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Anti-factors of Regular Bipartite Graphs

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Let $G = (X, Y; E)$ be a bipartite graph, where X and Y are color classes and E is the set of edges of G. Lovász and Plummer asked whether one can decide in polynomial time that a given bipartite graph $G = (X, Y; E)$ admits a 1-anti-factor, that is subset F of E such that $d_F(v) = 1$ for all $v \in X$ and $d_F(v) \neq 1$ for all $v \in Y$. Cornuéjols answered this question in the affirmative. Yu and Liu asked whether, for a given integer $k \geq 3$, every k-regular bipartite graph contains a 1-anti-factor. This paper answers this question in the affirmative.

Keywords: anti-factor, bipartite graph

1 Introduction

In this paper, we consider finite undirected graphs without loops and multiple edges. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A graph G' is called a *spanning subgraph* of G if $V(G) = V(G')$ and $E(G') \subseteq E(G)$. The degree of a vertex x in G is denoted by $d_G(x)$, and the set of vertices adjacent to x in G is denoted by $N_G(x)$. For $x \in V(G)$, we write $N_G[x] = N_G(x) \cup \{x\}$. For $xy \notin E(G)$, $G + xy$ denotes the graph with vertex set $V(G) \cup \{x, y\}$ and edge set $E(G) \cup \{xy\}$. For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$ and $G - S = G[V(G) - S]$. For two disjoint subsets $S, T \subseteq V(G)$, let $E_G(S, T)$ denote the set of edges of G joining S to T and let $e_G(S, T) = |E_G(S, T)|$. For a positive integer r, let $[r] = \{0, 1, \ldots, r\}$. Let $c(G)$ denote the number of connected components of G.

Let G be a graph, and for every vertex $x \in V(G)$, let $H(x)$ be a set of integers. An H-*factor* is a spanning graph F such that

$$d_F(x) \in H(x) \quad \text{for all } x \in V(G). \tag{1}$$

A *matching* of a graph is a set of edges such that no two edges share a vertex in common. A *perfect matching* of a graph is a matching covering all vertices. Clearly, a matching (or perfect matching) of a graph is also a $\{0, 1\}$-factor (1-factor, respectively). On 1-factors of bipartite graphs, Hall obtained the following result.

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Theorem 1.1 (Hall (1935)) *Let $k \geq 1$ be an integer. Every k-regular bipartite graph contains a 1-factor.*

A spanning subgraph F of bipartite graph $G = (X, Y; E)$ is called a *1-anti-factor* if $d_F(x) = 1$ for all $x \in X$ and $d_F(y) \neq 1$ for all $y \in Y$. Lovász and Plummer (see Lovász and Plummer (1986), Page 390) proposed the following problem: can one decide in polynomial time whether a given bipartite graph admits a 1-anti-factor?

A set $\{h_1, h_2, \ldots, h_m\}$ of increasing integers is called *allowed* (see Lovász (1972)) if $h_{i+1} - h_i \leq 2$ for all $1 \leq i \leq m-1$. Let $H: V(G) \to 2^{\mathbb{Z}}$ be a function. If $H(v)$ is allowed for each vertex v, then we call H an *allowed function.* The H-factor problem, i.e., determining whether a graph contains H-factors, is NP-complete in general. For the case when H is an allowed function, Lovász (1972) gave a structural description. In fact, Lovász introduced the definition of negative degree by giving a 2-end-coloring of edges. By defining the negative degree for a general graph G, Lovász may study the degree constrained factor problems of mixed graphs (including multiple edges, loops, directed edges, two way edges($\leftrightarrow$ or $\rightarrow\leftarrow$ (one edge))). Cornuéjols (1988) provided the first polynomial time algorithm for H-factor problem with H being allowed and so give an affirmative answer to the problem proposed by Lovász and Plummer.

A classical approach, due to Tutte, for studying f-factor problems is to look for reductions to the simpler matching problem. For studying H-factor problems, where every gap of $H(v)$ has the same parity, Szabó (2009) used a reduction to local K_2 and factor-critical subgraph packing problem of Cornuéjols et al. (1981/1982). The idea of reducing a degree prescription to other matching problems appeared in works of Cornuéjols (1988). Cornuéjols (1988) and Loebl (1993) considered reductions to the edge and triangle packing problem, which can be translated into 1-anti-factor problem. Let G be a graph, $U = V(G)$ and let W be the set of all edges and triangles of G. Let $G' = (U, W; E')$ be a bipartite graph, where $E' = \{xy \mid x \in U, y \in W \text{ and } x \in V(y)\}$. Then G' has a 1-anti-factor if and only if G contains a set of vertex-disjoint edges and triangles covering $V(G)$.

