X = \{v_0, v_1, \ldots, v_{i_1-1}\} - V_1$. Notice that, at the end of the procedure, $G[V_1]$ is connected, $G[X]$ is traceable, and $v_{i_1-1} \in X$. Now, if there exists $r \in \{k+1, k+2, \ldots, p\}$ such that $\sum_{j=k+1}^{r} n_j = |X|$, then a realization of τ in G under P is $(V_1, V_2, \ldots, V_p)$ where $(V_{k+1}, V_{k+2}, \ldots, V_r)$ is a realization of $(n_{k+1}, n_{k+2}, \ldots, n_r)$ in $G[X]$ and $(V_2, V_3, \ldots, V_k, V_{r+1}, V_{r+2}, \ldots, V_p)$ is a realization of $(n_2, n_3, \ldots, n_k, n_{r+1}, n_{r+2}, \ldots, n_p)$ in $G - \{v_0, v_1, \ldots, v_{i_1}\}$ under $\{v_{i_2}, v_{i_3} \ldots, v_{i_k}\}$ obtained using Theorem 5.

If such a value of r does not exist, then let r be such that $\sum_{j=k+1}^{r-1} n_j < |X|$ and $\sum_{j=k+1}^{r} n_j > |X|$. Let further $n_r' = |X| - \sum_{j=k+1}^{r-1} n_j$, $n_r'' = n_r - n_r'$, and $v_a \notin P$ be the nearest neighbour of v_{i_1-1} located on the right of v_{i_1}. Such a vertex necessarily exists since the opposite assumption would imply that our k prescribed vertices are located consecutively along G. Moreover, either v_a or v_{i_2} is the first vertex of $G - \{v_0, v_1, \ldots, v_{i_1}\}$. We then obtain a realization $(V_1, V_2, \ldots, V_{r-1}, V_r' \cup V_r'', V_{r+1}, V_{r+2}, \ldots, V_p)$ of τ in G under P, where $(V_r', V_{k+1}, V_{k+2}, \ldots, V_{r-1})$ is a realization of $(n_r', n_{k+1}, n_{k+2}, \ldots, n_{r-1})$ in $G[X]$ under (v_{i_1-1}), and $(V_2, V_3, \ldots, V_k, V_r'', V_{r+1}, V_{r+2}, \ldots, V_p)$ is a realization of $(n_2, n_3, \ldots, n_k, n_r'', n_{r+1}, n_{r+2}, \ldots, n_p)$ in $G[\{v_{i_1+1}, v_{i_1+2}, \ldots, v_{n-1}\}]$ under $(v_{i_2}, v_{i_3}, \ldots, v_{i_k}, v_a)$. These two realizations exist according to Lemma 6. □

We now strengthen Lemma 6 by showing that k^{th} powers of paths are partitionable under $(k+1)$-prescriptions when their endvertices are prescribed.

Lemma 8. *Let $P = (v_{i_1}, v_{i_2}, \ldots, v_{i_{k+1}})$ be a $(k+1)$-prescription of P_n^k with $k \geq 1$, $n \geq k$ and $0 \leq i_1 < i_2 < \ldots < i_{k+1} \leq n-1$. If $i_1 = 0$ and $i_{k+1} = n-1$, then every sequence $\tau = (n_1, n_2, \ldots, n_p)$ admissible for P_n^k with $p \geq k+1$ elements is realizable in P_n^k under P.*

Proof: We prove this claim by induction on k. For $k = 1$, the result is obvious. We thus now suppose that $k \geq 2$ and that the claim holds for every $k' < k$. Let $G = P_n^k$. If $n_1 \leq i_2$, then a realization of τ in G under P is $(V_1, V_2, \ldots, V_p)$ where $V_1 = \{v_0, v_1, \ldots, v_{n_1-1}\}$ and $(V_2, V_3, \ldots, V_p)$ is a realization of $(n_2, n_3, \ldots, n_p)$ in $G - V_1$ under $(v_{i_2}, v_{i_3}, \ldots, v_{i_{k+1}})$. This realization necessarily exists according to Lemma 6 since $v_{i_{k+1}}$ is the last vertex of $G - V_1$.

Suppose now that $n_1 > i_2$. Observe that $\{0, 1, \ldots, k-1\} - \bigcup_{j=2}^{k} \{i_j \mod k\}$ is not empty, so let us denote by r one of its elements. The subset V_1 of the realization is constructed as follows. It first contains all the vertices between v_0 and v_{i_2-1}, *i.e.* $\{v_0, v_1, \ldots, v_{i_2-1}\} \subseteq V_1$. We then add the vertex v_a to V_1, where $a \in \{i_2+1, i_2+2, \ldots, i_2+k-1\}$ is such that $a \equiv r \mod k$. Finally, as long as $|V_1| < n_1$, we repeatedly add to V_1 the vertex at distance k on the right from the last vertex added to V_1, unless it is equal to v_{n-1}, *i.e.* v_{a+k}, then v_{a+2k}, and so on. According to our choice of r, these vertices are not prescribed ones and, at any moment of the procedure, the subgraph $G - V_1$ is spanned by the $(k-1)^{th}$ power of a path, and the subgraph $G[V_1]$ is connected.

On the one hand, if $|V_1| = n_1$ holds after the procedure, then $(V_1, V_2, \ldots, V_p)$ is a realization of τ under P, where $(V_2, V_3, \ldots, V_p)$ is a realization of $(n_2, n_3, \ldots, n_p)$ in $G - V_1$ under the prescription

$(v_{i_2}, v_{i_3}, \ldots, v_{i_{k+1}})$ which necessarily exists by the induction hypothesis since v_{i_2} and $v_{i_{k+1}}$ are the end-vertices of $G - V_1$.

On the other hand, if $|V_1| < n_1$ holds once the procedure is achieved, then each vertex from $V(G) - V_1$ has a neighbour in V_1. Hence, we can obtain a realization $(V_1 \cup V_1', V_2, V_3, \ldots, V_p)$ of τ in G under P, where $(V_2, V_3, \ldots, V_p, V_1')$ is a realization of $(n_2, n_3, \ldots, n_p, n_1 - |V_1|)$ in $G - V_1$ under the prescription $(v_{i_2}, v_{i_3}, \ldots, v_{i_{k+1}})$. Once again, such a realization necessarily exists according to the induction hypothesis. □

We now prove an analogous result concerning cycles to the power of at least 2. Let $G = C_n^k$ for some $k \geq 2$ and $n \geq 2k$, the sequence $\tau = (n_0, n_1, \ldots, n_{p-1})$ be admissible for G, and $P = (v_{i_0}, v_{i_1}, \ldots, v_{i_{2k-1}})$ be a $2k$-prescription of G, with $p \geq 2k$ and $0 \leq i_0 < i_1 < \ldots < i_{2k-1} \leq n-1$. For every $j \in \{0, 1, \ldots, 2k-1\}$, we denote by D_j the set $\{v_{i_{j-1}}^+, (v_{i_{j-1}}^+)^+, \ldots, v_{i_j}^-, v_{i_j}\}$ containing the consecutive vertices of G lying between $v_{i_{j-1}}$ and v_{i_j}, including v_{i_j}. The size of every D_j is denoted d_j. In particular, we have $\sum_{j=0}^{2k-1} d_j = n$.

