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Detection number of bipartite graphs and cubic graphs

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For a connected graph G of order $|V(G)| \geq 3$ and a k-labelling $c : E(G) \to \{1, 2, \ldots, k\}$ of the edges of G, the *code* of a vertex v of G is the ordered k-tuple $(\ell_1, \ell_2, \ldots, \ell_k)$, where ℓ_i is the number of edges incident with v that are labelled i. The k-labelling c is *detectable* if every two adjacent vertices of G have distinct codes. The minimum positive integer k for which G has a detectable k-labelling is the *detection number* $det(G)$ of G. In this paper, we show that it is NP-complete to decide if the detection number of a cubic graph is 2. We also show that the detection number of every bipartite graph of minimum degree at least 3 is at most 2. Finally, we give some sufficient condition for a cubic graph to have detection number 3.

Keywords: some well classifying words

1 Introduction

For graph-theoretical terminology and notation, we in general follow the book of Balakrishnan and Ranganathan (2000). In this paper, we assume that the graphs G in discussion are finite, connected, undirected and simple with order $|V(G)| \geq 3$. Let $c : E(G) \to \{1, 2, \ldots, k\}$ be a labelling of the edges of G, where k is a positive integer. The *color code* of a vertex v of G is the ordered k-tuple $code_c(v) = (\ell_1, \ell_2, \ldots, \ell_k)$, where ℓ_i is the number of edges incident with v that are labelled i for $i \in \{1, 2, \ldots, k\}$. Therefore, $\ell_1 + \ell_2 + \cdots + \ell_k = d_G(v)$, the degree of v in G. The labelling c is called a *detectable coloring* of G if any pair of adjacent vertices of G have distinct color codes. The *detection number* or *detectable chromatic number* of G, denoted $det(G)$, is the minimum positive integer k for which G has a detectable k-coloring. We call G *k-detectable* if G has a detectable k-coloring.

The concept of detection number was introduced by Karoński et al. (2004), inspired by the basic problem in graph theory that concerns finding means to distinguish the vertices of a connected graph and to

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distinguish adjacent vertices of a graph, respectively, with the minimum number of colors. For a survey on vertex-distinguishing colorings of graphs, see Escuadro et al. (2007).

Karoński et al. (2004) conjectured that $det(G) \leq 3$. Addario-Berry et al. (2005) proved that: (i) $det(G) \leq 4$ and (ii) if $\chi(G) \leq 3$, then $det(G) \leq 3$. However, as observed by Khatirinejad et al. (2012), it seems NP-complete to decide if a graph is 2-detectable.

Conjecture 1.1 (Khatirinejad et al. (2012)) *It is NP-complete to decide whether a given graph is* 2*-detectable.*

As an evidence, Dudek and Wajc (2011) showed that closely related problems are NP-complete. In Section 2, we settle this conjecture by showing that deciding if a cubic graph is 2-detectable is an NP-complete problem.

On the other hand, Khatirinejad et al. (2012) believed that for a given bipartite graph, deciding if it is 2-detectable should be easy. For $m_1 \leq m_2 \leq \cdots \leq m_d$, let $\Theta(m_1, \ldots, m_d)$ be the graph constructed from d internally disjoint paths between distinct vertices x and y, in which the i-th path has length m_i. Such a graph is called a *Theta* and the two vertices $\{x, y\}$ are its *poles*. It is *bad* if $m_1 = 1$ and $m_i \equiv 1 \mod 4$ for all $2 \leq i \leq d$. Khatirinejad et al. (2012) proved that a Theta is 2-detectable if and only if it is not bad, and asked whether all bipartite graphs except the bad Thetas were 2-detectable. This was answered in the negative by Davoodi and Omoomi who gave a new family of non-2-detectable bipartite graphs, the *Theta trees*. A *Theta tree* is a graph obtained from a tree T by replacing each vertex t of $V(T)$ by a bad Theta with poles u_t and v_t and every edge st of $E(T)$ by a path P_{st} of length p_{st} between u_t and u_s and a path Q_{st} of length q_{st} between v_t and v_s such that p_{st} and q_{st} are odd and $p_{st} + q_{st} \equiv 0 \mod 4$. Hence, they raised the following question.