Shirazi and Verstraëte (2008) showed that every graph G contains an H-factor when $|\{1, \ldots, d_G(v)\} - H(v)| = 1$ holds for all $v \in V(G)$. Addario-Berry et al. (2007) showed that every graph G contains a factor F such that $d_F(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1,\}$ for all $v \in V(G)$, where $d_G(v)/3 \leq a_v^- \leq d_G(v)/2 - 1$ and $d_G(v)/2 \leq a_v^+ \leq 2d_G(v)/3$. Addario-Berry et al. (2008) slightly improved the result in Addario-Berry et al. (2007) and obtained a similar result for bipartite graphs. For more results on non-consecutive H-factor problems of graphs, we refer readers to Lu (2016); Lu et al. (2013); Thomassen et al. (2016).

However, there is no nice formula to determine whether a bipartite graph contains a 1-anti-factor. So it is interesting to classify bipartite graphs with 1-anti-factors. Yu and Liu (see Yu and Liu (2009), Page 76) asked whether every connected r-regular bipartite graph contains a 1-anti-factor. In this paper, we give an affirmative answer to Yu and Liu's problem and obtain the following result.

Theorem 1.2 *Let $k \geq 3$ be an integer. Every k-regular bipartite graph contains a 1-anti-factor.*

The rest of the paper is organized as follows. In Section 2, we introduce Lovász's H-Factor Structure Theorem that is needed in the proof of Theorem 1.3. The proof of Theorem 1.2 will be presented in Section 3.

2 Lovász's H-Factor Structure Theorem

Let F be a spanning subgraph of $G = (V, E)$ and let $H : V(G) \to 2^{\mathbb{Z}}$ be an allowed function. Following Lovász (1972), one may measure the "deviation" of F from the condition (1) by $\boldsymbol{\nabla_H(F, G)} :=$

$\sum_{v\in V(G)} \min\{|d_F(v) - h| : h \in H(v)\}$. Moreover, let $\boldsymbol{\nabla_H(G)} = \min\{\nabla_H(F, G) :\ F$ is a spanning subgraph of $G\}$. $\nabla_H(G)$ is called *deficiency* of G with respect to the function H. The subgraph F is said to be H-*optimal* if $\nabla_H(F, G) = \nabla_H(G)$. It is clear that F is an H-factor if and only if $\nabla_H(F, G) = 0$, and any H-factor (if exists) is H-optimal. We study H-factors of graphs based on Lovász's structural description to the degree prescribed factor problem.

For $v \in V$, we denote by $\boldsymbol{IH(v)}$ the set of degrees of v in all H-optimal spanning subgraphs of G, i.e., $IH(v) := \{d_F(v) :$ F is an H-optimal spanning subgraph of $G\}$. Based on the relation of the sets $IH(v)$ and $H(v)$, one may partition the vertex set V into four classes:

$$\begin{aligned}
C_H(G) &:= \{v \in V : IH(v) \subseteq H(v)\},\\
A_H(G) &:= \{v \in V - C_H(G) : \min IH(v) \geq \max H(v)\},\\
B_H(G) &:= \{v \in V - C_H(G) : \max IH(v) \leq \min H(v)\},\\
D_H(G) &:= V - C_H(G) - A_H(G) - B_H(G).
\end{aligned}$$

When there is no confusion, we omit the reference to G. It is clear that the 4-tuple (A_H, B_H, C_H, D_H) is a partition of V. A graph G is said to be H-*critical* if it is connected and $D_H = V$. By the definition of A_H, B_H, C_H the following observations hold:

(*) for every $v \in A_H$, there exists an H-optimal graph F such that $d_F(v) > \max H(v)$;

(**) for every $v \in B_H$, there exists an H-optimal graph F such that $d_F(v) < \min H(v)$.

We will need the following results of Lovász (1972).

Lemma 2.1 (Lovász (1972)) *Let G be a simple graph and let $H : V(G) \to 2^{\mathbb{Z}}$ be an allowed function. Let $v \in D_H$.*

(a) *$IH(v)$ consists of consecutive integers.*

(b) *$IH(v) \cap H(v)$ contains no consecutive integers.*

Let R be a connected induced subgraph of G. Let $H_R : V(R) \to 2^{\mathbb{Z}}$ be a set function such that $H_R(x) = H(x)$ for all $x \in V(R)$.

