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Inversion sequences avoiding pairs of patterns

Chunyan Yan[1] Zhicong Lin[2*†]

[1] *School of Science, Jimei University, Xiamen 361021, P.R. China*
[2] *Research Center for Mathematics and Interdisciplinary Sciences, Shandong University, Qingdao 266237, P.R. China*

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The enumeration of inversion sequences avoiding a single pattern was initiated by Corteel, Martinez, Savage and Weselcouch, and Mansour and Shattuck, independently. Their work has sparked various investigations of generalized patterns in inversion sequences, including patterns of relation triples by Martinez and Savage, consecutive patterns by Auli and Elizalde, and vincular patterns by Lin and Yan. In this paper, we carried out the systematic study of inversion sequences avoiding two patterns of length 3. Our enumerative results establish further connections to the OEIS sequences and some classical combinatorial objects, such as restricted permutations, weighted ordered trees and set partitions. Since patterns of relation triples are some special multiple patterns of length 3, our results complement the work by Martinez and Savage. In particular, one of their conjectures regarding the enumeration of $(021, 120)$-avoiding inversion sequences is solved.

Keywords: Inversion sequences, Pattern avoidance, Restricted permutations, Weighted ordered trees, Set partitions

1 Introduction

A *permutation* of $[n] := \{1, 2, \ldots, n\}$ is a word $\pi_1\pi_2\cdots\pi_n$ such that $\{\pi_1, \pi_2, \ldots, \pi_n\} = [n]$. An *inversion sequence* of length n is an integer sequence $(e_1, e_2, \ldots, e_n)$ with the restriction $0 \le e_i \le i-1$ for all $1 \le i \le n$. There are several interesting bijections between $\mathfrak{S}_n$, the set of permutations of $[n]$, and $\mathbf{I}_n$, the set of inversion sequences of length n, which are known as *codings* of permutations (see [15, 22, 27, 32] and the references therein). Perhaps, the most natural coding in the literature is the so-called *Lehmer code* $\Theta : \mathfrak{S}_n \to \mathbf{I}_n$ of permutations define for $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ as

$$\Theta(\pi) = (e_1, e_2, \ldots, e_n), \quad \text{where } e_i := |\{j : j < i \text{ and } \pi_j > \pi_i\}|.$$

Permutations and inversion sequences are viewed as words over $\mathbb{N}$. A word $W = w_1w_2\cdots w_n$ is said to avoid the word (or *pattern*) $P = p_1p_2\cdots p_k$ ($k \le n$) if there does not exist $i_1 < i_2 < \cdots < i_k$ such that

*Corresponding author. E-mail address: linz@sdu.edu.cn.
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the subword $w_{i_1}w_{i_2}\cdots w_{i_k}$ of W is order isomorphic to P. For example, the word $W = 32421$ contains the pattern 231, because the subword $w_2w_3w_5 = 241$ of W has the same relative order as 231. However, W avoids both 101 and 123. For a set of words $\mathcal{W}$ and a sequence of patterns $P_1, \ldots, P_r$, let us denote $\mathcal{W}(P_1, \ldots, P_r)$ the set of words in $\mathcal{W}$ which avoid all patterns P_i for $i = 1, \ldots, r$.

The enumerative aspect of pattern avoiding permutations has been the focus for over a half century since MacMahon's investigation of word statistics and Knuth's consideration of stack-sorting algorithm (see Kitaev's monograph [19]). Nevertheless, the systematic study of patterns in inversion sequences was initiated only around 2015 by Corteel–Martinez–Savage–Weselcouch [15] and Mansour–Shattuck [26], where the enumeration of inversion sequences avoiding a single pattern of length 3 were nearly completed. Two remarkable connections are

$$|\mathbf{I}_n(021)| = S_n \quad \text{and} \quad |\mathbf{I}_n(000)| = E_n,$$

where S_n is the n-th *large Schröder number* and E_n is the n-th *Euler up-down number*. Their work has inspired lots of further investigations [2, 3, 5, 9, 12, 22, 20, 23, 24, 27, 25, 35] of patterns in inversion sequences from the perspective of enumeration. Especially, it has sparked various studies of generalized patterns in inversion sequences, including patterns of relation triples by Martinez and Savage [27], consecutive patterns by Auli and Elizalde [2, 3], and vincular patterns by Lin and Yan [25].

Motivated by the above works, the objective of this paper is to investigate systematically inversion sequences avoiding two patterns of length 3. Our study complements the work by Martinez and Savage [27] on enumeration of inversion sequences avoiding a fixed triple of binary relations $(\rho_1, \rho_2, \rho_3) \in \{<, >, \leq, \geq, =, \neq, -\}^3$. Following [27], consider the set $\mathbf{I}_n(\rho_1, \rho_2, \rho_3)$ consisting of those $e \in \mathbf{I}_n$ with no $i < j < k$ such that

$$e_i \,\rho_1\, e_j, \quad e_j \,\rho_2\, e_k \quad \text{and} \quad e_i \,\rho_3\, e_k.$$

Here the relation $''-''$ on a set S is all of $S \times S$, i.e., $x \,''-''\, y$ for all $x, y \in S$. For example, we have $\mathbf{I}_n(\geq, \neq, \geq) = \mathbf{I}_n(101, 110, 201, 210), \mathbf{I}_n(<, -, <) = \mathbf{I}_n(012, 021, 011)$ and $\mathbf{I}_n(>, -, \leq) = \mathbf{I}_n(101, 102)$. In general, patterns of relation triples are some special multiple patterns of length 3, so our study complements the work [27] in this sense.

Our work is also parallel to the work by Baxter and Pudwell [4] on pairs of patterns of length 3 in *ascent sequences*, one of the most important subsets of inversion sequences introduced by Bousquet-Mélou et al. [8] to encode the unlabeled (2+2)-free posets. It should be noted that the enumeration of pattern avoiding ascent sequences was first carried out by Duncan and Steingrímsson in [16] and followed by other researchers in [4, 29, 34], even a few years earier than the study on inversion sequences, to our surprise.

The enumerative results obtained in this paper are summarized in Table 1, which establish connections to the OEIS sequences and some classical combinatorial objects, such as restricted permutations, weighted ordered trees and irreducible set partitions. In particular, a conjecture of Martinez and Savage [27] regarding the enumeration of $(021, 120)$-avoiding inversion sequences is solved. Our work together with [9, 27, 12] completely classify all the Wilf-equivalences for inversion sequences avoiding pairs of length-3 patterns. Moreover, two related *unbalanced* Wilf-equivalence conjectures, presented as Conjectures 8.3 and 9.1, arise in this work.

Since all the Wilf-equivalences for pattern pairs p of length 3 can be distinguished by the number $|\mathbf{I}_8(p)|$, we shall denote the class of p-avoiding inversion sequences by class $|\mathbf{I}_8(p)|$. Within a Wilf class, equivalence classes are labeled A,B,C, etc.; see the last column of Table 1 or 2. For example, class 128B represents the class of $(001, 101)$-avoiding inversion sequences.