Problem 1.2 *Except from bad Thetas and Theta trees, is there any bipartite graph which is not* 2*-detectable?*

We partially answer to this question by showing (Theorem 3.1) that every bipartite graph with minimum degree at least 3 is 2-detectable. In particular, every cubic bipartite graph is 2-detectable.

We then restrict our attention to cubic graphs. For such graphs, by Brooks' theorem, if $G \neq K_4$, then $\chi(G) \leq 3$, and hence by the result of Addario-Berry et al. (2005) that $det(G) \leq 3$. Escuadro et al. (2008) observed for some cubic graphs that: $det(K_4) = 3$; $det(K_{3,3}) = 2$, where $K_{r,s}$ is the complete bipartite graph with partite sizes r and s; $det(C_3 \,\square\, K_2) = 3$, $det(C_4 \,\square\, K_2) = 2$, $det(C_5 \,\square\, K_2) = 3$ and if $n \geq 6$ is an integer, then $det(C_n \,\square\, K_2) = 2$, where $\square$ denotes the Cartesian product, and C_n denotes the cycle of length n. We then exhibit some infinite families of cubic graphs with detection number 3. This allow us to characterize all cubic graphs up to ten vertices according to their detection number.

2 NP-completeness for cubic graphs

The aim of this section is to prove the following theorem.

Theorem 2.1 *The following problem is NP-complete.*
Input: A cubic graph G.
Question: Is G 2-detectable?

The proof of this theorem is a reduction from MONOTONE NOT-ALL-EQUAL 3SAT, which is defined as follows:

Input: A set of clauses each having three non-negated literals.
Question: Does there exists a *suitable* truth assignment, that is such that each clause has at least one true and at least one false literal?

This problem was recently shown NP-complete by Berg and Khosrav (2012).

In order to construct gadgets and proceed with the reduction, we need some preliminaries.

The *halter* is the graph depicted Figure 1. The vertices a and b are the *ends* of the halter, and the edges aa' and bb' are its *reins*.

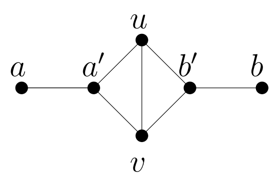

Fig. 1: The halter

Lemma 2.2 *If a halter is a subgraph of a cubic graph G and if G has a detectable 2-coloring, then the edges of the halter are colored as shown in Figure 2.*

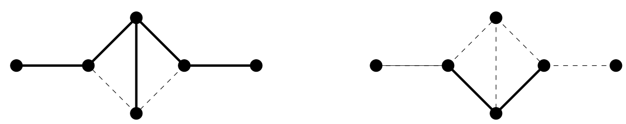

Fig. 2: The two possible colorings of a halter (Bold edges are colored 1 and dashed edges are colored 2.)

Proof: Let c be a detectable 2-coloring of G. Without loss of generality assume that $c(uv) = 1$.

If $c(ua') = c(va') = c(ub') = c(vb')$, then $code(u) = code(v)$, a contradiction.

Out of the four edges ua', va', ub' and vb', assume that exactly three are of same color. By symmetry, assume that $c(va') = c(ub') = c(vb')$. Suppose $c(ua') = 1$. If $c(aa') = 1$, then $code(a') = code(u)$, a contradiction; if $c(aa') = 2$, then $code(a') = code(v)$, a contradiction. Hence, $c(ua') = 2$. If $c(bb') = 1$, then $code(b') = code(v)$, a contradiction; if $c(bb') = 2$, then $code(b') = code(u)$, a contradiction.

Consequently, among the four edges ua', va', ub', and vb', two are of color 1 and the remaining two are of color 2.