Lemma 9. *Let $P = (v_{i_0}, v_{i_1}, \ldots, v_{i_{2k-1}})$ be a $2k$-prescription of C_n^k with $k \geq 2$, $n \geq 2k$ and $0 \leq i_0 < i_1 < \ldots < i_{2k-1} \leq n-1$. If the prescribed vertices are not organized into two maximal prescribed blocks with size k, then every sequence $\tau = (n_0, n_1, \ldots, n_{p-1})$ admissible for C_n^k with $p \geq 2k$ elements is realizable in C_n^k under P.*

Proof: Let $k \geq 2$ be fixed, and $G = C_n^k$ for some value of $n \geq 2k$. We prove that every partition $\tau = (n_0, n_1, \ldots, n_{p-1})$ of n with $p \geq 2k$ elements is realizable in G under every $2k$-prescription $P = (v_{i_0}, v_{i_1}, \ldots, v_{i_{2k-1}})$ with $0 \leq i_0 < i_1 < \ldots < i_{2k-1} \leq n-1$ when the prescribed vertices do not form two maximal prescribed blocks with size k. For every $j \in \{0, 1, \ldots, 2k-1\}$, let $q_j = \sum_{\ell=j}^{j+k-1} d_\ell$ and $s_j = \sum_{\ell=j}^{j+k-1} n_\ell$, where the indices are counted modulo $2k$. In other words, the value q_j is the order of the graph $v_{i_{j-1}}^+ G v_{i_{j+k-1}} = G[\{i_{j-1}^+, (i_{j-1}^+)^+, \ldots, i_{j+k-1}\}]$ including the k prescribed vertices v_{i_j}, $v_{i_{j+1}}, \ldots, v_{i_{j+k-1}}$, and s_j is the amount of vertices needed by the subgraphs containing these prescribed vertices in a realization of τ in G under P. Note that there necessarily exists a j such that $s_j \leq q_j$ since having $\sum_{j=0}^{2k-1} s_j \geq \sum_{j=0}^{2k-1} (q_j + 1)$ implies $k \sum_{\ell=0}^{2k-1} n_\ell \geq k \sum_{\ell=0}^{2k-1} d_\ell + 2k$, which is impossible since $n = \sum_{\ell=0}^{2k-1} d_\ell$ and $n \geq \sum_{\ell=0}^{2k-1} n_\ell$. To prove the claim, we distinguish several cases depending on the relationship between s_j's and q_j's.

Case 1. *$s_j = q_j$ for some $j \in \{0, 1, \ldots, 2k-1\}$.*

In this situation, a realization of τ in G under P is deduced as follows. Assume $j = 0$ without loss of generality, and set $G_0 = G[\bigcup_{\ell=0}^{k-1} D_\ell]$. Note that G_0 is the k^{th} power of a path. If $\sum_{\ell=0}^{k-1} d_\ell \geq k+1$, then G_0 is k-connected and thus admits a realization $(V_0, V_1, \ldots, V_{k-1})$ of $(n_0, n_1, \ldots, n_{k-1})$ under $(v_{i_0}, v_{i_1}, \ldots, v_{i_{k-1}})$ according to Theorem 1. Otherwise, if $\sum_{\ell=0}^{k-1} d_\ell = k$, then $(V_0, V_1, \ldots, V_{k-1}) = (\{v_{i_0}\}, \{v_{i_1}\}, \ldots, \{v_{i_{k-1}}\})$ is a realization of $(n_0, n_1, \ldots, n_{k-1}) = (1, 1, \ldots, 1)$ in G_0 under $(v_{i_0}, v_{i_1}, \ldots, v_{i_{k-1}})$. On the other hand, the graph $G - \bigcup_{\ell=0}^{k-1} D_\ell$ is the k^{th} power of a path whose last vertex is $v_{i_{2k-1}}$. Therefore, there exists a realization $(V_k, V_{k+1}, \ldots, V_{p-1})$ of $(n_k, n_{k+1}, \ldots, n_{p-1})$ under $(v_{i_k}, v_{i_{k+1}}, \ldots, v_{i_{2k-1}})$ in this graph by Lemma 6. The partition $(V_0, V_1, \ldots, V_{p-1})$ is then a realization of τ in G under P.

Case 2. *We are not in **Case 1** and $s_j > q_j$ for some $j \in \{0, 1, \ldots, 2k-1\}$.*
In particular, there exists a value of j for which $s_j > q_j$ and $s_{j+1} < q_{j+1}$. Suppose $j = 0$ without loss of generality.

Case 2.1. *There exists a set $X = \{v^+_{i_{2k-1}}, (v^+_{i_{2k-1}})^+, \ldots, v_a\}$ with $a \in \{i_{k-1}+1, i_{k-1}+2, \ldots, i_k - 1\}$ such that $|X| = s_0$.*
A realization of τ in G under P can be obtained as follows. Firstly, let $(V_0, V_1, \ldots, V_{k-1})$ be a realization of $(n_0, n_1, \ldots, n_{k-1})$ in $G[X]$ under $(v_{i_0}, v_{i_1}, \ldots, v_{i_{k-1}})$. Such a realization exists by Theorem 1 since $G[X]$ is the k^{th} power of a path. Secondly, let $(V_k, V_{k+1}, \ldots, V_{p-1})$ be a realization of $(n_k, n_{k+1}, \ldots, n_{p-1})$ in $G - X$ under $(v_{i_k}, v_{i_{k+1}}, \ldots, v_{i_{2k-1}})$ which necessarily exists according to Lemma 6 since $v_{i_{2k-1}}$ is the last vertex of $G - X$. The partition $(V_0, V_1, \ldots, V_{p-1})$ is then a realization of τ in G under P.