Pattern pair p	$a_n = \lvert\mathbf{I}_n(p)\rvert$ counted by:	solved?	OEIS	a_8, equiv class
$(001, 010)$	n	Sec. 2.1	A000027	8,A
$(001, 011)$	n	Sec. 2.1	A000027	8,B
$(001, 012)$	n	Sec. 2.1	A000027	8,C
$(001, 110)$	Lazy caterer sequence	Sec. 2.2	A000124	29,A
$(001, 021)$	Lazy caterer sequence	Sec. 2.2	A000124	29,B
$(001, 120)$	Lazy caterer sequence	Sec. 2.2	A000124	29,C
$(000, 001)$	Fibonacci number F_{n+1}	Sec. 2.3	A000045	34
$(001, 100)$	$F_{n+2} - 1$	Sec. 2.4	A000071	54
$(001, 210)$	Cake number $\binom{n}{3} + n$	Sec. 2.5	A000125	64
$(000, 011)$	2^{n-1}	Sec. 2.6	A000079	128,A
$(001, 101)$	2^{n-1}	Sec. 2.6	A000079	128,B
$(001, 102)$	2^{n-1}	Sec. 2.6	A000079	128,C
$(001, 201)$	2^{n-1}	Sec. 2.6	A000079	128,D
$(010, 012)$	2^{n-1}	Sec. 2.6	A000079	128,E
$(011, 012)$	2^{n-1}	Sec. 2.6	A000079	128,F
$(110, 012)$	$2^n - n$	Sec. 2.7	A000325	248,A
$(012, 021)$	$2^n - n$	Sec. 2.7	A000325	248,B
$(012, 201)$	$\lvert\mathfrak{S}_n(321, 2143)\rvert$	Sec. 2.8	A088921	411,A
$(012, 210)$	$\lvert\mathfrak{S}_n(321, 2143)\rvert$	Sec. 2.8	A088921	411,B
$(011, 102)$	F_{2n-1}	Sec. 3	A001519	610,A
$(012, 102)$	F_{2n-1}	Sec. 3	A001519	610,B
$(012, 120)$	F_{2n-1}	Sec. 3	A001519	610,C
$(010, 011)$	$\sum_{k=0}^{n-1}(n-k)^k$	Sec. 2.9	A026898	733
$(011, 021)$	Catalan number C_n	Sec. 2.10	A000108	1430,A
$(010, 021)$	Catalan number C_n	Sec. 2.10	A000108	1430,B
$(011, 201)$	$\lvert\mathbf{I}_n(-,>,\geq)\rvert = \lvert\mathbf{I}_n(\neq,\geq,\geq)\rvert$	Open	A279555	3091,A
$(011, 210)$	$\lvert\mathbf{I}_n(-,>,\geq)\rvert = \lvert\mathbf{I}_n(\neq,\geq,\geq)\rvert$	Open	A279555	3091,B
$(021, 120)$	$1 + \sum_{i=1}^{n-1}\binom{2i}{i-1}$	Sec. 4.1	A279561	4082,A
$(102, 120)$	$1 + \sum_{i=1}^{n-1}\binom{2i}{i-1}$	Sec. 4.2	A279561	4082,B
$(110, 102)$	$1 + \sum_{i=1}^{n-1}\binom{2i}{i-1}$	Sec. 4.3	A279561	4082,C
$(010, 100)$	Bell number B_n	Sec. 5	A000110	4140,D
$(011, 101)$	Bell number B_n	Sec. 5	A000110	4140,E
$(011, 110)$	Bell number B_n	Sec. 5	A000110	4140,F
$(101, 021)$	$\lvert\mathfrak{S}_n(4123, 4132, 4213)\rvert$	Sec. 6	A106228	5798,B
$(021, 201)$	Large Schröder number S_n	Sec. 2.11	A006318	8558,A
$(021, 210)$	Large Schröder number S_n	Sec. 2.11	A006318	8558,B
$(101, 110)$	indecomposable set partitions	Sec. 7	A074664	11624

Tab. 1: Pattern pairs whose avoidance sequences appear to match sequences in the OEIS.

Pattern pair p	$a_n = \|\mathbf{I}_n(p)\|$ counted by:	calculated?	in OEIS?	a_8, equiv class
$(000, 012)$	$1, 2, 4, 5, 21, 0, 0, \ldots$	Ultimately zero	new	0
$(100, 012)$	$1, 2, 5, 12, 27, 56, 110, \ldots$	Open	New	207
$(101, 012)$	$\|\mathfrak{S}_n(321, 2413, 3142)\|$	In [27, Sec. 2.8]	A034943	351
$(000, 021)$	$1, 2, 5, 14, 39, 111, 317, \ldots$	Open	New	911
$(000, 102)$	$1, 2, 5, 14, 40, 121, 373, \ldots$	Open	New	1181
$(000, 010)$	$1, 2, 4, 10, 29, 95, 345, \ldots$	Open	A279552	1376
$(011, 120)$	$1, 2, 5, 14, 42, 132, 431, \ldots$	Open	New	1452
$(100, 011)$	Nexus numbers	In [27, Sec. 2.15]	A047970	2048
$(021, 102)$	$1, 2, 6, 20, 66, 213, 683, \ldots$	Open	New	2211
$(010, 102)$	$1, 2, 5, 15, 51, 186, 707, \ldots$	Open	New	2763
$(000, 120)$	$1, 2, 5, 15, 50, 185, 737, \ldots$	Open	New	3126
$(010, 120)$	$1, 2, 5, 15, 52, 201, 845, \ldots$	Open	A279559	3801
$(010, 110)$	$1, 2, 5, 15, 52, 201, 847, \ldots$	Open	New	3836
$(010, 101)$	Bell number B_n	In [27, Sec. 2.17]	A000110	4140,A
$(000, 101)$	Bell number B_n	In [12, Sec. 6]	A000110	4140,B
$(000, 110)$	Bell number B_n	In [12, Sec. 6]	A000110	4140,C
$(100, 021)$	$1, 2, 6, 21, 78, 297, 1144, \ldots$	Open	New	4433,A
$(110, 021)$	Wilf-equiv to 4433,A (Sec. 8)	Open	New	4433,B
$(010, 201)$	$1, 2, 5, 15, 53, 214, 958, \ldots$	Open	New	4650,A
$(010, 210)$	Wilf-equiv to 4650,A (Sec. 8)	Open	New	4650,B
$(100, 102)$	$1, 2, 6, 21, 80, 318, 1305, \ldots$	Open	New	5487
$(102, 210)$	$1, 2, 6, 22, 87, 351, 1416, \ldots$	Open	New	5681
$(101, 102)$	$\|\mathfrak{S}_n(4123, 4132, 4213)\|$	In [12, Sec. 4]	A106228	5798,A
$(102, 201)$	$1, 2, 6, 22, 87, 354, 1465, \ldots$	Open	A279566	6154
$(000, 201)$	$1, 2, 5, 16, 60, 257, 1218, \ldots$	Open	New	6270,A
$(000, 210)$	Wilf-equiv to 6270,A (Sec. 8)	Open	New	6270,B
$(000, 100)$	$1, 2, 5, 16, 60, 260, 1267, \ldots$	Open	A279564	6850
$(101, 120)$	$1, 2, 6, 22, 90, 397, 1859, \ldots$	Open	New	9145
$(100, 120)$	$1, 2, 6, 22, 92, 421, 2062, \ldots$	Open	New	10646
$(110, 120)$	$1, 2, 6, 22, 92, 423, 2091, \ldots$	Open	A279570	10950
$(100, 110)$	$1, 2, 6, 22, 93, 437, 2233, \ldots$	Open	New	12227
$(100, 101)$	$1, 2, 6, 22, 93, 439, 2267, \ldots$	Open	New	12628
$(120, 201)$	$1, 2, 6, 23, 102, 498, 2607, \ldots$	Open	New	14386
$(120, 210)$	$1, 2, 6, 23, 102, 499, 2625, \ldots$	Open	A279573	14601
$(110, 201)$	$1, 2, 6, 23, 103, 512, 2739, \ldots$	Open	New	15464
$(101, 210)$	$1, 2, 6, 23, 103, 513, 2763, \ldots$	Open	New	15816
$(101, 201)$	$\|\mathfrak{S}_n(21\bar{3}54)\|$	In [27, Sec. 2.27]	A117106	17734,A
$(100, 210)$	$\|\mathfrak{S}_n(21\bar{3}54)\|$	In [9, Sec. 3.3]	A117106	17734,B
$(100, 201)$	$\|\mathfrak{S}_n(21\bar{3}54)\|$	In [27, Sec. 2.27]	A117106	17734,C
$(110, 210)$	$\|\mathfrak{S}_n(21\bar{3}54)\|$	In [27, Sec. 2.27]	A117106	17734,D
$(201, 210)$	$\|\mathfrak{S}_n(MMP(0, 2, 0, 2))\|$	In [27, Sec. 2.29]	A212198	23072

Tab. 2: Pattern pairs whose avoidance sequences either have been studied in [9, 27, 12] or appear to be new in the OEIS.