If $c(ua') \neq c(ub')$ and $c(va') \neq c(vb')$, then $code(u) = code(v)$, a contradiction.

By symmetry, assume that $c(ua') = c(ub')$. So $c(va') = c(vb')$ and $c(ua') \neq c(va')$. Assume without loss of generality that $c(ua') = 1$. Since $code(v) = (1, 2)$, $c(aa') = c(bb') = 1$. Consequently, we have $c(aa') = c(ua') = c(uv) = c(ub') = c(bb') = 1$ and $c(va') = c(vb') = 2$. See Figure 2 (a).

Similarly, if $c(uv) = 2$, then we have $c(aa') = c(ua') = c(uv) = c(ub') = c(bb') = 2$ and $c(va') = c(vb') = 1$. See Figure 2 (b). $\square$

Lemma 2.3 *Let G be a cubic graph. If a vertex x is the end of two halters in G, then in any detectable 2-coloring of G, x has code $(3,0)$ or $(0,3)$.*

Proof: Assume for a contradiction, that the code of x is neither $(3,0)$ nor $(0,3)$. By symmetry, we may assume that x has code $(2,1)$. Therefore x is incident to two edges colored 1 and thus at least one of the rein e incident to it is colored 1. Therefore by Lemma 2.2, the neighbor of x through e has code $(2,1)$, a contradiction. □

We will now give the proof of Theorem 2.1, which stated that it is an NP-complete problem to decide if a cubic graph is 2-detectable.

Proof of Theorem 2.1: Let $\mathcal{C}$ be a collection of clauses with three non-negated variables over a set U of variables. We construct a cubic graph $G = G(\mathcal{C}, U)$ as follows.

For every clause $C \in \mathcal{C}$, we create a vertex $v(C)$.

For every variable $u \in U$, let $\mathcal{C}_u$ be the set of clauses in which one of the two literals u and $\bar{u}$ appears. We construct a *variable gadget* associated to u, by considering a cycle on the $|\mathcal{C}_u|$ vertices $\{p(u,C) \mid C \in \mathcal{C}_u\}$ and replacing each edge ab of this cycle by a halter with ends a and b.

Now for each variable u and clause $C \in \mathcal{C}_u$, we connect $v(C)$ and $p(u,C)$ with an edge if the literal u appears in C.

Clearly, the resulting graph G is cubic. Let us now prove that G is 2-detectable if and only if $\mathcal{C}$ admits a suitable assignment.

Suppose first that G admits a detectable 2-coloring. Let us establish few claims. The first one follows directly from Lemmas 2.2 and 2.3.

Claim 1 *In the variable gadget of every variable u, all the $p(u,C)$ have the same code, which is either $(3,0)$ or $(0,3)$.*

Claim 2 *For every clause C, the three neighbours of $v(C)$ do not have the same code.*

Proof: By construction, the neighbours of $v(C)$ are all ends of two halters, and so have code in $\{(3,0),(0,3\}$ by Lemma 2.3. Assume for a contradiction, that they all have the same code, say $(3,0)$, then the three edges incident to $v(C)$ are colored 1 and the code of $v(C)$ is also $(3,0)$, a contradiction. □

With these claims in hand, we can now prove that $\mathcal{C}$ admits a suitable assignment. Let ϕ be the truth assignment defined by $\phi(u) = true$ if all the $p(u,C)$ of its variable gadget have code $(3,0)$, and $\phi(u) = false$ if all the $p(u,C)$ of its variable gadget have code $(0,3)$. This assignment is well-defined because of Claim 1. Now, by Claim 2 the three neighbours of $v(C)$, which corresponds to the three literals of C do not have the same code. This implies that the corresponding literals do not have the same value. Therefore the truth assignment ϕ is suitable.