Case 2.2. *Such a set X does not exist.*
In such a situation, we have $s_0 > q_0 + d_k - 1$, *i.e.* $\sum_{\ell=0}^{k-1} n_\ell > \sum_{\ell=0}^{k} d_\ell - 1$. Besides, since n_ℓ's and d_ℓ's are strictly greater than 0, we get $\sum_{\ell=0}^{k} n_\ell \geq 1 + \sum_{\ell=1}^{k} d_\ell$. Since $s_1 < q_1$, *i.e.* $\sum_{\ell=1}^{k} n_\ell < \sum_{\ell=1}^{k} d_\ell$, it follows that there exists a n'_0 such that $1 \leq n'_0 \leq n_0$ and $n'_0 + \sum_{\ell=1}^{k} n_\ell = 1 + \sum_{\ell=1}^{k} d_\ell = |\{v_{i_0}, v^+_{i_0}, \ldots, v_{i_k}\}|$. A realization of τ in G under P is then obtained as follows. On the one hand, let $(V'_0, V_1, V_2, \ldots, V_k)$ be a realization of $(n'_0, n_1, n_2, \ldots, n_k)$ in $G[\{v_{i_0}, v^+_{i_0}, \ldots, v_{i_k}\}]$ under $(v_{i_0}, v_{i_1}, \ldots, v_{i_k})$, which exists according to Lemma 8 since v_{i_0} and v_{i_k} are the endvertices of $G[\{v_{i_0}, v^+_{i_0}, \ldots, v_{i_k}\}]$. On the other hand, let $n''_0 = (n_0 - n'_0) + 1$ (note that $n''_0 \geq 1$), and let $(V''_0, V_{k+1}, V_{k+2}, \ldots, V_{p-1})$ be a realization of $(n''_0, n_{k+1}, n_{k+2}, \ldots, n_{p-1})$ in $G[\{v^+_{i_k}, (v^+_{i_k})^+, \ldots, v_{i_0}\}]$ under $(v_{i_0}, v_{i_{k+1}}, v_{i_{k+2}}, \ldots, v_{i_{2k-1}})$, which exists according to Lemma 6 since $G[\{v^+_{i_k}, (v^+_{i_k})^+, \ldots, v_{i_0}\}]$ is the k^{th} power of a path with last vertex v_{i_0}, and k prescribed vertices are specified. The partition $(V'_0 \cup V''_0, V_1, V_2, \ldots, V_{p-1})$ is then a realization of τ in G under P since $G[V'_0]$ and $G[V''_0]$ are connected and both contain the vertex v_{i_0} (which is actually the only vertex appearing in both these subgraphs).

Case 3. *$s_j < q_j$ for every $j \in \{0, 1, \ldots, 2k-1\}$.*
We distinguish two subcases.

Case 3.1. *There are two consecutive prescribed vertices.*
Assume $v_{i_0} = v^+_{i_{2k-1}}$ without loss of generality, with $i_0 = 0$ and $i_{2k-1} = n - 1$.

Case 3.1.1. *There exists $r \in \{2k, 2k+1, \ldots, p-1\}$ such that $s_0 + \sum_{\ell=2k}^{r} n_\ell = q_0$.*
In this situation, we can deduce a realization of τ in G under P as follows. Firstly, let

$$(V_0, V_1, \ldots, V_{k-1}, V_{2k}, V_{2k+1}, \ldots, V_r)$$

be a realization of

$$(n_0, n_1, \ldots, n_{k-1}, n_{2k}, n_{2k-1}, \ldots, n_r)$$

in $G[\bigcup_{\ell=0}^{k-1} D_\ell]$ under $(v_{i_0}, v_{i_1}, \ldots, v_{i_{k-1}})$ which exists according to Lemma 6 since v_{i_0} is the first vertex of $G[\bigcup_{\ell=0}^{k-1} D_\ell]$, this graph being the k^{th} power of some path. Secondly, let

$$(V_k, V_{k+1}, \ldots, V_{2k-1}, V_{r+1}, V_{r+2}, \ldots, V_{p-1})$$

be a realization of

$$(n_k, n_{k+1}, \ldots, n_{2k-1}, n_{r+1}, n_{r+2}, \ldots, n_{p-1})$$

in $G - \bigcup_{\ell=0}^{k-1} D_\ell$ under $(v_{i_k}, v_{i_{k+1}}, \ldots, v_{i_{2k-1}})$ which exists for the same reason as previously since $v_{i_{2k-1}}$ is the last vertex of $G - \bigcup_{\ell=0}^{k-1} D_\ell$. The partition $(V_0, V_1, \ldots, V_{p-1})$ is then a realization of τ in G under P.

Case 3.1.2. *Such r does not exist.*
Let $r \in \{2k, 2k+1, \ldots, p-1\}$ be the value for which we have $s_0 + \sum_{\ell=2k}^{r-1} n_\ell < q_0$ and $s_0 + \sum_{\ell=2k}^{r} n_\ell > q_0$. Such a value exists since $s_0 < q_0$ and $s_k < q_k$. So let further $n'_r = q_0 - (s_0 + \sum_{\ell=2k}^{r-1} n_\ell)$ and $n''_r = n_r - n'_r$. Denote by v_a the last non-prescribed vertex of $G[\bigcup_{\ell=0}^{k-1} D_\ell]$, and by v_b the first non-prescribed vertex of $G - \bigcup_{\ell=0}^{k-1} D_\ell$.

Case 3.1.2.1. *The vertices v_a and v_b are adjacent in G.*
Let $v_b = v_{i_q}^+$ for some $q \in \{k+1, k+2, \ldots, 2k-2\}$. Then we obtain a realization of τ in G under P as follows. Firstly, let

$$(V_0, V_1, \ldots, V_{k-1}, V'_r, V_{2k}, V_{2k+1}, \ldots, V_{r-1})$$

be a realization of $(n_0, n_1, \ldots, n_{k-1}, n'_r, n_{2k}, n_{2k+1}, \ldots, n_{r-1})$ in $G[\bigcup_{\ell=0}^{k-1} D_\ell]$ under $(v_{i_0}, v_{i_1}, \ldots, v_{i_{k-1}}, v_a)$, which exists by Lemma 8 since $G[\bigcup_{\ell=0}^{k-1} D_\ell]$ is the k^{th} power of a path whose endvertices are v_{i_0} and $v_{i_{k-1}}$. Secondly, let

$$(V_k, V_{k+1}, \ldots, V_q, V''_r, V_{q+1}, V_{q+2}, \ldots, V_{2k-1}, V_{r+1}, V_{r+2}, \ldots, V_{p-1})$$

be a realization of

$$(n_k, n_{k+1}, \ldots, n_q, n''_r, n_{q+1}, n_{q+2}, \ldots, n_{2k-1}, n_{r+1}, n_{r+2}, \ldots, n_{p-1})$$

in $G - \bigcup_{\ell=0}^{k-1} D_\ell$ under $(v_{i_k}, v_{i_{k+1}}, \ldots, v_{i_q}, v_b, v_{i_{q+1}}, v_{i_{q+2}}, \ldots, v_{i_{2k-1}})$.
This realization exists according to Lemma 8 since $G - \bigcup_{\ell=0}^{k-1} D_\ell$ is the k^{th} power of a path, either v_b or v_{i_k} is the first vertex of $G - \bigcup_{\ell=0}^{k-1} D_\ell$, and $v_{i_{2k-1}}$ is the last vertex of $G - \bigcup_{\ell=0}^{k-1} D_\ell$. It follows that

$$(V_0, V_1, \ldots, V_{r-1}, V'_r \cup V''_r, V_{r+1}, V_{r+2}, \ldots, V_{p-1})$$

is a realization of τ in G under P since $G[V'_r \cup V''_r]$ is connected thanks to the edge $v_a v_b$.