Statistics on inversion sequences

For an inversion sequence $e = (e_1, e_2, \dots, e_n) \in \mathbf{I}_n$, define the following five classical statistics:

- $\mathrm{asc}(e) := |\{i \in [n-1] : e_i < e_{i+1}\}|$, the number of *ascents* of e;
- $\mathrm{dist}(e) := |\{e_1, e_2, \dots, e_n\} \setminus \{0\}|$, the number of *distinct positive entries* of e;
- $\mathrm{rmin}(e) := |\{i \in [n] : e_i < e_j \text{ for all } j > i\}|$, the number of *right-to-left minima* of e;
- $\mathrm{zero}(e) := |\{i \in [n] : e_i = 0\}|$, the number of *zero entries* in e;
- $\mathrm{satu}(e) := |\{i \in [n] : e_i = i - 1\}|$, the number of *saturated entries* of e.

Note that, over inversion sequences, the first two statistics are *Eulerian*, while the last three statistics are *Stirling* (See for example [22, 17]). These five statistics on restricted inversion sequences have been extensively studied in the literature (cf. [12, 15, 16, 17, 22, 20, 25, 27]) and will play important roles in solving some enumerative problems in this paper.

Notations

Throughout this paper, we use the notation

$$\chi(\mathsf{S}) = \begin{cases} 1, & \text{if the statement } \mathsf{S} \text{ is true;} \\ 0, & \text{otherwise.} \end{cases}$$

We will also use the Kronecker delta function: $\delta_{i,j}$ equals 1 if $i = j$; and 0, otherwise.

2 Some simple results

This section contains some results that are obtained by elementary discussions. Some of the avoiding classes are even proved to be the same as these that have been studied in the literature. For clarity, all of them are listed below:

$$\begin{array}{lll} \mathbf{I}_n(001,110) = \mathbf{I}_n(\geq,\neq,-), & \mathbf{I}_n(000,001) = \mathbf{I}_n(=,\leq,-), & \mathbf{I}_n(001,100) = \mathbf{I}_n(\geq,\leq,\neq), \\ \mathbf{I}_n(000,011) = \mathbf{I}_n(\leq,=,-), & \mathbf{I}_n(011,012) = \mathbf{I}_n(<,\leq,-), & \mathbf{I}_n(012,201) = \mathbf{I}_n(\neq,<,\neq), \\ \mathbf{I}_n(001,101) = \mathbf{I}_n(001), & \mathbf{I}_n(001,102) = \mathbf{I}_n(001), & \mathbf{I}_n(001,201) = \mathbf{I}_n(001), \\ \mathbf{I}_n(021,210) = \mathbf{I}_n(021), & \mathbf{I}_n(021,201) = \mathbf{I}_n(021). & \end{array}$$

The reason why they are equal will be given throughout this section.

2.1 Classes 8(A,B,C): n

Theorem 2.1. *For any $n \geq 1$ and a pattern pair $p \in \{(001,010),(001,011),(001,012)\}$, we have $|\mathbf{I}_n(p)| = n$.*

This result is an immediate consequence of the following three simple observations.

Observation 2.2. *An inversion sequence $e \in \mathbf{I}_n$ is $(001,010)$-avoiding if and only if for some $t \geq 1$,*

$$e_1 < e_2 < \cdots < e_t = e_{t+1} = \cdots = e_n.$$

Observation 2.3. *An inversion sequence* $e \in \mathbf{I}_n$ *is* $(001, 011)$*-avoiding if and only if either* $e = (0, 0, \ldots, 0)$ *or for some* $t \geq 2$*,*

$$e_1 < e_2 < \cdots < e_t > e_{t+1} = e_{t+2} = \cdots = e_n = 0.$$

Observation 2.4. *An inversion sequence* $e \in \mathbf{I}_n$ *is* $(001, 012)$*-avoiding if and only if either* $e = (0, 0, \ldots, 0)$ *or for some* $t \geq 2$*,*

$$e_1 < e_2 = e_3 = \cdots = e_t > e_{t+1} = e_{t+2} = \cdots = e_n = 0.$$

2.2 Classes 29(A,B,C): Lazy caterer sequence

The integer sequence

$$\{n(n-1)/2 + 1\}_{n \geq 1} = \{1, 2, 4, 7, 11, 16, 22, 29, 37, 46, \ldots\}$$

appear as [28, A000124] and is called the *Lazy caterer sequence* or the sequence of *central polygonal numbers*, which enumerates $(132, 321)$-avoiding permutations [33]. This sequence also counts three inequivalent classes $\mathbf{I}_n(<, \neq, -)$, $\mathbf{I}_n(<, -, <)$ and $\mathbf{I}_n(\geq, \neq, -)$ of relation triple avoiding inversion sequences [27, Sec. 2.4].

Theorem 2.5. *For any* $n \geq 1$ *and a pattern pair* $p \in \{(001, 110), (001, 021), (001, 120)\}$*, we have* $|\mathbf{I}_n(p)| = \binom{n}{2} + 1$*, the* n*-th central polygonal number.*

This result is clear from the three characterizations below.

Observation 2.6. *An inversion sequence* $e \in \mathbf{I}_n$ *is* $(001, 110)$*-avoiding if and only if for some* $t \geq 1$*,*

$$e_1 < e_2 < \cdots < e_t \geq e_{t+1} = e_{t+2} = \cdots = e_n.$$

Remark 2.7. Comparing Observation 2.6 with [27, Observation 11] we see that the two sets $\mathbf{I}_n(001, 110)$ and $\mathbf{I}_n(\geq, \neq, -)$ are the same.

Observation 2.8. *An inversion sequence* $e \in \mathbf{I}_n$ *is* $(001, 021)$*-avoiding if and only if for some* $t \geq 1$ *and* $s \geq t$*,*

$$e_1 < e_2 < \cdots < e_t = e_{t+1} = \cdots = e_s > e_{s+1} = e_{s+2} = \cdots = e_n = 0.$$

Observation 2.9. *An inversion sequence* $e \in \mathbf{I}_n$ *is* $(001, 120)$*-avoiding if and only if for some* $t \geq 1$ *and* $s \geq t$*,*

$$e_1 < e_2 < \cdots < e_t = e_{t+1} = \cdots = e_s > e_{s+1} = e_{s+2} = \cdots = e_n = e_{t-1}.$$

2.3 Class 34 and Fibonacci numbers

The n*-th Fibonacci number* F_n can be defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and the initial values $F_0 = 0$ and $F_1 = 1$.

Theorem 2.10. *For* $n \geq 1$*, we have* $\mathbf{I}_n(000, 001) = \mathbf{I}_n(=, \leq, -)$*. Consequently, the cardinality of* $\mathbf{I}_n(000, 001)$ *is* F_{n+1}*.*

Proof: It is obvious that $\mathbf{I}_n(000, 001) = \mathbf{I}_n(=, \leq, -)$. The second statement then follows from [27, Theorem 4]. □

2.4 Class 54: $F_{n+2} - 1$

Theorem 2.11. *For $n \geq 1$, we have $\mathbf{I}_n(001, 100) = \mathbf{I}_n(\geq, \leq, \neq)$. Consequently, the cardinality of $\mathbf{I}_n(001, 100)$ is $F_{n+2} - 1$.*

Proof: Observe that $e \in \mathbf{I}_n$ is $(001, 100)$-avoiding if and only if for some t, s with $1 \leq t \leq s \leq n$,

$$e_1 < e_2 < \cdots < e_t = e_{t+1} = \cdots = e_s > e_{s+1} > e_{s+2} > \cdots > e_n.$$

Comparing this with the characterization of $(\geq, \leq, \neq)$-avoiding inversion sequences in the proof of [27, Theorem 13], we have $\mathbf{I}_n(001, 100) = \mathbf{I}_n(\geq, \leq, \neq)$ and thus $|\mathbf{I}_n(001, 100)| = F_{n+2} - 1$. ☐

2.5 Class 64 and Cake numbers

The n-th *Cake number*, $\binom{n}{3} + n$, is the maximal number of pieces resulting from $n - 1$ planar cuts through a cake. The sequence of Cake numbers is registered as [28, A000125] and has a number of geometric or combinatorial interpretations. We have the following interpretation of Cake numbers as $(001, 210)$-avoiding inversion sequences.