Conversely, suppose that $\mathcal{C}$ admits a suitable truth assignment ϕ. For each variable u, color the edges incident to each $p(u,C)$ with 1 if $\phi(u) = true$ and with 2 if $\phi(u) = false$. Similarly, color the edges incident to $p(\bar{u},C)$ with 1 if $\phi(\bar{u}) = true$ and with 2 if $\phi(\bar{u}) = false$. It can easily be seen that such a coloring extends using the colorings of halter shown in Figure 2 to variable gadgets, so that no two adjacent vertices in these gadget have the same code. It remains to show that every vertex $v(C)$ has a code distinct from its neighbours. But since ϕ was suitable, at least one literal is false so the edge between $v(C)$ and the vertex corresponding to this literal is colored 2, and at least one literal is true so the edge between $v(C)$ and the vertex corresponding to this literal is colored 1. This implies that the code of $v(C)$ is in $\{(2,1),(1,2)\}$. But in our coloring the code of the neighbours of $v(C)$ are either $(3,0)$ or $(0,3)$. Hence we have a detectable 2-coloring. □

3 Bipartite graphs

In this section, our aim is to prove the following theorem.

Theorem 3.1 *Every bipartite graph with minimum degree at least* 3 *is* 2*-detectable.*

If any one of the parts of the bipartite graph have even number of vertices, Theorem 3.1 is an immediate consequence of Theorem 3.3 of Khatirinejad et al. (2012). For sake of completeness, we give its proof here.

Theorem 3.2 (Khatirinejad et al. (2012)) *If $G = ((A,B),E)$ is a connected bipartite graph with $|B|$ even, then G admits an edge labelling $c : E(G) \to \{1,2\}$ such that every vertex in A is incident to an even number of edges labelled* 1 *and every vertex in B is incident to an odd number of edges labelled* 1*. In particular, $det(G) = 2$.*

Proof: Set $B = \{b_1, b_2, \ldots, b_{2p}\}$. For every i, $1 \le i \le p$, let P_i be a path joining b_{2i-1} to b_{2i}.

We start with all edges labelled 2. Then, for each i, $1 \le i \le p$, one after antoher, we exchange the labels along P_i. Hence at the end of this process, every vertex of A is incident to an even number of edges labelled 1 and every vertex in B is incident to an odd number of edges labelled 1. □

To complete the proof of Theorem 3.1, we need some preliminaries.

Let G be a graph. The *closed neighborhood* of vertex v is the set $N[v] := N(v) \cup \{v\}$. For a set S of vertices, we set $N(S) := \bigcup_{s \in S} N(s)$ and $N[S] := \bigcup_{s \in S} N[s]$; and $G[S]$ denotes the subgraph induced by S. Two vertices x and y are *twins* if $N(x) = N(y)$. Hence *a set of twins* is a set S such that all vertices in S are pairwise twins. In particular, any singleton is a set of twins.

Lemma 3.3 *Let $G = ((A,B),E)$ be a connected bipartite graph. Then there exists a nonempty set of twins S such that $G - N[S]$ is connected.*

Proof: If for some vertex v of G, $G - N[v]$ is connected, then we have the result with $S = \{v\}$. So assume that for every vertex v of G, $G - N[v]$ is not connected. Let us choose a vertex v of G such that $G - N[v]$ has a component of largest possible size. Moreover, we choose v with largest possible degree among such vertices. Without loss of generality assume that $v \in A$.

By assumption, $G - N[v]$ is not connected and let C be the vertex set of a component of $G - N[v]$ of largest size. Then every vertex u in $N(v)$ is adjacent to a vertex in C, for otherwise $N[C]$ is included in a component of $G - N[u]$, which contradicts our choice of v since $|E(G[N[C]])| > |E(G[C])|$ as G is connected. Hence $N[C] = C \cup N(v)$.