Lemma 2.2 (Lovász (1972)) *Let G be a graph and let $H : V(G) \to 2^{\mathbb{Z}}$ be an allowed function.*

(a) $\nabla_H(G) = c(G[D_H]) + \sum_{v\in B_H}(\min H(v) - d_{G-A_H}(v)) - \sum_{v\in A_H} \max H(v)$.

(b) *If $B_H = \emptyset$, then every connected component R of $G[D_H]$ is H_R-critical.*

(c) $E_G(C_H, D_H) = \emptyset$.

(d) *If G is H-critical, then $\nabla_H(G) = 1$.*

3 The Proof of Theorem 1.2

Lemma 3.1 *Let $p \geq 2$ be an integer. Let $G = (X, Y; E)$ be a bipartite graph. Let $H : V(G) \to 2^{\mathbb{Z}}$ such that $H(y) = [\max\{d_G(y), p\}] - \{1\}$ for all $y \in Y$ and $H(x) = \{-1, 1\}$ for all $x \in X$. Then $A_H \subseteq X$ and $B_H = \emptyset$.*

Proof: Firstly, we show that $B_H = \emptyset$. Suppose that $B_H \neq \emptyset$ and let $v \in B_H$. By the definition of B_H, if $v \in X$, then $\max IH(v) \leq \min H(v) = -1$, which is impossible. Thus we may assume that $v \in Y$. This implies that $0 \leq \max IH(v) \leq \min H(v) = 0$. Hence $IH(v) = \{0\} \subseteq H(v)$, which implies $v \in C_H$, a contradiction.

Next we show that $A_H \subseteq X$ by contradiction. Suppose that there exists a vertex $y \in A_H - X$. Since $p \geq 2$, by the definition of set A_H, we have that $d_G(y) \geq \max IH(y) \geq \min IH(y) \geq \max H(y) \geq d_G(y)$. Thus we may infer that $IH(y) = \{d_G(y)\} \subseteq H(y)$, which implies that $y \in C_H$ by the definition, a contradiction. This completes the proof. $\square$

Lemma 3.2 *Let $p \geq 2$ be an integer. Let $G = (X, Y; E)$ be a bipartite graph and let $H : V(G) \to 2^{\mathbb{Z}}$ such that $H(y) = [\max\{d_G(y), p\}] - \{1\}$ for all $y \in Y$ and $H(x) = \{-1, 1\}$ for all $x \in X$. If G is H-critical, then the following properties hold.*

- (i) *$G - x$ contains an H_{G-x}-factor for all $x \in X$;*
- (ii) *$IH(u) \subseteq \{0, 1, 2\}$ for all $u \in V(G)$;*
- (iii) *$|X|$ is odd;*
- (iv) *Let $y \in Y$ such that $d_G(y) \geq 3$. Then there exist three vertices $x_1, x_2, x_3 \in N_G(y)$ such that $\nabla_{H_{G'}}(G') = 2$, where $G' = G - \{x_1, x_2, x_3, y\}$.*

Proof: Let G be H-critical. By the definition of H-critical graph and Lemma 2.2 (d), we have that $D_H = V(G)$ and $\nabla_H(G) = 1$. For any $x \in X$, by the definition of D_H, there exists an H-optimal subgraph F of G such that $d_F(x) = 0$ and $d_F(w) \in H(w)$ for all $w \in V(G) - \{x\}$. Hence $G - x$ contains an H_{G-x}-factor. This completes the proof of (i).

Next we show (ii). Suppose that there exists a vertex $u \in V(G)$ and an integer $r \geq 3$ with $r \in IH(u)$. Since $\nabla_H(G) = 1$ and $H(x) = \{-1, 1\}$ for any $x \in X$, we have $u \in Y$. From the definition of D_H, we may infer that $IH(u) - H(u) \neq \emptyset$. Recall that $H(u) = [\max\{d_G(u), p\}] - \{1\}$. Thus we have $1 \in IH(u)$. By Lemma 2.1 (a), $IH(u)$ is an interval, which implies $\{2, 3\} \subseteq IH(u)$. Then we have $\{2, 3\} \subseteq IH(u) \cap H(u)$, contradicting to Lemma 2.1 (b). This completes the proof of (ii).