Theorem 2.12. *For any $n \geq 1$, we have $|\mathbf{I}_n(001, 210)| = \binom{n}{3} + n$.*

Proof: Observe that $e \in \mathbf{I}_n$ is $(001, 210)$-avoiding if and only if for some $t \geq 1$ and $t \leq s \leq n$,

$$e_1 < e_2 < \cdots < e_t = e_{t+1} = \cdots = e_s > e_{s+1} = e_{s+2} = \cdots = e_n.$$

It follows that for a fixed t, $1 \leq t \leq n$, there are $(n - t)(t - 1) + 1$ many inversion sequences with the above form. Summing over t from 1 to n gives the desired result. ☐

2.6 Classes 248(A,B,C,D,E,F): Powers of 2

Simion and Schmidt [33] showed that $|\mathfrak{S}_n(132, 231)| = 2^{n-1}$. Corteel et al. [15] showed that the coding $\Theta : \mathfrak{S}_n \to \mathbf{I}_n$ restricts to a bijection from $\mathfrak{S}_n(132, 231)$ to $\mathbf{I}_n(001)$ and thus $|\mathbf{I}_n(001)|$ has cardinality 2^{n-1}. Martinez and Savage [27, Section 2.6] proved that the three classes $\mathbf{I}_n(<, \leq, -)$, $\mathbf{I}_n(<, \geq, -)$ and $\mathbf{I}_n(\leq, =, -)$ also have cardinality 2^{n-1}. In this section, we prove more interpretations for 2^{n-1}.

Theorem 2.13. *For any $n \geq 1$ and a pattern pair*

$$p \in \{(000, 011), (011, 012), (001, 101), (001, 102), (001, 201), (010, 012)\},$$

we have $|\mathbf{I}_n(p)| = 2^{n-1}$.

Proof: Since $\mathbf{I}_n(000, 011) = \mathbf{I}_n(\leq, =, -)$ and $\mathbf{I}_n(011, 012) = \mathbf{I}_n(<, \leq, -)$, the result for p equals $(000, 011)$ or $(011, 012)$ was proved in [27, Section 2.6].

It has already been characterized in [15] that $e \in \mathbf{I}_n(001)$ if and only if for some $t \in [n]$,

$$e_1 < e_2 < \cdots < e_t \geq e_{t+1} \geq e_{t+2} \geq \cdots \geq e_n.$$

Thus, an inversion sequence e avoids 001 must necessarily avoid 101, 102 and 201. It follows that $\mathbf{I}_n(p) = \mathbf{I}_n(001)$ for any $p \in \{(001, 101), (001, 102), (001, 201)\}$ and the result for these three cases are true.

For $p = (010, 012)$, we observe that inversion sequences $e \neq (0, 0, \ldots, 0)$ in $\mathbf{I}_n(p)$ are those satisfying

$$e_1 = e_2 = \cdots = e_t < e_{t+1} \geq e_{t+2} \geq \cdots \geq e_n > 0$$

for some $t \in [n-1]$. On the other hand, $(011, 012)$-avoiding inversion sequences are those whose positive entries are decreasing. For any $e \in \mathbf{I}_n(011, 012)$ with e_t as the leftmost positive entry (if any), define $e' = (0, \ldots, 0, e'_t, \ldots, e'_n)$ where for $t \leq i \leq n$, e'_i equals the nearest positive entry to the left of e_i if $e_i = 0$; otherwise $e'_i = e_i$. For instance, if $e = (0, 0, 0, 3, 0, 2, 0, 0, 1) \in \mathbf{I}_9(011, 012)$, then $e' = (0, 0, 0, 3, 3, 2, 2, 2, 1)$. It is routine to check that the mapping $e \mapsto e'$ sets up a bijection between $\mathbf{I}_n(011, 012)$ and $\mathbf{I}_n(010, 012)$. This completes the poof of the theorem. □

2.7 Classes 248(A,B): $2^n - n$

Permutations with at most one descent are known as *Grassmannian permutations* after Lascoux and Schützenberger [21]. Grassmannian permutations of length n are enumerated by $2^n - n$, which is also the cardinality of three equivalence classes of relation triples for inversion sequences as shown in [27]. We relate $2^n - n$ to two pairs of patterns in inversion sequences.

Theorem 2.14. *For any $n \geq 1$ and a pattern pair $p \in \{(012, 021), (110, 012)\}$, we have $|\mathbf{I}_n(p)| = 2^n - n$.*

Proof: The case of $p = (012, 021)$: Observe that $e \in \mathbf{I}_n(012, 021)$ are those satisfying for some $t \in [n]$,

$$0 = e_1 = e_2 = \cdots e_{t-1} < e_t \text{ and either } e_j = 0 \text{ or } e_j = e_t \text{ when } t + 1 \leq j \leq n.$$

The number of inversion sequences satisfying the above condition is $(t-1)2^{n-t}$ for $t \geq 2$. Thus,

$$|\mathbf{I}_n(p)| = 1 + \sum_{t=2}^{n} (t-1)2^{n-t} = 1 + 2^{n-2} \sum_{k=1}^{n-1} k \left(\frac{1}{2}\right)^{k-1} = 2^n - n,$$

where the last equality follows from

$$\sum_{k=1}^{n-1} kx^{k-1} = \left(\sum_{k=1}^{n-1} x^k\right)' = \frac{nx^{n-1}(x-1) + 1 - x^n}{(1-x)^2}.$$

The case of $p = (110, 012)$: It is sufficient to show that for $n \geq 2$,

$$a_n - a_{n-1} = 2^{n-1} - 1,$$

where $a_n = |\mathbf{I}_n(p)|$. Note that $\mathbf{I}_n(110, 012)$ consists of inversion sequences $e \in \mathbf{I}_n$ whose positive entries are weakly decreasing and whenever $e_i = e_j = \ell > 0$ for some $i < j$, then $e_{j+1} = e_{j+2} = \cdots = e_n = \ell$. For $e \in \mathbf{I}_{n-1}(110, 012)$, let $e' = (e_1, e_2, \ldots, e_{n-1}, a) \in \mathbf{I}_n(110, 012)$, where a equals the least positive entry of e, if $e \neq (0, 0, \ldots, 0)$; and 0, otherwise. It is clear that the mapping $e \mapsto e'$ is a one-to-one correspondence between $\mathbf{I}_{n-1}(110, 012)$ and the set $\mathbf{I}_n(110, 012) \setminus \widetilde{\mathbf{I}}_n(110, 012)$, where $\widetilde{\mathbf{I}}_n(110, 012)$ consists of inversion sequences $e \in \mathbf{I}_n \setminus \{(0, 0, \ldots, 0)\}$ whose positive entries are decreasing. So it remains to show that $|\widetilde{\mathbf{I}}_n(110, 012)| = 2^{n-1} - 1$. Since for any fixed t and k, $2 \leq t \leq n$ and $1 \leq k$, the number of inversion sequences $e \in \widetilde{\mathbf{I}}_n(110, 012)$ with $\min\{i : e_i > 0\} = t$ and $|\{i : e_i > 0\}| = k$ is $\binom{t-1}{k}\binom{n-t}{k-1}$, we have

$$|\widetilde{\mathbf{I}}_n(110, 012)| = \sum_{t=2}^{n} \sum_{k \geq 1} \binom{t-1}{k}\binom{n-t}{k-1} = \sum_{t=2}^{n} \binom{n-1}{t-2} = 2^{n-1} - 1,$$

where the second equality follows from the Vandermonde identity. This completes the proof of the theorem. □