Set $S = (V(G) \setminus N[C]) \cap A$. Let w be a vertex in $S \setminus \{v\}$. Then $N(v) \subseteq N(w)$, for otherwise $N(v) \setminus N(w)$ would be nonempty and in the same component as $G[C]$ in $G - N[w]$, contradicting our choice of v. Hence, $G[C]$ is a component of $G - N[w]$, and so by our choice of v, $d(v) \ge d(w)$. Thus $N(v) = N(w)$, that is w and v are twins.

Therefore S is a set of twins and $G - N[S]$ is the component $G[C]$. □

Proof of Theorem 3.1: It is clearly enough to prove it for connected graphs. Let $G = ((A, B), E)$ be a connected bipartite graph with minimum degree at least 3.

If $|B|$ is even, then we have the result by Theorem 3.2. Symmetrically, we have the result if $|A|$ is even. Thus we may assume that $|A|$ and $|B|$ are odd.

By Lemma 3.3, there is a set S of twins such that $G - N[S]$ is connected. Free to rename A and B, we may assume that $S \subseteq A$. Set $k := |N(S)|$. If k is odd, then set $H := G - N[S]$ and $X := N[S]$. If k is even, let u be a vertex in $N[S]$ which is adjacent to a vertex t in $G - N[S]$; then let H be the graph obtained from $G - N[S]$ by adding the vertex u and the edge ut, and set $X := N[S] \setminus \{u\}$.

In both cases, H is bipartite and $V(H) \cap B$ is $B \setminus X$ so has even size. Therefore, by Theorem 3.2, H admits an edge labelling $c : E(H) \to \{1, 2\}$ such that every vertex in $A \setminus S$ is incident to an even number of edges labelled 1 and every vertex in $B \setminus X$ is incident to an odd number of edges labelled 1. Observe moreover that, when k is even, the edge ut is necessarily labelled 1.

Pick a vertex $v \in S$ and extend c by labelling 1 to all the edges from v to X and all remaining edges incident to a vertex in $N[S]$ with 2. Then, for every vertex b in $B \setminus N[S]$, $code_c(b) = (\alpha, d_G(b) - \alpha)$, $\alpha \equiv 1 \mod 2$, for every vertex a in $A \setminus \{v\}$, $code_c(a) = (\beta, d_G(a) - \beta)$, $\beta \equiv 0 \mod 2$, for every vertex x in $N(S)$, $code_c(x) = (1, d_G(x) - 1)$, and $code_c(v)$ equals $(k, 0)$ if k is odd and equals $(k-1, 1)$ if k is even. Hence c is a detectable 2-coloring because $k \geq 3$. □

4 Cubic graphs with detection number 3

In this section, our aim is to exhibit some infinite families of cubic graphs with detection number 3.

The proofs of all statement in this section are careful case analysis. Therefore, we omit them in the journal version. They can be found in the full version: Havet et al. (2012).

Lemma 4.1 *Let I' be the graph depicted in Figure 3. If I' is a subgraph of a cubic graph G and if G has a detectable 2-coloring, then the edges of I receive both colors and* $\{code(x), code(y), code(z)\}$ *is either* $\{(3, 0), (2, 1), (1, 2)\}$ *or* $\{(0, 3), (2, 1), (1, 2)\}$ *(See Figure 4).*

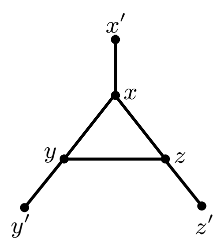

Fig. 3: The graph I'

Fig. 4: Possible colorings of I'

Lemma 4.2 *Let M be the graph depicted in Figure 5. If M is a subgraph of a cubic graph G and if G has a detectable 2-coloring, then the edges of M receive colors shown in any one of the Figure 6.*