Given $x \in X$, since $x \in D_H = V(G)$, we may choose an H-optimal subgraph F of G such that $d_F(x) = 0$. Note that $\nabla_H(G) = 1$. Thus we have $d_F(w) \in H(w)$ for all $w \in V(G) - \{x\}$. Since F is bipartite,

$$\sum_{y \in Y} d_F(y) = e_F(X, Y) = \sum_{x \in X} d_F(x) = |X| - 1. \tag{2}$$

By (ii), we have that $d_F(y) \in \{0, 2\}$ for all $y \in Y$. So we have that $\sum_{y \in Y} d_F(y)$ is even. By (2), $|X|$ is odd. This completes the proof of (iii).

Now we show that (iv) holds. Let F be an H-optimal subgraph of G such that $d_F(y) = 1$ and let $N_F(y) = \{x\}$. Since $\nabla_H(G) = 1$ and $d_F(y) = 1 \notin H(y)$, we have $d_F(w) \in H(w)$ for all $w \in V(G) - \{y\}$. Let $x_2, x_3 \in N_G(y) - x$. Then we have that $d_{F+x_2y+x_3y}(y) = 3 \in H(y)$. One can see that $d_{F+x_2y+x_3y}(w) = d_F(w) \in H(w)$ for all $w \in V(G) - \{x_2, x_3, y\}$ and $d_{F+x_2y+x_3y}(x_i) = 2$ for $i \in \{2, 3\}$. Set $G' = G - \{y, x, x_2, x_3\}$. Let $y_i \in N_F(x_i) - \{y\}$ for $i \in \{2, 3\}$. (Note that $y_2 = y_3$ is possible.) Thus we have $d_{F-\{y,x,x_2,x_3\}}(w) \in H(w) = H_{G'}(w)$ for all $w \in V(G') - y_2 - y_3$. Recall that $d_F(y_i) \in H_{G'}(y_i)$ for $i \in \{2, 3\}$. One can see that

$$\nabla_{H_{G'}}(F - \{y, x, x_2, x_3\}; G') \leq 2.$$

Hence we have

$$\nabla_{H_{G'}}(G') \leq 2.$$

Since G contains no H-factors, we have

$$\nabla_{H_{G'}}(G') \geq 1.$$

If $\nabla_{H_{G'}}(G') = 1$, let F' be an $H_{G'}$-optimal subgraph of G', then $F' \cup \{xy, x_2y, x_3y\}$ is also an H-optimal subgraph of G, which implies $3 \in IH(y)$, contradicting to (ii). This completes the proof. $\square$

Theorem 3.3 *Let $p \geq 2$ be an integer. Let $G = (X, Y, E)$ be a bipartite graph and let $H : V(G) \to 2^{\mathbb{Z}}$ such that $H(y) = [\max\{d_G(y), p\}] - \{1\}$ for all $y \in Y$ and $H(x) = \{-1, 1\}$ for all $x \in X$. Then G contains an H-factor if and only if for any subset $S \subseteq X$, we have*

$$q(G - S) \leq |S|, \tag{3}$$

where $q(G - S)$ denotes the number of connected components R of $G - S$, such that R is H_R-critical.

Proof: Firstly, we prove the necessity. Suppose that G contains an H-factor F. Let $R_1, \ldots, R_q$ denote these H_R-critical components of $G - S$. Since R_i contains no H_{R_i}-factors, every H-factor of G contains at least an edge from R_i to S. Thus

$$q(G - S) \leq \sum_{x \in S} d_F(x) = |S|,$$

which implies $q(G - S) \leq |S|$.

Next, we prove the sufficiency. Suppose that G contains no H-factors. Let A_H, B_H, C_H, D_H be defined as in Section 2. By Lemma 3.1, $A_H \subseteq X$ and $B_H = \emptyset$.

By Lemma 2.2 (a), we have

$$\begin{aligned} 0 < \nabla_H(G) &= c(G[D_H]) + \sum_{v \in B_H} (\min H(v) - d_{G-A_H}(v)) - \sum_{v \in A_H} \max H(v) \\ &= c(G[D_H]) - |A_H|, \end{aligned}$$

i.e.,

$$c(G[D_H]) > |A_H|. \tag{4}$$

By Lemma 3.1, we have $B_H = \emptyset$. By Lemma 2.2 (b), every connected component R of $G[D_H]$ is also H_R-critical. Then, by (5),

$$q(G - A_H) \geq c(G[D_H]) > |A_H|.$$

This completes the proof. □

From the proof of Theorem 3.3 and Lemma 2.2 (b), one can see the following result.