2.8 Classes 441(A,B): $(321, 2143)$-avoiding permutations

The 2143-avoiding permutations are called *vexillary permutations* [21] and the 321-avoiding vexillary permutations are related to the combinatorics of Schubert polynomials by the work of Billey, Jockush and Stanley [6]. It was shown that

$$|\mathfrak{S}_n(321, 2143)| = 2^{n+1} - \binom{n+1}{3} - 2n - 1,$$

the sequence A088921 in the OEIS. In [27, Sec. 2.9], Martinez and Savage showed that $(\neq, <, \neq)$-avoiding inversion sequences are counted by the same function as 321-avoiding vexillary permutations. Regarding pairs of patterns in inversion sequences, there has the following enumerative result.

Theorem 2.15. *For any $n \geq 1$ and a pattern pair $p \in \{(012, 201), (012, 210)\}$, we have $|\mathbf{I}_n(p)| = 2^{n+1} - \binom{n+1}{3} - 2n - 1$.*

Proof: Since $\mathbf{I}_n(\neq, <, \neq) = \mathbf{I}_n(102, 012, 201)$ and every inversion sequence contains the pattern 102 must also contain 012, we have $\mathbf{I}_n(012, 201) = \mathbf{I}_n(\neq, <, \neq)$ and so $|\mathbf{I}_n(012, 201)| = 2^{n+1} - \binom{n+1}{3} - 2n - 1$.

Observe that a sequence $e \in \mathbf{I}_n$ is $(012, 210)$-avoiding if and only if

- positive entries of e are weakly decreasing and
- $\mathrm{dist}(e) \leq 2$, and if $\mathrm{dist}(e) = 2$ with e_ℓ as the leftmost least positive entry, then $e_\ell = e_{\ell+1} = \cdots = e_n$.

Let $\mathbf{I}_{n,t}(012, 210)$ be the set of all inversion sequences $e \in \mathbf{I}_n(012, 210)$ with e_t as the leftmost positive entry. Recall that $\mathrm{dist}(e)$ denotes the number of distinct positive entries of e. The cardinality of $\{e \in \mathbf{I}_{n,t}(012, 210) : \mathrm{dist}(e) = 1\}$ is easily seen to be $(t-1)2^{n-t}$, while

$$|\{e \in \mathbf{I}_{n,t}(012, 210) : \mathrm{dist}(e) = 2\}| = \sum_{\ell=1}^{n-t} \binom{t-1}{2} 2^{n-t-\ell} = \binom{t-1}{2}(2^{n-t} - 1),$$

where each term $\binom{t-1}{2}2^{n-t-\ell}$ counts the number of $e \in \mathbf{I}_{n,t}(012, 210)$ with $\mathrm{dist}(e) = 2$ whose leftmost least positive entry is $e_{n-\ell-1}$. It then follows that

$$|\mathbf{I}_n(012, 210)| = 1 + \sum_{t=2}^{n} \left((t-1)2^{n-t} + \binom{t-1}{2}(2^{n-t} - 1) \right),$$

which can be simplified to $2^{n+1} - \binom{n+1}{3} - 2n - 1$. This completes the proof of the theorem. □

2.9 Class 733: $\sum_{k=0}^{n-1}(n-k)^k$

Theorem 2.16. *For $n \geq 1$, $|\mathbf{I}_n(010, 011)| = \sum_{k=0}^{n-1}(n-k)^k$.*

Proof: Note that $\mathbf{I}_n(010, 011)$ is the set of $e \in \mathbf{I}_n$ satisfying for some $t \in [n]$,

$$0 = e_1 = e_2 = \cdots = e_t < e_{t+1} \text{ and } e_i \neq e_j \text{ for } t \leq i < j \leq n.$$

So for fixed $t \in [n]$, the number of inversion sequences satisfying the above condition is t^{n-t}. The result then follows. □

2.10 Classes 1430(A,B) and Catalan numbers

Recall that a *Dyck path* of length n is a path in $\mathbb{N}^2$ from $(0, 0)$ to (n, n) using the east step $(1, 0)$ and the north step $(0, 1)$, which does not pass above the diagonal $y = x$. It is well known that the n-th *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts the length n Dyck paths. One can represent a Dyck path of length n by a weakly increasing inversion sequence e with e_i as the height of its i-th east step.

Theorem 2.17. *For any $n \geq 1$ and a pattern pair $p \in \{(011, 021), (010, 021)\}$, we have $|\mathbf{I}_n(p)| = C_n$.*

Proof: Note that $\mathbf{I}_n(010, 021)$ is exactly the set of weakly increasing inversion sequences of length n and so $|\mathbf{I}_n(010, 021)| = C_n$. The mapping $e \mapsto e'$, where e' is constructed from e by replacing each positive entry by 0 except for all the first occurrences of positive values, is a bijection between $\mathbf{I}_n(010, 021)$ and $\mathbf{I}_n(011, 021)$. The proof is complete. □

2.11 Classes 8558(A,B) and Schröder numbers

A *Schröder n-path* is a path in $\mathbb{N}^2$ from $(0, 0)$ to $(2n, 0)$ using only the steps $(1, 1)$ (up step), $(1, -1)$ (down step) and (2,0) (flat step). The n-th Large Schröder number S_n counts the Schröder n-paths, as well as the $(2413, 4213)$-avoiding permutations known as *Separable permutations*. One of the most remarkable results (see [15, 27, 22, 20, 26]) in the enumeration of pattern avoiding inversion sequences is

$$|\mathbf{I}_n(021)| = S_n.$$

Since every inversion sequence contains a pattern 210 or 201 must also contain the pattern 021, we have $\mathbf{I}_n(p) = \mathbf{I}_n(021)$ for $p \in \{(021, 210), (021, 201)\}$ and consequently the following result holds.

Theorem 2.18. *For any $n \geq 1$ and a pattern pair $p \in \{(021, 210), (021, 201)\}$, we have $|\mathbf{I}_n(p)| = S_n$.*

3 Classes 610(A,B,C): Boolean permutations

The bisection of the Fibonacci sequence

$$\{F_{2n-1}\}_{n\geq 1} = \{1, 2, 5, 13, 34, 89, 233, 610, 1597, \ldots\}$$

appears as A001519 in [28] and has many combinatorial interpretations, among which is the number of $(321, 3412)$-avoiding permutations known as *Boolean permutations*. Note that $a_n = F_{2n-1}$ satisfies the recurrence

$$a_n = 3a_{n-1} - a_{n-2}.$$

Corteel et al. [15] showed that $|\mathbf{I}_n(012)|$ shares the same recurrence relation above as a_n and thus proving

$$|\mathbf{I}_n(012)| = F_{2n-1}. \tag{1}$$

The following result provides more interpretations of a_n in terms of pattern avoiding inversion sequences.

Theorem 3.1. *For any $n \geq 1$ and a pattern pair $p \in \{(012, 102), (012, 120), (011, 102)\}$, we have $|\mathbf{I}_n(p)| = F_{2n-1}$.*

Proof: Since an inversion sequence contains the pattern 102 or 120 must also contain the pattern 012, we have $\mathbf{I}_n(p) = \mathbf{I}_n(012)$ for $p \in \{(012, 102), (012, 120)\}$. The result for these two cases then follows from (1).