Case 3.1.2.2. *The vertices v_a and v_b are not adjacent in G.*
In this situation, either $v_{i_{k-1}}$ or v_{i_k} belongs to a prescribed block with size at least k. Then one can relabel the prescribed vertices so that v_{i_0} and $v_{i_{2k-1}}$ correspond to two consecutive prescribed vertices from this prescribed block, and use the procedures from **Case 3.1**. Since $s_j < q_j$ for every $j \in \{0, 1, \ldots, 2k-1\}$, note that this time the two vertices v_a and v_b (if these vertices are needed) have to be adjacent since otherwise it would mean that the prescribed vertices form another prescribed block with size at least k, implying that there are two prescribed blocks with size k, contradicting the assumption of the lemma.

Case 3.2. *There are no two consecutive prescribed vertices.*

Case 3.2.1. *There exists a set X of consecutive vertices of G such that $X \cap P = \{v_{i_j}, v_{i_{j+1}}, \dots, v_{i_{j+k-1}}\}$ and $|X| = s_j$ for some $j \in \{0, 1, \dots, 2k-1\}$.*

In this situation, we obtain a realization of τ in G under P as follows. Assume $j = 0$ without loss of generality. Firstly, let $(V_0, V_1, \dots, V_{k-1})$ be a realization of $(n_{i_0}, n_{i_1}, \dots, n_{i_{k-1}})$ in $G[X]$ under $(v_{i_0}, v_{i_1}, \dots, v_{i_{k-1}})$, which exists by Theorem 1 since $G[X]$ is the k^{th} power of some path. Secondly, let $(V_k, V_{k+1}, \dots, V_{p-1})$ be a realization of $(n_k, n_{k+1}, \dots, n_{p-1})$ in $G-X$ under $(v_{i_k}, v_{i_{k+1}}, \dots, v_{i_{2k-1}})$ obtained thanks to Lemma 7 since $G - X$ is the k^{th} power of a path and there are no consecutive prescribed vertices. Then $(V_0, V_1, \dots, V_{p-1})$ is a realization of τ in G under P.

Case 3.2.2. $s_j < q_j - d_j + 1$ *for every* $j \in \{0, 1, \dots, 2k-1\}$.

Case 3.2.2.1. *There are two prescribed vertices v_{i_ℓ} and $v_{i_{\ell+1}}$ such that $n_\ell + n_{\ell+1} \geq d_{\ell+1} + 1$.*

Assume $\ell = 2k-1$ without loss of generality. Then there exist two sets of consecutive vertices $X = \{v_{i_{2k-1}}, v^+_{i_{2k-1}}, \dots, v_a\}$ and $Y = \{v^+_a, (v^+_a)^+, \dots, v_{i_0}\}$, with $a \in \{i_{2k-1}, i_{2k-1}+1 \mod n, \dots, i_0 - 1 \mod n\}$, $|X| \leq n_{2k-1}$ and $|Y| \leq n_0$. A realization of τ in G under P can be then obtained as in **Case 3.1** by doing as if $v_{i_{2k-1}}$ and v_{i_0} were consecutive prescribed vertices (this is straightforward due to the notation we have adopted herein), but requesting $v_{i_{2k-1}}$ and v_{i_0} to belong to subgraphs with order $n_{2k-1} - |X| + 1$ and $n_0 - |Y| + 1$, respectively. Recall that we are under the assumption that there are no two consecutive prescribed vertices. For the resulting parts V'_{2k-1} and V'_0, the graphs $G[V'_{2k-1} \cup X]$ and $G[V'_0 \cup Y]$ are connected, and have order n_{2k-1} and n_0, respectively.

Case 3.2.2.2. $n_j + n_{j+1} < d_{j+1} + 1$ *for every* $j \in \{0, 1, \dots, 2k-1\}$.

In particular, $n_0 + n_1 < d_1 + 1 = |\{v_{i_0}, v^+_{i_0}, \dots, v_{i_1}\}|$. We cannot have both $n_0 \geq \lceil \frac{d_1+1}{2} \rceil$ and $n_1 \geq \lceil \frac{d_1+1}{2} \rceil$, since otherwise we would get $n_0 + n_1 \geq d_1 + 1$, a contradiction. Let us thus suppose that $n_0 < \lceil \frac{d_1+1}{2} \rceil$ without loss of generality. Then note that the graph induced by $V_0 = \{v_{i_0}, v_{i_0+2}, v_{i_0+4}, \dots, v_{i_0+2(n_0-1)}\}$ has order n_0 and contains v_{i_0}, and the graph $G[\{v_{i_{2k-1}}, v^+_{i_{2k-1}}, \dots, v_{i_1}\} - V_0]$ is traceable with endvertices $v_{i_{2k-1}}$ and v_{i_1}.

Let $t_1 = |\{v_{i_1}, v^+_{i_1}, \dots, v_{i_k}\}| - \sum_{\ell=1}^{k} n_\ell$ and $t_2 = |\{v^+_{i_{2k-1}}, (v^+_{i_{2k-1}})^+, \dots, v^-_{i_1}\}| - n_0$. From τ, we define three sequences τ_1, τ_2 and τ_3.

First, let $\tau_1 = (n_1, n_2, \dots, n_k, n_{2k}, n_{2k+1}, \dots, n_{r_1-1})$, where r_1 is the unique index in $\{2k, 2k+1, \dots, p-1\}$ such that $\sum_{\ell=2k}^{r_1-1} n_\ell \leq t_1$ and $\sum_{\ell=2k}^{r_1} n_\ell > t_1$. Now, if $t_1 - \sum_{\ell=2k}^{r_1-1} n_\ell > 0$, then add $n'_{r_1} = t_1 - \sum_{\ell=2k}^{r_1-1} n_\ell$ as the $(k+1)^{th}$ element of τ_1. Note that the elements of τ_1 sum up to $|\{v_{i_1}, v^+_{i_1}, \dots, v_{i_k}\}|$.

Let $n''_{r_1} = n_{r_1} - n'_{r_1}$. If $n''_{r_1} \geq t_2$, then let $\tau_2 = (t_2)$, and set $r_2 = r_1$ and $n''_{r_2} = n''_{r_1} - t_2$. Otherwise, let r_2 be the index in $\{r_1+1, r_1+2, \dots, p-1\}$ for which $n''_{r_1} + \sum_{\ell=r_1+1}^{r_2-1} n_\ell \leq t_2$ and $n''_{r_1} + \sum_{\ell=r_1+1}^{r_2} n_\ell > t_2$. Now let $\tau_2 = (n''_{r_1}, n_{r_1+1}, n_{r_1+2}, \dots, n_{r_2-1})$. Set $n'_{r_2} = t_2 - (n''_{r_1} + \sum_{\ell=r_1+1}^{r_2-1} n_\ell)$ and $n''_{r_2} = n_{r_2} - n'_{r_2}$, and add n'_{r_2} as the second element of τ_2 if $n'_{r_2} > 0$. Once τ_2 is constructed, note that its elements sum up to $|\{v^+_{i_{2k-1}}, (v^+_{i_{2k-1}})^+, \dots, v^-_{i_1}\}| - n_0$.