Lemma 3.4 *Let $p \geq 2$ be an integer. Let $G = (X, Y, E)$ be a bipartite graph and let $H : V(G) \to 2^{\mathbb{Z}}$ such that $H(y) = [\max\{d_G(y), p\}] - \{1\}$ for all $y \in Y$ and $H(x) = \{-1, 1\}$ for all $x \in X$. If G contains no H-factors, then*

$$\nabla_H(G) = c(G[D_H]) - |A_H|, \tag{5}$$

where every connected component R of $G[D_H]$ is H_R-critical and also a connected component of $G - A_H$.

Lemma 3.5 *Let $k \geq 2$ be an integer. Let $G = (X, Y; E)$ be a connected k-regular bipartite graph and let $H : V(G) \to 2^{\mathbb{Z}}$ such that $H(y) = [k] - \{1\}$ for all $y \in Y$ and $H(x) = \{-1, 1\}$ for all $x \in X$. Then either G contains an H-factor or G is H-critical.*

Proof: Suppose that G contains no H-factors and is not H-critical. By Lemma 3.1, we have that

$$B_H = \emptyset \text{ and } A_H \subseteq X. \tag{6}$$

Since G is not H-critical, we have $D_H \neq V(G)$. Thus we infer that $A_H \neq \emptyset$, otherwise, $C_H = V(G) - D_H \neq \emptyset$ and by Lemma 2.2 (c), $E_G(C_H, D_H) = \emptyset$, a contradiction since G is connected.

Recall that H contains no H-factors. By Lemmas 3.1 and 3.4, we have $B_H = \emptyset$, $A_H \subseteq X$ and

$$0 < \nabla_H(G) = c(G[D_H]) - |A_H|. \tag{7}$$

Let $R_1, \ldots, R_q$ denote connected components of $G - A_H$, where $q = c(G - A_H)$. Since G is a connected regular bipartite graph and $A_H \subseteq X$, we have $|X| = |Y|$ and every connected component R of $G - A_H$ satisfies $|V(R) \cap X| < |V(R) \cap Y|$. So we have

$$qk \leq k \sum_{i=1}^{q} (|V(R_i) \cap Y| - |V(R_i) \cap X|) = \sum_{i=1}^{q} e_G(V(R_i), A_H) = \sum_{x \in A_H} d_G(x) = k|A_H|,$$

which implies

$$c(G[D_H]) \leq q = c(G - A_H) \leq |A_H|,$$

contradicting to (7). This completes the proof. □

Let $\mathcal{H}$ be the set of graphs G, which satisfies the following properties:

(a) G is a connected bipartite graph with color classes X, Y;

(b) $|X| = |Y| - 1$;

(c) $d_G(x) = 3$ for every vertex $x \in X$ and $d_G(y) \leq 3$ for every vertex $y \in Y$.

Lemma 3.6 *If $G \in \mathcal{H}$, then G is not H-critical, where $H : V(G) \to 2^{\mathbb{Z}}$ is a function such that $H(x) = \{-1, 1\}$ for all $x \in X$ and $H(y) = \{0, 2, 3\}$ for all $y \in Y$.*

Proof: Suppose that the result does not hold. Let $G \in \mathcal{H}$ be an H-critical graph with the smallest order. By Lemma 3.2 (iii), $|X|$ is odd. Recall that $|X| = |Y| - 1$ and $d_G(x) = 3$ for all $x \in X$. Hence $|X| + 1 = |Y| \geq 4$ and there exists $y \in Y$ such that $d_G(y) = 3$. If $|Y| = 4$, then $|X| = 3$ and the spanning subgraph of G with edge set $\{xy \mid x \in N_G(y)\}$ is an H-factor, a contradiction. Hence we may assume that $|X| \geq 5$.

Let $N(y) = \{x_1, x_2, x_3\}$ and $G' = G - N[y]$. Let $H' = H_{G'}$. By Lemma 3.2 (iv), we have $\nabla_{H'}(G') = 2$. Let $A' := A_{H'}(G')$, $B' := B_{H'}(G')$, $C' := C_{H'}(G')$ and $D' = D_{H'}(G')$. By Lemma 3.1, $B' = \emptyset$. By Lemma 3.4, we have