Next, we deal with the case $p = (011, 102)$. Consider the triangle

$$\mathcal{A}_{n,k} := \{e \in \mathbf{I}_n(012) : \mathrm{last}(e) = k\},$$

where $\mathrm{last}(e)$ is the last entry of e. We claim that for $n \geq 2$,

$$|\mathcal{A}_{n,0}| = |\mathcal{A}_{n,1}| = a_{n-1} \tag{2}$$

and

$$|\mathcal{A}_{n,k}| = |\mathcal{A}_{n-1,k-1}| \text{ for } 2 \leq k \leq n-1. \tag{3}$$

Notice that the mapping $\delta_k : (e_1, e_2, \ldots, e_{n-1}) \mapsto (e_1, e_2, \ldots, e_{n-1}, k)$ is a one-to-one correspondence between $\mathbf{I}_{n-1}(012)$ and $\mathcal{A}_{n,k}$ for $k = 0, 1$ and so Eq. (2) follows. For fixed k, $2 \leq k \leq n-1$, the mapping $(e_1, e_2, \ldots, e_{n-1}) \mapsto (0, e'_1, e'_2, \ldots, e'_{n-1})$ with $e'_i = e_i + \chi(e_i > 0)$ is a one-to-one correspondence between $\mathcal{A}_{n-1,k-1}$ and $\mathcal{A}_{n,k}$, and thus Eq. (3) holds. For any $e \in \mathbf{I}_n$, we recall from the introduction that $\mathrm{satu}(e)$ is the number of indices $i \in [n]$ such that $e_i = i - 1$. We aim to show that

$$\mathcal{B}_{n,k} := \{e \in \mathbf{I}_n(011, 102) : \mathrm{satu}(e) = k + 1\}$$

has the same carnality as $\mathcal{A}_{n,k}$, which would finish the proof. We will do this by showing

$$|\mathcal{B}_{n,0}| = |\mathcal{B}_{n,1}| = |\mathbf{I}_{n-1}(011, 102)| \tag{4}$$

and

$$|\mathcal{B}_{n,k}| = |\mathcal{B}_{n-1,k-1}| \text{ for } 2 \leq k \leq n-1. \tag{5}$$

Comparing (4) and (5) with (2) and (3) we have $|\mathcal{B}_{n,k}| = |\mathcal{A}_{n,k}|$ by induction.

It remains to prove (4) and (5). Clearly, the mapping $(e_1, \ldots, e_{n-1}) \mapsto (0, e_1, \ldots, e_{n-1})$ is a one-to-one correspondence between $\mathbf{I}_{n-1}(011, 102)$ and $\mathcal{B}_{n,0}$ and so $|\mathbf{I}_{n-1}(011, 102)| = |\mathcal{B}_{n,0}|$. For $e \in \mathcal{B}_{n,1}$, there exist only one index $j > 1$ such that $e_j = j - 1$ and we introduce the mapping $\rho : \mathcal{B}_{n,1} \to \mathbf{I}_{n-1}(011, 102)$ by

- if $j - 2$ does not appear as the entry of e, then let $\rho(e) = (e_2, e_3, \ldots, e_j - 1, e_{j+1}, \ldots, e_n)$;
- otherwise, let $\rho(e)$ be constructed by removing the entry e_j from e directly.

For instance, the mapping reads $\rho(0,0,0,3)=(0,0,2)$, $\rho(0,0,1,3)=(0,1,2)$, $\rho(0,0,2,0)=(0,1,0)$, $\rho(0,0,2,1)=(0,0,1)$ and $\rho(0,1,0,0)=(0,0,0)$. The crucial observation about ρ is that e_j is the largest entry of e and $\rho(e)$ is constructed by the second bullet if and only if $\rho(e)\in\mathcal{B}_{n-1,0}$. To see that ρ is a bijection, we introduce its inverse ρ^{-1} explicitly. For $e\in\mathbf{I}_{n-1}(011,102)$ with $e_j=\ell$ as the largest entry, define $\rho^{-1}:\mathbf{I}_{n-1}(011,102)\rightarrow\mathcal{B}_{n,1}$ as

- if $e\notin\mathcal{B}_{n-1,0}$, then $\rho^{-1}(e)=(0,e_1,e_2,\dots,e_j+1,e_{j+1},\dots,e_n)$;
- otherwise, $e\in\mathcal{B}_{n-1,0}$ and let $\rho^{-1}(e)=(e_1,e_2,\dots,e_{\ell+1},\ell+1,e_{\ell+2},\dots,e_{n-1})$.

It is routine to check that ρ^{-1} is the inverse of ρ, which proves $|\mathcal{B}_{n,1}|=|\mathbf{I}_{n-1}(011,102)|$ and (4) is established. To see (5), for any $e\in\mathcal{B}_{n,k}$ ($2\leq k\leq n-1$) with e_ℓ as the largest entry, consider the mapping $e\mapsto e'$ where e' is obtained from e by removing e_ℓ directly. Since e_ℓ is the rightmost saturated entry of e, it is easy to see that $e\mapsto e'$ sets up a one-to-one correspondence between $\mathcal{B}_{n,k}$ and $\mathcal{B}_{n-1,k-1}$, which completes the proof of the theorem. □

4 Classes 4082(A,B,C): $1+\sum_{i=1}^{n-1}\binom{2i}{i-1}$

In [27, Sec. 3.1.3], Martinez and Savage showed that

$$|\mathbf{I}_n(>,\neq,-)|=1+\sum_{i=1}^{n-1}\binom{2i}{i-1}$$

and conjectured the following Wilf-equivalence.

Conjecture 4.1 (Martinez and Savage [27])**.** For $n\geq 1$, we have

$$|\mathbf{I}_n(>,\neq,-)|=|\mathbf{I}_n(<,>,\neq)|.$$

The integer sequence $\{1+\sum_{i=1}^{n-1}\binom{2i}{i-1}\}_{n\geq1}$, whose generating function is

$$1+\sum_{n\geq1}\left(1+\sum_{i=1}^{n-1}\binom{2i}{i-1}\right)x^n=\frac{1-4x+\sqrt{-16x^3+20x^2-8x+1}}{2(x-1)(4x-1)},\tag{6}$$

has been registered as A279561 in the OEIS [28]. In this section, we will prove Conjecture 4.1 and two more interpretations for this integer sequence.

4.1 *Pattern pair* $(021,120)$*: proof of Conjecture 4.1*

Recall that a Dyck path D can be represented as $D=h_1h_2\cdots h_n$, where h_i is the height of its i-th east step satisfying $0\leq h_i<i$. Let $\mathfrak{D}_n$ be the set of Dyck paths of length n. Note that $|\mathfrak{D}_n|=C_n$, the n-th Catalan number, whose generating function is

$$C(x):=1+\sum_{n\geq1}C_nx^n=\frac{1-\sqrt{1-4x}}{2x}.\tag{7}$$

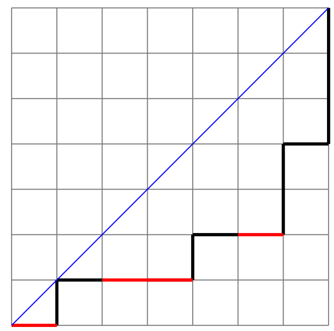

Fig. 1: The outline of an inversion sequence

For our purpose, we color each east step of a Dyck path by black or red and call such a Dyck path a *colored Dyck path*. If the i-th east step has height k and color red, then we write $h_i = \bar{k}$. It was observed in [15] that an inversion sequence is 021-avoiding if and only if its positive entries are weakly increasing. By this characterization, each sequence $e \in \mathbf{I}_n(021)$ can be represented by a colored Dyck path $\mathcal{H}(e) = h_1 h_2 \cdots h_n$, called the *outline of* e in [22, Def. 2.1], where

$$h_i = \begin{cases} e_i, & \text{if } e_i > 0, \\ \bar{k}, & \text{if } e_i = 0 \text{ and } k = \max\{e_1, \ldots, e_i\}. \end{cases}$$

For example, the outline of $(0,1,0,0,2,0,4)$ is the colored Dyck path $\bar{0}1\bar{1}\bar{1}2\bar{2}4$ drawn in Fig. 1. Let us denote by $\mathcal{A}_n$ the set of colored Dyck paths of length n satisfying

(a) all the east step of height 0 are colored red, and

(b) the first east step of each positive height is colored black.