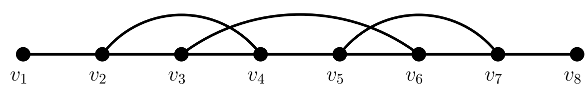

Fig. 5: The graph M

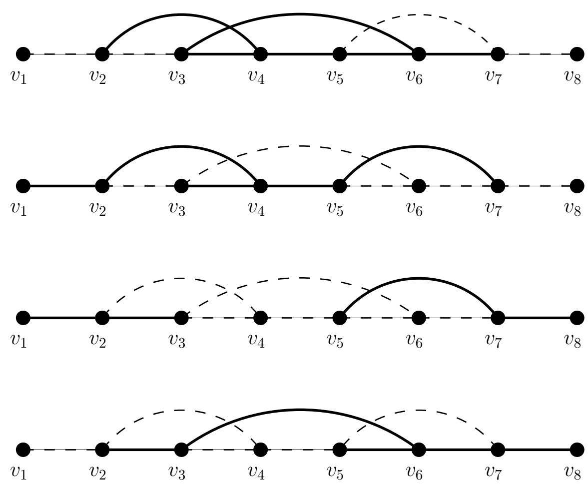

Fig. 6: Possible colorings of M

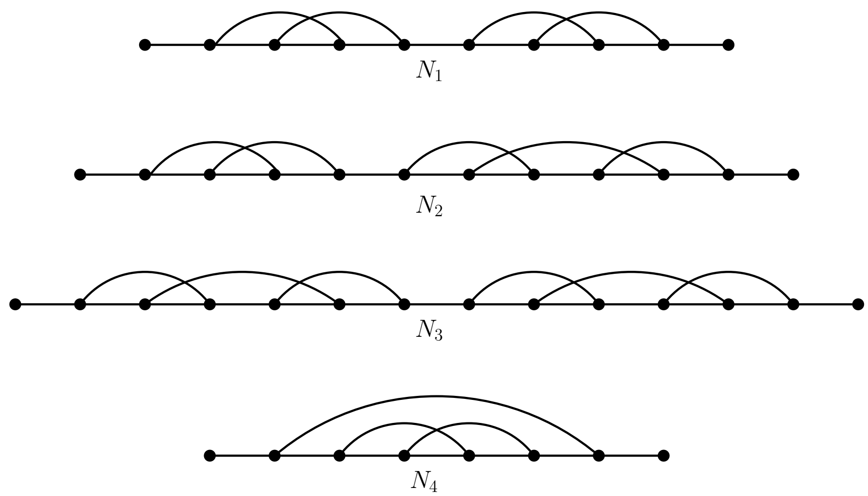

Fig. 7: The graphs $N_i, 1 \leq i \leq 4$

Theorem 4.3 *If a cubic graph G contains one of the N_i for $i \in \{1, 2, 3, 4\}$ depicted Figure 7, then $det(G) = 3$.*

Theorem 4.4 *Let w_1w_2 be an edge of a connected cubic graph G. Suppose $G - \{w_1, w_2\}$ contains four disjoint subgraphs J_1, J_2, J_3, J_4, where $J_i \in \{K_4 - e, M - \{v_1, v_8\}\}$ for $i \in \{1, 2, 3, 4\}$, and if w_1 is adjacent to a degree-2 vertex z_1 of J_1 and a degree-2 vertex z_2 of J_2, and w_2 is adjacent to a degree-2 vertex z_3 of J_3 and a degree-2 vertex z_4 of J_4, in G, then $det(G) = 3$.*

Now we construct a family of cubic graphs L_n, $n \geq 2$, with $det(L_n) = 3$ as follows: Begin with C_{5n}, the cycle of length $5n$, say, $v_0v_1v_2 \ldots v_{5n-1}v_0$; add chords of distance 2, $v_{5r+1}v_{5r+3}$ and $v_{5r+2}v_{5r+4}$ for $r \in \{0, 1, 2, \ldots, n-1\}$. If n is even, pair the vertices in $\{v_0, v_5, v_{10}, \ldots, v_{5n-5}\}$ in any order and join these pairs as edges; if n is odd, except three vertices in $\{v_0, v_5, v_{10}, \ldots, v_{5n-5}\}$, pair the remaining vertices in any order and join these pairs as edges and add a new vertex v and join v to the omitted three vertices.