Finally, assuming $n''_{r_2} > 0$ (otherwise, remove this element from the sequence), let $\tau_3 =$

$(n_{k+1}, n_{k+2}, \dots, n_{2k-1}, n''_{r_2}, n_{r_2+1}, n_{r_2+2}, \dots, n_{p-1})$. Note that the elements of τ_3 sum up to $|\{v_{i_k}^+, (v_{i_k}^+)^+, \dots, v_{i_{2k-1}}\}|$.
Remark that every element of τ has been associated with one of τ_1, τ_2 and τ_3, with at most two non-prescribed elements being split so that the τ_i's sum up exactly to the orders of some subgraphs of G. In the case where τ contains a "big" non-prescribed element, it is even possible that this element was split into three integers among τ_1, τ_2 and τ_3. To obtain the realization of τ in G under P, we realize τ_1, τ_2 and τ_3 in vertex-disjoint subgraphs of G, and this in such a way that if an original element of τ was dispatched in several of the τ_i's, then the resulting connected subgraphs perform a whole connected subgraph when unified.
The three realizations R_1, R_2 and R_3 are obtained as follows.

- Let R_1 be a realization of τ_1 in $G[\{v_{i_1}, v_{i_1}^+, \dots, v_{i_k}\}]$ under $(v_{i_1}, v_{i_2}, \dots, v_{i_k}, v_{i_1}^+)$, which exists according to Lemma 8 since v_{i_1} and v_{i_k} are the endvertices of $G[\{v_{i_1}, v_{i_1}^+, \dots, v_{i_k}\}]$ and there are $k+1$ prescribed vertices.
- Let R_2 be a realization of τ_2 in $G[\{v_{i_{2k-1}}^+, (v_{i_{2k-1}}^+)^+, \dots, v_{i_1}^-\} - V_0]$, which is traceable by our choice of V_0. Additionally request the realization to satisfy the prescription $(v_{i_1}^-, v_{i_{2k-1}}^+)$ when τ_2 has at least two elements. Such a requirement is allowed according to Lemma 8.
- Let R_3 be a realization of τ_3 in $G[\{v_{i_k}^+, (v_{i_k}^+)^+, \dots, v_{i_{2k-1}}\}]$ under $(v_{i_{k+1}}, v_{i_{k+2}}, \dots, v_{i_{2k-1}}, v_{i_{2k-1}}^-)$. The existence of such a realization follows from Lemma 6 since $G[\{v_{i_k}^+, (v_{i_k}^+)^+, \dots, v_{i_{2k-1}}\}]$ is the k^{th} power of some path whose last vertex is $v_{i_{2k-1}}$.

The realization of τ in G under P is obtained by considering V_0 and the parts from R_1, R_2 and R_3, and unifying those parts whose sizes result from the split of a single element of τ, if necessary. By our choice of the prescribed vertices, these parts have neighbouring vertices (this follows from the facts that $k \geq 2$, and that the prescribed vertices of P are not consecutive), and thus induce connected subgraphs. This completes the proof.

□

4 Partitioning Harary graphs under prescriptions

Harary graphs are trivially AP according to Corollary 3. We here show that we can always prescribe the largest possible number of vertices (with respect to their connectivity) while partitioning these graphs, except for 3-connected Harary graphs. We consider the three kinds of Harary graphs for this purpose.

4.1 Construction 1: k is even

The Harary graph $H_{k,n}$ with k even is isomorphic to $C_n^{k/2}$ which is AP+$(k-1)$ according to Theorem 5 for every $k \geq 2$ and $n \geq 2k$. We thus derive the following result.

Corollary 10. *For every even $k \geq 2$ and $n \geq 2k$, the Harary graph $H_{k,n}$ is AP+$(k-1)$.*

4.2 Construction 2: k is odd and n is even

Let $k \geq 2$ and $n \geq 2k+1$ be two integers such that n is even. By construction, the Harary graph $H_{2k+1,n}$ is spanned by $H_{2k,n}$ and is thus AP+$(2k-1)$ according to Corollary 10. However, regarding the connectivity of $H_{2k+1,n}$, one could wonder whether $H_{2k+1,n}$ is AP+$2k$.

Before proving that $H_{2k+1,n}$ is indeed AP+$2k$, we first introduce the following lemma which deals with the traceability of a graph composed by two linked squares of paths.

Lemma 11. *If G is a graph such that $V(G) = V_1 \cup V_2$, the subgraphs $G[V_1]$ and $G[V_2]$ are both spanned by the square of a path, and there exists an edge joining one vertex of V_1 and one of V_2, then G is traceable.*

Proof: Let $v_1, v_2, \ldots, v_\ell$ and $u_1, u_2, \ldots, u_{\ell'}$ denote the consecutive vertices of $G[V_1]$ and $G[V_2]$, and $v_a \in V_1$ and $u_b \in V_2$ be two vertices of G such that $v_a u_b \in E(G)$. Consider the following subpaths of G:

- $P = v_1 v_2 \ldots v_{a-1}$;
- $Q = \begin{cases} v_{a+1}v_{a+3}\ldots v_{\ell-1}v_\ell v_{\ell-2}v_{\ell-4}\ldots v_{a+2} \text{ if } \ell - a \text{ is even,} \\ v_{a+1}v_{a+3}\ldots v_\ell v_{\ell-1}v_{\ell-3}\ldots v_{a+2} \text{ otherwise;} \end{cases}$
- $R = \begin{cases} u_{b+2}u_{b+4}\ldots u_{\ell'}u_{\ell'-1}u_{\ell'-3}\ldots u_{b+1} \text{ if } \ell' - b \text{ is even,} \\ u_{b+2}u_{b+4}\ldots u_{\ell'-1}u_{\ell'}u_{\ell'-2}u_{\ell'-4}\ldots u_{b+1} \text{ otherwise;} \end{cases}$
- $S = u_{b-1}u_{b-2}\ldots u_1$.

It is then easy to check that $PQv_a u_b RS$ is a Hamiltonian path of G. □

We are now ready to prove our main result.

Theorem 12. *For every $k \geq 2$ and even $n \geq 2k+1$, the Harary graph $H_{2k+1,n}$ is AP+$2k$.*

Proof: Let $k \geq 2$ and even $n \geq 2k+1$ be fixed, and $G = H_{2k+1,n}$ be the $(2k+1)$-connected Harary graph on n vertices. We prove that every sequence $\tau = (n_0, n_1, \ldots, n_{p-1})$, admissible for G with $p \geq 2k+1$ elements, is realizable in G under every $2k$-prescription $P = (v_{i_0}, v_{i_1}, \ldots, v_{i_{2k-1}})$ with $0 \leq i_0 < i_1 < \ldots < i_{2k-1} \leq n-1$. We distinguish two main cases.