It is clear that the geometric representation $\mathcal{H}$ is a bijection between $\mathbf{I}_n(021)$ and $\mathcal{A}_n$. Let $\mathcal{B}_n$ be the set of colored Dyck paths in $\mathcal{A}_n$ such that only east steps with smallest positive height can have two colors. For example, the colored Dyck path in Fig. 1 is an element in $\mathcal{A}_7 \setminus \mathcal{B}_7$. Note that $\mathbf{I}_n(<,>,\neq) = \mathbf{I}_n(021,120)$ and the geometric representation $\mathcal{H}$ induces a bijection between $\mathbf{I}_n(021,120)$ and $\mathcal{B}_n$, as any $(021,120)$-avoiding inversion sequence can not have zero entry after the first appearance of the entry with second smallest positive value. Let

$$S(x) := 1 + \sum_{n\geq 1} |\mathcal{B}_n| x^n = 1 + x + 2x^2 + 6x^3 + 21x^4 + \cdots.$$

In view of (6), Conjecture 4.1 is equivalent to the following result.

Theorem 4.2. *We have*

$$S(x) = \frac{1 - 4x + \sqrt{-16x^3 + 20x^2 - 8x + 1}}{2(x-1)(4x-1)}.$$

We decompose the proof of the above theorem into the following two lemmas. Let $\mathfrak{C}_n$ be the set of colored Dyck paths of length n such that only east steps with height 0, except the first step (which is colored black), can have two colors. Introduce the generating function

$$H(x) := 1 + \sum_{n\geq 0} |\mathfrak{C}_n| x^n = 1 + x + 3x^2 + 10x^3 + 35x^4 + \cdots .$$

Lemma 4.3. *We have the following functional equation for $S(x)$:*

$$S(x) = 1 + xH(x)/(1-x) + xC(x)(S(x) - 1/(1-x)). \tag{8}$$

Proof: For a colored Dyck path $D = h_1 \cdots h_n \in \mathcal{B}_n$, let

$$k = \min\{i \in [n] : h_{i+1} = i \text{ or } i = n\}.$$

Then, D can be decomposed uniquely into a pair (D_1, D_2) of Dyck paths, where

$$D_1 := h_2 h_3 \dots h_k \quad \text{and} \quad D_2 := (h_{k+1} - k)(h_{k+2} - k) \cdots (h_n - k).$$

We need to distinguish two cases:

- if $D_1 = \bar{0} \cdots \bar{0}$, the Dyck path consisting of $k-1$ red east steps of height 0, then $D_2 \in \mathfrak{C}_{n-k}$.
- otherwise $D_1 \neq \bar{0} \cdots \bar{0}$ and $D_1 \in \mathcal{B}_{k-1}$, then $D_2 \in \mathfrak{D}_{n-k}$.

Turning this decomposition into generating functions then yields

$$S(x) = 1 + x\left(\sum_{k\geq 0} x^k\right) H(x) + x\left(S(x) - \sum_{k\geq 0} x^k\right) C(x),$$

which is equivalent to (8). □

Lemma 4.4. *We have the following functional equation for $H(x)$:*

$$H(x) = 1 + x(2H(x) - 1)C(x). \tag{9}$$

Proof: *We will use similar decomposition of the colored Dyck paths in $\mathfrak{C}_n$ as in the proof of Lemma 4.3. Let $\tilde{\mathfrak{C}}_n$ be the set of colored Dyck paths of length n such that only east steps of height 0 can have two colors. Clearly, we have*

$$1 + \sum_{n\geq 1} |\tilde{\mathfrak{C}}_n| x^n = 2H(x) - 1. \tag{10}$$

For a colored Dyck path $D = h_1 \cdots h_n \in \mathfrak{C}_n$, let

$$k = \min\{i \in [n] : h_{i+1} = i \text{ or } i = n\}.$$

Then, D can be decomposed uniquely into a pair (D_1, D_2) of Dyck paths, where

$$D_1 := h_2 h_3 \dots h_k \in \tilde{\mathfrak{C}}_{k-1} \quad \text{and} \quad D_2 := (h_{k+1} - k)(h_{k+2} - k) \cdots (h_n - k) \in \mathfrak{D}_{n-k}.$$

Turning this decomposition into generating function gives (9). □

Proof Proof of Theorem 4.2: By (9), we have

$$H(x) = \frac{1 - xC(x)}{1 - 2xC(x)}. \tag{11}$$

Solving (8) for $S(x)$ gives

$$S(x) = \frac{1 + x(H(x) - C(x))/(1-x)}{1 - xC(x)}. \tag{12}$$

Substituting (7) and (11) into (12) then proves Theorem 4.2 after simplification. □

4.2 Pattern pair $(110, 102)$

For a Dyck path $D = h_1 \cdots h_n \in \mathfrak{D}_n$, let $\mathrm{last}(D) = h_n$ be the height of the last east step of D. Let us define $d_{n,m}$ the number of Dyck paths $D \in \mathfrak{D}_n$ with $\mathrm{last}(D) = m$ and introduce $d_n(u) := \sum_{m=0}^{n-1} d_{n,m} u^m$. Let

$$D(u,x) := \sum_{n \geq 1} d_n(u) x^n = x + (1+u)x^2 + (2u^2 + 2u + 1)x^3 + \cdots$$

be the enumerator of Dyck paths by the height of their last steps. It is known (cf. [12, Lem. 5.2]) that $D(u,x)$ is algebraic and

$$D(u,x) = \frac{2x}{1 - 2x + \sqrt{1 - 4ux}}. \tag{13}$$

For any $0 \leq m < \ell \leq n$, denote by $\mathcal{A}_{n,m,\ell}$ the set of $(110, 102)$-avoiding inversion sequences of length n in which the largest entry is m with the left-most occurrence of m in position ℓ. Let $a_{n,m,\ell} = \#\mathcal{A}_{n,m,\ell}$. Then, the first few values of the arrays $[a_{n,m,\ell}]_{0 \leq m < \ell \leq n}$ are:

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 2 & 1 \\ 0 & 0 & 12 & 8 & 4 \\ 0 & 0 & 0 & 19 & 9 \\ 0 & 0 & 0 & 0 & 14 \end{bmatrix}.$$

We have the following recurrence relation for $a_{n,m,\ell}$.

Lemma 4.5. *For $1 \leq m < \ell \leq n$, we have*

$$a_{n,m,\ell} = d_{n-1,m} - d_{\ell-1,m} + d_{\ell-1,m-1} - \chi(m=1, \ell=2) + \sum_{j=0}^{m} \sum_{k=\ell-1}^{n-1} a_{n-1,j,k}, \tag{14}$$

where $a_{n,0,\ell} = \delta_{\ell,1}$ for $1 \leq \ell \leq n$.

Proof: For an element $e \in \mathcal{A}_{n,m,\ell}$, we distinguish two cases:

(a) If there are at least two m appearing as entries of e, then consider the sequence $\hat{e}$ obtained from e by removing e_ℓ. Since e is 110-avoiding, e_n must equal m, which forces the entries of $\hat{e}$ to be weakly increasing, for otherwise there will be a 102 pattern occurs in $\hat{e}$ (this would contradict the fact that $\hat{e}$ is 102-avoiding). Thus, $\hat{e}$ can be considered as a Dyck path whose last east step has height m and the $(\ell-1)$-th east step has height smaller than m. There are exactly $d_{n-1,m} - d_{\ell-1,m}$ many members of $\mathcal{A}_{n,m,\ell}$ arise in this case.