$$\nabla_{H'}(G') = c(G'[D']) - |A'| = 2. \tag{8}$$

Now we show that $G'[D']$ contains a connected component R such that $R \in \mathcal{H}$, which contradicts to the choice of G since R is H_R-critical and $|V(R)| < |V(G)|$. Let $q := c(G' - A')$. Let $R_1, \ldots, R_q$ denote the connected components of $G' - A'$. Note that for every connected component R of $G - A'$, $d_R(x) = 3$ for all $x \in V(R) \cap X$. So we have $|V(R) \cap X| < |V(R) \cap Y|$. Recall that $|X| = |Y| - 1$. Moreover, one can see that $|X| = \sum_{i=1}^{q} |V(R_i) \cap X| + |A'| + 3$ and $|Y| = \sum_{i=1}^{q} |V(R_i) \cap Y| + 1$. So we may infer that

$$\sum_{i=1}^{q} |V(R_i) \cap X| + |A'| + 3 = \sum_{i=1}^{q} |V(R_i) \cap Y| \geq \sum_{i=1}^{q} |V(R_i) \cap X| + q, \tag{9}$$

i.e.,

$$q \leq |A'| + 3. \tag{10}$$

Since $E_{G'}(C', D') = \emptyset$, combining (8), we have $q \geq c(G[D']) = |A'|+2 \geq 2$. So $q \in \{|A'|+2, |A'|+3\}$. By (9), each connected component R of $G'-A'$ except at most one satisfies $|V(R)\cap X| = |V(R)\cap Y|-1$. Since $c(G[D']) \geq 2$, we have $G[D']$ contains an H_R-critical component R such that $|V(R) \cap X| = |V(R) \cap Y| - 1$. By Lemma 3.2 (iii), $|V(R) \cap X|$ is odd and so $V(R) \cap X \neq \emptyset$. Hence we have $R \in \mathcal{H}$. This completes the proof. □

Proof of Theorem 1.2: Let G be a k-regular bipartite graph with bipartition (X, Y). Let $H : V(G) \to 2^{\mathbb{Z}}$ such that $H(x) = \{-1, 1\}$ for all $x \in X$ and $H(y) = \{0, 2, 3\}$ for all $y \in Y$. Clearly, if G has an H-factor, then G has a 1-anti-factor. By Hall's Theorem, G contains a 3-factor. Thus it is sufficient for us to show that every connected 3-regular bipartite graph contains an H-factor. So we may assume that G is a connected 3-regular bipartite graph. By contradiction, suppose that G contains no H-factors.

By Lemma 3.5, we may assume that G is H-critical. Let $y \in Y$ and $G' = G - N[y]$. Let $H' := H_{G'}$, $D' := D_{H'}(G')$, $A' := A_{H'}(G')$, $B' := B_{H'}(G')$ and $C' := C_{H'}(G')$. By Lemma 3.2 (ii) and (iv), we have that $IH(y) \subseteq \{0, 1, 2\}$ and $\nabla_{H'}(G') = 2$. By Lemma 3.4, we have

$$2 = \nabla_{H'}(G') = c(G'[D']) - |A'|, \tag{11}$$

By Lemma 3.1, we have $B' = \emptyset$. Let $q := c(G' - A')$. Let $R_1, \ldots, R_q$ be the connected components of $G' - A'$.

Now we will show that $G'[D']$ contains a connected component R such that R is H_R-critical and $R \in \mathcal{H}$, which contradicts to Lemma 3.6. (The proof is completely similar with that of Lemma 3.6.) Note that $|X| = |Y|$, $|X| = \sum_{i=1}^{q} |V(R_i) \cap X| + 3 + |A'|$ and

$$|Y| = \sum_{i=1}^{q} |V(R_i) \cap Y| + 1 \geq \sum_{i=1}^{q} |V(R_i) \cap X| + q + 1. \tag{12}$$

So we have $q \leq |A'| + 2$. By (11), we have $q \geq c(G'[D']) = |A'| + 2$. Thus $q = |A'| + 2$ and so the equality holds for (12), which implies that for every connected component R of $G' - A'$, it is H_R-critical and $|V(R) \cap X| = |V(R) \cap Y| - 1$. So every connected component of $G'[D']$ belongs to $\mathcal{H}$. This completes the proof of Theorem 1.2. □

Remark 1. The bound that $k \geq 3$ in Theorem 1.2 is sharp. Let $m \in \mathrm{N}$ be a positive integer. For example, C_{4m+2} is a 2-regular graph and contains no H-factors. However, it is easy to show that C_{4m} contains an H-factor.

Remark 2. Theorem 1.2 does not hold for multi-graphs. By doubling every second edge in C_{4m+2}, we get a 3-regular bipartite multi-graph G. But, as one sees in Remark 1 that C_{4m+2} does not contain an H-factor, one sees that neither does G.

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