Case 1. If the prescribed vertices are not organized into two maximal prescribed blocks with size k, then, because $k \geq 2$, we can deduce a realization of τ in the spanning C_n^k of G under P, thanks to Lemma 9. Such a realization is naturally a realization of τ in G under P.

Case 2. Suppose now that the prescribed vertices form two maximal prescribed blocks B_1 and B_2 with size exactly k in G. In this situation, note that $G - P$ only remains connected thanks to some diagonal edges. Indeed, assume $B_1 = \{v_{i_0}, v_{i_1}, \ldots, v_{i_{k-1}}\}$ and $B_2 = \{v_{i_k}, v_{i_{k+1}}, \ldots, v_{i_{2k-1}}\}$ without loss of generality. Then the antipodal neighbours of $v_{i_0}^-$ and $v_{i_{k-1}}^+$ cannot both belong to P: since $n \geq 2k+2$, if this were the case then these two antipodal neighbours would belong to B_2, and similarly for all antipodal neighbours of $v_{i_0}, v_{i_1}, \ldots, v_{i_{k-1}}$ (according to our assumptions on the maximal prescribed blocks). We would then get that B_2 has size at least $k+2$, a contradiction. Let us thus denote by v_a and v_b two antipodal neighbours of G such that $v_a, v_b \notin B_1 \cup B_2$. In particular, we may suppose $a \in \{i_{k-1}+1, i_{k-1}+2, \ldots, i_k - 1\}$ and $b \in \{i_{2k-1}+1, i_{2k-1}+2, \ldots, i_0 - 1\}$. Let further $a_1 = a - i_{k-1} - 1$, $a_2 = i_k - a - 1$, $a_3 = i_0 - b - 1$ and $a_4 = b - i_{2k-1} - 1$ denote the number of consecutive vertices between B_1, B_2 and the two vertices v_a and v_b according to the natural ordering of G.

Case 2.1. $\sum_{j=0}^{k-1} n_j \leq a_1 + a_3 + k$ *and* $\sum_{j=k}^{2k-1} n_j \leq a_2 + a_4 + k$.

In this situation, we can find two subsets X and Y of consecutive vertices of G such that $|X| = \sum_{j=0}^{k-1} n_j$, $|Y| = \sum_{j=k}^{2k-1} n_j$, $\{v_{i_0}, v_{i_1}, \dots, v_{i_{k-1}}\} \subseteq X$, $\{v_{i_k}, v_{i_{k+1}}, \dots, v_{i_{2k-1}}\} \subseteq Y$, and $v_a, v_b \notin X \cup Y$. Since $G[X]$ and $G[Y]$ are both isomorphic to the k^{th} power of a path, by Theorem 1 we know that we can deduce two realizations $(V_0, V_1, \dots, V_{k-1})$ and $(V_k, V_{k+1}, \dots, V_{2k-1})$ of $(n_0, n_1, \dots, n_{k-1})$ and $(n_k, n_{k+1}, \dots, n_{2k-1})$, respectively, in $G[X]$ and $G[Y]$, respectively, under $(v_{i_0}, v_{i_1}, \dots, v_{i_k-1})$ and $(v_{i_k}, v_{i_{k+1}}, \dots, v_{i_{2k-1}})$, respectively. Now, since $k \geq 2$, the graph $G-(X \cup Y)$ is traceable according to Lemma 11 and thus admits a realization $(V_{2k}, V_{2k+1}, \dots, V_{p-1})$ of $(n_{2k}, n_{2k+1}, \dots, n_{p-1})$. Finally, the partition $(V_0, V_1, \dots, V_{p-1})$ is a realization of τ in G under P.

Case 2.2. $\sum_{j=0}^{k-1} n_j > a_1 + a_3 + k$ *without loss of generality.*

Note that we have $\sum_{j=0}^{2k-1} n_j \geq \min\{a_1 + a_2 + 2k + 1, a_3 + a_4 + 2k + 1\}$, since otherwise

$$a_1 + a_3 + 2k + 1 \leq \sum_{j=0}^{k-1} n_j + \sum_{j=k}^{2k-1} n_j = \sum_{j=0}^{2k-1} n_j < \frac{1}{2}(a_1 + a_2 + a_3 + a_4) + 2k + 1,$$

which implies $a_1 + a_3 < a_2 + a_4$, a contradiction. Then we consider two new cases.

Case 2.2.1. $\sum_{j=0}^{2k-1} n_j \geq a_1 + a_2 + 2k + 1$.

Under this assumption, we can find two subsets of consecutive vertices $X, Y \subseteq V(G)$ such that $\{v_{i_0}, v_{i_1}, \dots, v_{i_{k-1}}\} \subseteq X$, $\{v_{i_k}, v_{i_{k+1}}, \dots, v_{i_{2k-1}}\} \subseteq Y$, $|X| = \sum_{j=0}^{k-1} n_j$, $|Y| = \sum_{j=k}^{2k-1} n_j$, and the last vertex of $G[X]$ is the vertex preceding the first vertex of $G[Y]$. By Theorem 1, we know that we can deduce realizations $(V_0, V_1, \dots, V_{k-1})$ and $(V_k, V_{k+1}, \dots, V_{2k-1})$ of $(n_0, n_1, \dots, n_{k-1})$ and $(n_k, n_{k+1}, \dots, n_{2k-1})$, respectively, in $G[X]$ and $G[Y]$, respectively, under $(v_{i_0}, v_{i_1}, \dots, v_{i_{k-1}})$ and $(v_{i_k}, v_{i_{k+1}}, \dots, v_{i_{2k-1}})$, respectively. Finally, since the graph $G-(X \cup Y)$ is isomorphic to the k^{th} power of a path, there exists a realization $(V_{2k}, V_{2k+1}, \dots, V_p)$ of the remaining sequence $(n_{2k}, n_{2k+1}, \dots, n_{p-1})$ in it. We get that $(V_0, V_1, \dots, V_{p-1})$ is a realization of τ in G under P.

Case 2.2.2. $\sum_{j=0}^{2k-1} n_j \geq a_3 + a_4 + 2k + 1$.

In this case, we proceed similarly as in **Case 2.2.1**, but the last vertex of $G[Y]$ has to be the vertex preceding the first vertex of $G[X]$.

□

4.3 Construction 3: k and n are odd

Since two Harary graphs $H_{2k+1,n}$ and $H_{2k+1,n'}$, with $k \geq 2$, and $n \geq 2k+1$ and $n' \geq 2k+1$ being even and odd, respectively, are both spanned by C_n^k, **Case 1** from the proof of Theorem 12 also holds directly regarding Harary graphs with odd connectivity and order. Despite $H_{2k+1,n}$ and $H_{2k+1,n'}$ slightly differ by their diagonal edges, it is easy to realize that if the assumptions of **Case 2** from the proof of Theorem 12 are fulfilled, that proof can be adapted for considering Harary graphs of odd connectivity and order.