By Theorem 4.3 with $i = 4$ and Theorem 4.4, for $n \geq 4$, $det(L_n) = 3$. One can also show that $det(L_2) = 3$, and $det(L_3) = 3$.

Theorem 4.5 *For each n, there exists a cubic graph of order $5n$ satisfying $det(G) = 3$.*

There are 5 nonisomorphic cubic graphs on 8 vertices (see Meringer (1999)). Three of them have detection number 3 (see Khatirinejad et al. (2012)).

There are 19 nonisomorphic cubic graphs on 10 vertices (see Meringer (1999)). Out of these, exactly 6 have detection number 3. It is known that $det(C_5 \Box K_2) = 3$, see Escuadro et al. (2008). The remaining 5 graphs are shown in Figure 8. Observe that the graph in Figure 8 (a) is L_2.

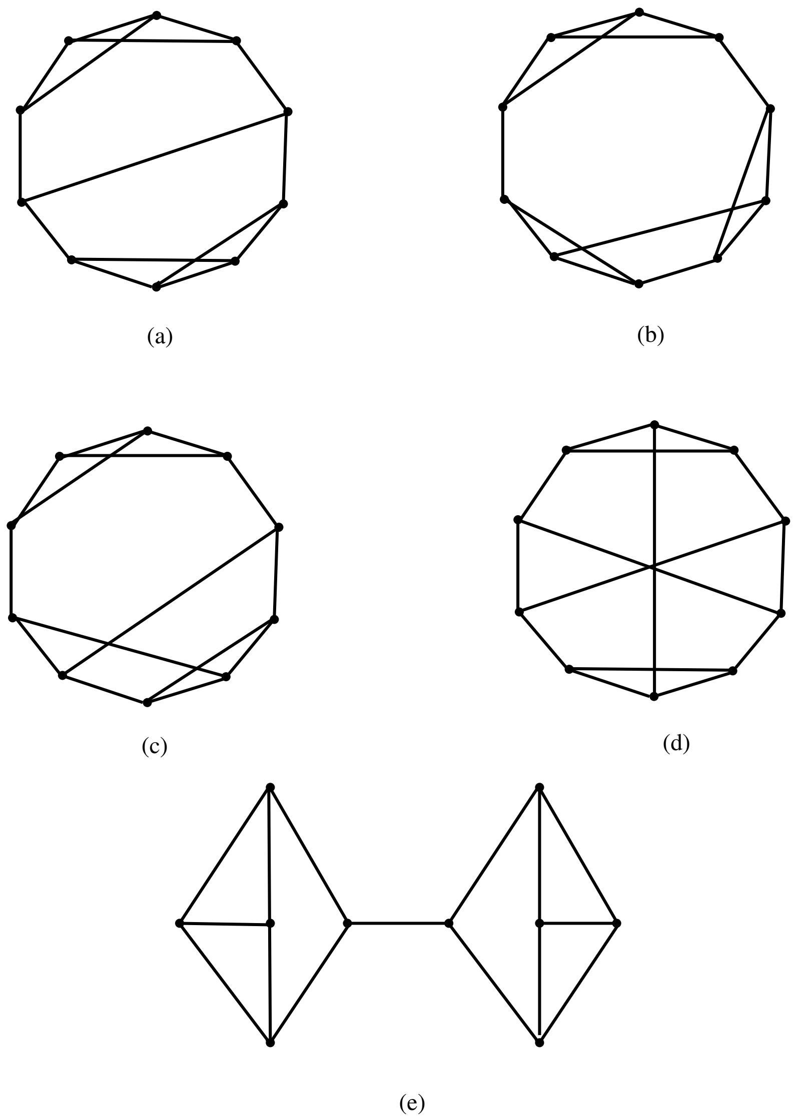

Fig. 8: Cubic graphs on 10 vertices with detection number 3

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