Theorem 13. *For every $k \geq 2$ and odd $n \geq 2k+1$, the Harary graph $H_{2k+1,n}$ is AP+$2k$.*

5 On the existence of optimal AP+2 graphs

Recall that Theorems 12 and 13 exclude 3-connected Harary graphs, mainly because some of their subgraphs do not satisfy the traceability property exhibited in Lemma 11. Therefore, our proof cannot be used to prove that 3-connected Harary graphs are AP+2.

Besides, it turns out that 3-connected Harary graphs are not all AP+2 anyway. A straight argument for that claim follows from the fact that an *unbalanced* bipartite graph $G = (A \cup B, E)$, *i.e.* such that $|A| \neq |B|$, with even order does not admit a perfect matching.

Lemma 14. *If a bipartite graph $G = (A \cup B, E)$ has even (resp. odd) order, then, assuming G has enough vertices, the graph G cannot be AP+k for every even (resp. odd) $k \geq 2$ (resp. $k \geq 1$).*

Proof: We prove the claim for bipartite graphs with even order, but the proof is analogous for bipartite graphs with odd order. Let $k \geq 2$ be even and fixed. For such a value of k, we can find two subsets $X \subset A$ and $Y \subset B$ such that $X \cap Y = \emptyset$, $|X| + |Y| = k$, and $|A - X| \neq |B - Y|$. Let $A' = A - X$ and $B' = B - X$. Then since $|A'| + |B'|$ is even and $|A'| \neq |B'|$, the graph $G[A' \cup B']$ cannot admit a perfect matching. It follows that the sequence $(1, 1, \dots, 1, 2, 2, \dots, 2)$, with the value 1 appearing k times, is not realizable in G under $(v_1, v_2, \dots, v_k)$, where $\{v_1, v_2, \dots, v_k\} = X \cup Y$. □

Corollary 15. *For every $n \equiv 2 \mod 4$, the Harary graph $H_{3,n}$ is not AP+2.*

Proof: This follows from Lemma 14 since every such Harary graph is a *balanced* bipartite graph. □

In order to prove that there actually exist optimal AP+2 graphs on n vertices and $\lceil \frac{3n}{2} \rceil$ edges for every $n \geq 4$, we introduce another class of 3-connected graphs. Let $n \geq 4$. The graph Pr_n is constructed as follows.

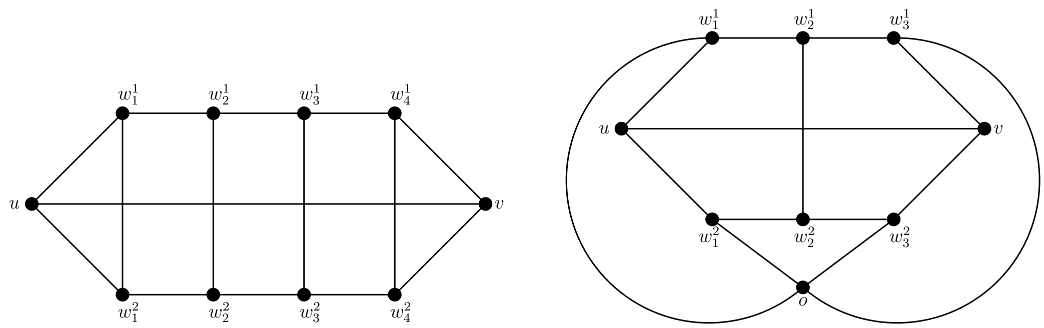
- If n is even, then Pr_n is obtained from the cycle C_n, whose vertices are successively denoted by $u, w_1^1, w_2^1, \dots, w_{\frac{n-2}{2}}^1, v, w_{\frac{n-2}{2}}^2, w_{\frac{n-2}{2}-1}^2 \dots, w_1^2$, by adding to it the edge uv, and the edge $w_i^1 w_i^2$ for every $i \in \{1, 2, \dots, \frac{n-2}{2}\}$.
- If n is odd, then Pr_n is obtained by first removing the edges $w_1^1 w_1^2$ and $w_{\frac{n-3}{2}}^1 w_{\frac{n-3}{2}}^2$ from Pr_{n-1}, and then adding to it a new vertex o and the edges ow_1^1, ow_1^2, $ow_{\frac{n-3}{2}}^1$, and $ow_{\frac{n-3}{2}}^2$.

Two examples of such graphs are drawn in Figure 2. For every $n \geq 4$, the graph Pr_n is an edge-minimal 3-connected graph since it has size $\lceil \frac{3n}{2} \rceil$. To prove that Pr graphs are AP+2, we consider the following sufficient condition for a graph to be AP+2. Recall that a graph G is *Hamiltonian-connected* if G admits a Hamiltonian path with endvertices u and v for every two vertices u and v of G.

Lemma 16. *If a graph G is Hamiltonian-connected, then G is AP+2.*

Proof: The statement follows from Lemma 8 since every path P_n can be partitioned under every 2-prescription (u, v) as long as u and v are the endvertices of P_n. □

Before showing that $G = Pr_n$ is Hamiltonian-connected for every $n \geq 4$, we first introduce some notation. Let $q = \frac{n-2}{2}$ (resp. $q = \frac{n-3}{2}$) if n is even (resp. odd). Given two integers x and y in $\{1, 2, \dots, q\}$ (resp. $\{2, 3, \dots, q-1\}$) such that $x \leq y$, we denote by $P_{x,y}^{\nearrow}(G)$ and $P_{x,y}^{\searrow}(G)$ the following paths of G.



(a) Pr_{10}. (b) Pr_9.

Fig. 2: Two examples of Pr graphs.

$$P^{\nearrow}_{x,y}(G) = \begin{cases} w_x^2 w_x^1 \text{ if } x = y, \\ w_x^2 w_x^1 P^{\searrow}_{x+1,y}(G) \text{ otherwise.} \end{cases}$$

$$P^{\searrow}_{x,y}(G) = \begin{cases} w_x^1 w_x^2 \text{ if } x = y, \\ w_x^1 w_x^2 P^{\nearrow}_{x+1,y}(G) \text{ otherwise.} \end{cases}$$

The paths $P^{\nwarrow}_{x,y}(G)$ and $P^{\swarrow}_{x,y}(G)$ of G are defined analogously from right to left when $x \geq y$. For every $\alpha \in \{1, 2\}$, we additionally define $P^{\alpha,\rightarrow}_{x,y}(G)$ (resp. $P^{\alpha,\leftarrow}_{x,y}(G)$) for $x < y$ (resp. $x > y$) to be the path $w_x^\alpha w_{x+1}^\alpha \dots w_y^\alpha$ (resp. $w_x^\alpha w_{x-1}^\alpha \dots w_y^\alpha$) of G . For convenience, let us assume that $P^{\nearrow}_{x,y}(G) = P^{\searrow}_{x,y}(G) = P^{\alpha,\rightarrow}_{x,y}(G) = \emptyset$ (resp. $P^{\nwarrow}_{x,y}(G) = P^{\swarrow}_{x,y}(G) = P^{\alpha,\leftarrow}_{x,y}(G) = \emptyset$) whenever x or y does not belong to the interval above or when $x > y$ (resp. $x < y$). According to our terminology, note *e.g.* that $uP^{1,\rightarrow}_{1,4}(Pr_{10})vP^{2,\leftarrow}_{4,1}(Pr_{10})$ and $uP^{\nearrow}_{1,4}(Pr_{10})v$ are Hamiltonian paths of Pr_{10}.

We are now ready to prove that every Pr_n graph is Hamiltonian-connected, and thus AP+2 according to Lemma 16.

Theorem 17. *For every $n \geq 4$, the graph Pr_n is Hamiltonian-connected.*

Proof: Let $G = Pr_n$, and $q = \frac{n-2}{2}$ if n is even, or $q = \frac{n-3}{2}$ otherwise. Table 1 (resp. Table 2) exhibits, given two distinct vertices s and t of G, a Hamiltonian path P of G whose endvertices are s and t when n is even (resp. n is odd). In Table 1 (resp. Table 2), it is assumed that $1 \leq i \leq q$ when j is not defined (resp. $1 < i < q$), and $0 \leq i < j \leq q$ otherwise (resp. $1 < i < j < q$). Every Hamiltonian path which does not appear in these two tables can be deduced from another Hamiltonian path using the symmetries of G. □

Corollary 18. *For every $n \geq 4$, the graph Pr_n is AP+2.*

s	t	P
u	v	$uP_{1,q}^{\nearrow}(G)v$
u	w_i^1	$uP_{1,i-1}^{\nearrow}(G)w_i^2P_{i+1,q}^{2,\rightarrow}(G)vP_{q,i}^{1,\leftarrow}(G)$ if $i-1$ is even $uP_{1,i-1}^{\searrow}(G)w_i^2P_{i+1,q}^{2,\rightarrow}(G)vP_{q,i}^{1,\leftarrow}(G)$ otherwise
w_i^1	w_j^1	$P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,1}^{\nwarrow}(G)uvP_{q,j}^{\nwarrow}(G)$ if $q-j$ is even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,1}^{\nwarrow}(G)uvP_{q,j}^{\swarrow}(G)$ otherwise
w_i^1	w_j^2	$P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,1}^{\nwarrow}(G)uvP_{q,j}^{\swarrow}(G)$ if $q-j$ is even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,1}^{\nwarrow}(G)uvP_{q,j}^{\nwarrow}(G)$ otherwise

Tab. 1: Proof that Pr_n is Hamiltonian-connected for every even $n \geq 4$.

s	t	P
o	u	$oP_{1,q}^{1,\rightarrow}(G)vP_{q,1}^{2,\leftarrow}(G)u$
o	w_1^1	$ow_q^1vw_q^2P_{q-1,2}^{\nwarrow}w_1^2uw_1^1$ if q is even $ow_q^2vw_q^1P_{q-1,2}^{\swarrow}w_1^2uw_1^1$ otherwise
o	w_i^1	$ow_1^1uw_1^2P_{2,i-1}^{\nearrow}(G)w_i^2P_{i+1,q}^{2,\rightarrow}(G)vP_{q,i}^{1,\leftarrow}(G)$ if i is even $ow_1^2uw_1^1P_{2,i-1}^{\searrow}(G)w_i^2P_{i+1,q}^{2,\rightarrow}(G)vP_{q,i}^{1,\leftarrow}(G)$ otherwise
u	v	$uP_{1,q}^{2,\rightarrow}(G)oP_{1,q}^{1,\rightarrow}(G)v$
u	w_1^1	$uvP_{q,1}^{2,\leftarrow}(G)oP_{q,1}^{1,\leftarrow}(G)$
u	w_q^1	$uvP_{q,1}^{2,\leftarrow}(G)oP_{1,q}^{1,\rightarrow}(G)$
u	w_i^1	$uP_{1,i-1}^{1,\rightarrow}(G)P_{i-1,1}^{2,\leftarrow}(G)ow_q^2vw_q^1P_{q-1,i}^{\swarrow}(G)$ if $q-i$ is even $uP_{1,i-1}^{1,\rightarrow}(G)P_{i-1,1}^{2,\leftarrow}(G)ow_q^1vw_q^2P_{q-1,i}^{\nwarrow}(G)$ otherwise
w_1^1	w_1^2	$w_1^1uvP_{q,2}^{1,\leftarrow}(G)P_{2,q}^{2,\rightarrow}(G)ow_1^2$
w_1^1	w_q^1	$P_{1,q-1}^{1,\rightarrow}(G)P_{q-1,1}^{2,\leftarrow}(G)uvw_q^2ow_q^1$
w_1^1	w_q^2	$w_1^1ow_1^2uvP_{q,2}^{1,\leftarrow}(G)P_{2,q}^{2,\rightarrow}(G)$
w_i^1	w_j^1	$P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^2uw_1^1ow_q^2vw_q^1P_{q-1,j}^{\swarrow}(G)$ if i and $q-j$ are even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^1uw_1^2ow_q^2vw_q^1P_{q-1,j}^{\swarrow}(G)$ if i is odd and $q-j$ is even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^2uw_1^1ow_q^1vw_q^2P_{q-1,j}^{\nwarrow}(G)$ if i is even and $q-j$ is odd $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^1uw_1^2ow_q^1vw_q^2P_{q-1,j}^{\nwarrow}(G)$ if i and $q-j$ are odd
w_i^1	w_j^2	$P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^2uw_1^1ow_q^1vw_q^2P_{q-1,j}^{\nwarrow}(G)$ if i and $q-j$ are even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^1uw_1^2ow_q^1vw_q^2P_{q-1,j}^{\nwarrow}(G)$ if i is odd and $q-j$ is even $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^2uw_1^1ow_q^2vw_q^1P_{q-1,j}^{\swarrow}(G)$ if i is even and $q-j$ is odd $P_{i,j-1}^{1,\rightarrow}(G)P_{j-1,i}^{2,\leftarrow}(G)P_{i-1,2}^{\nwarrow}(G)w_1^1uw_1^2ow_q^2vw_q^1P_{q-1,j}^{\swarrow}(G)$ if i and $n-j$ are odd

Tab. 2: Proof that Pr_n is Hamiltonian-connected for every odd $n \geq 5$.

6 Conclusion

We summarize Corollaries 10 and 18 and Theorems 12 and 13 in this concluding theorem.

Theorem 19. *For every $k \geq 1$ and $n \geq k$, there exists an optimal AP+k graph on n vertices and $\lceil \frac{n(k+1)}{2} \rceil$ edges.*

This result does not tell much about the number of optimal AP+k graphs on n vertices for some fixed values of k and n. However, this number is upper bounded by the number of edge-minimal $(k+1)$-connected graphs with order n according to Observation 4.

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