(b) Otherwise, the letter m occurs only once within e. Suppose that j is the second largest letter of e, then we need further to consider two cases:

 (b1) The left-most occurrence of j within e is in positions 1 through $\ell-2$. Since e is 102-avoiding, each entry between the left-most j and m must be a j. This forces all entries after m must also equal j because of e is 110-avoiding. Moreover, as e is 102-avoiding, the subsequence $(e_1, e_2, \ldots, e_{\ell-2})$ is weakly increasing and so members of $\mathcal{A}_{n,m,\ell}$ arising in this case (with j as second largest entry) are in bijection with Dyck paths of length $\ell-2$ with last east step of height j. Summing over all $0 \le j \le m-1$, there are totally $d_{\ell-1,m-1} - \chi(m=1, \ell=2)$ many members of $\mathcal{A}_{n,m,\ell}$ arise in this case.

 (b2) The left-most occurrence of j within e is either in position $\ell-1$ or in positions $\ell+1$ through n. In either case, the entry $e_\ell = m$ is extraneous since the first $\ell-2$ entries of e are governed by a j either coming just before m or at some position after m. The deletion of e_ℓ from e then results in a member of $\mathcal{A}_{n,j,k}$ for some $0 \le j \le m-1$ and $\ell-1 \le k \le n-1$ and summing over these indices gives the last term in (14).

All the above cases together gives (14), as desired. □

For any $n \ge 2$, let

$$a_n(u,v) := \sum_{\ell=2}^{n} \sum_{m=1}^{\ell-1} a_{n,m,\ell} u^m v^\ell.$$

Introduce the generating function $a(x;u,v)$ by

$$a(x;u,v) := \sum_{n\ge 2} a_n(u,v)x^n = uv^2x^2 + (2uv^2 + uv^3 + 2u^2v^3)x^3 + \cdots .$$

Next, we show that recurrence relation (14) can be translated into the following functional equation for $a(x;u,v)$.

Lemma 4.6. *The functional equation for* $a(x;u,v)$ *is*

$$\begin{aligned} a(x;u,v) = \frac{vx^2}{x-1} + \frac{vx}{1-v}D(uv,x) + \left(\frac{uvx-vx}{1-x} - \frac{v^2x}{1-v}\right)D(u,xv) + \\ + \frac{uv^2x}{(1-v)(1-uv)}a(x;uv,1) + \frac{u^2v^2x}{(1-u)(1-uv)}a(x;1,uv) - \frac{uv^2x}{(1-u)(1-v)}a(x;u,v). \quad (15) \end{aligned}$$

Proof: Multiplying both sides of (14) by $x^n u^m v^\ell$ and summing over all $1 \le m < \ell \le n$ will yield a functional equation for $a(x;u,v)$. More precisely, we need to simplify the following terms coming from the right hand side of (14):

$$\begin{aligned}
&\sum_{n\geq 2} x^n \sum_{\ell=2}^{n} v^\ell \sum_{m=1}^{\ell-1} (d_{\ell-1,m-1} - d_{\ell-1,m}) u^m \\
&= \sum_{n\geq 2} x^n \sum_{\ell=2}^{n} v^\ell (u d_{\ell-1}(u) - d_{\ell-1}(u) + 1) \\
&= (u-1) \sum_{\ell\geq 2} d_{\ell-1}(u) v^\ell \sum_{n\geq \ell} x^n + \sum_{\ell \geq 2} v^\ell \sum_{n\geq \ell} x^n \\
&= \frac{(u-1)vx}{1-x} D(u,vx) + \frac{v^2x^2}{(1-x)(1-vx)},
\end{aligned}$$

$$\begin{aligned}
&\sum_{n\geq 2} x^n \sum_{\ell=2}^{n} v^\ell \sum_{m=1}^{\ell-1} d_{n-1,m} u^m \\
&= \sum_{n\geq 2} x^n \sum_{m=1}^{n-1} d_{n-1,m} u^m \sum_{\ell=m+1}^{n} v^\ell \\
&= \frac{v}{v-1} \sum_{n\geq 2} \Big(d_{n-1}(uv) - 1 - v^n (d_{n-1}(u) - 1) \Big) x^n \\
&= \frac{v}{v-1} \left(xD(uv,x) - \frac{x^2}{1-x} - vxD(u,vx) + \frac{v^2x^2}{1-vx} \right),
\end{aligned}$$

$$-\sum_{n\geq 2} x^n \sum_{\ell=2}^{n} \sum_{m=1}^{\ell-1} \chi(m=1,\ell=2) u^m v^\ell = -\sum_{n\geq 2} uv^2 x^n = \frac{uv^2x^2}{x-1},$$

and

$$\begin{aligned}
\sum_{n\geq 2} x^n \sum_{\ell=2}^{n} v^\ell \sum_{m=1}^{\ell-1} u^m \sum_{j=0}^{m} \sum_{k=\ell-1}^{n-1} a_{n-1,j,k} &= \frac{uv^2x^2}{1-x} + \frac{uv^2 x}{(1-v)(1-uv)} a(x;uv,1) \\
&\quad + \frac{u^2v^2x}{(1-u)(1-uv)} a(x;1,uv) - \frac{uv^2x}{(1-u)(1-v)} a(x;u,v),
\end{aligned}$$

where the last equality follows from the same manipulation as in [26, Lem. 3.5]. Summing over all the above expressions gives the functional equation (15). □

Finally, we will solve the functional equation (15) using the kernel method to obtain the following expression for $a(x;1,1)$.

Theorem 4.7. *We have*

$$a(x;1,1) = \frac{x^2 + x^2\sqrt{1-4x}}{(x-1)((3x-1)\sqrt{1-4x} - 4x^2 + 5x - 1)}. \tag{16}$$

Equivalently,

$$|\mathbf{I}_n(110,102)| = 1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}.$$

Proof: We apply the kernel method and set the coefficients of $a(x;u,v)$ on both sides of (15) equal:

$$1 = -\frac{uv^2x}{(1-u)(1-v)}.$$

Solving this equation for u gives

$$u = u(x,v) = \frac{1-v}{1-v-v^2x} \tag{17}$$

Let us introduce

$$\alpha = \alpha(x,v) = uv = \frac{v-v^2}{1-v-v^2x}.$$

Substituting the expression for u given by (17) into (15), and simplifying yields

$$a(x;\alpha,1) = \frac{\alpha(v-1)}{v-\alpha}a(x;1,\alpha) + \frac{x(1-v)(1-\alpha)}{\alpha(1-x)} + \frac{\alpha-1}{\alpha}D(\alpha,x) + \frac{(1-\alpha)(v-\alpha-v\alpha-v^2x)}{v\alpha(1-x)}D(\alpha/v,vx). \tag{18}$$

Observe that $\alpha(x,w) = v$ whenever $w = w(x,v)$ is a root of the quadratic equation

$$(1-vx)w^2 - (1+v)w + v = 0. \tag{19}$$

Replacing v by w in (18) gives

$$a(x;v,1) = E_1(v,w)a(x;1,v) + E_2(v,w) + \frac{(v-1)D(v,x)}{v} + E_3(v,w)D(v/w,wx), \tag{20}$$

where

$$E_1(v,w) = \frac{v(w-1)}{w-v}, \quad E_2(v,w) = \frac{x(1-w)(1-v)}{v(1-x)}$$

and

$$E_3(v,w) = \frac{(1-v)(w-v-vw-w^2x)}{wv(1-x)}.$$

Let w_1 and w_2 denote the two roots of the equation (19). Substituting w_1 and w_2 into (20) then gives

$$a(x;v,1) = E_1(v,w_1)a(x;1,v) + E_2(v,w_1) + \frac{(v-1)D(v,x)}{v} + E_3(v,w_1)D(v/w_1,w_1x)$$

and

$$a(x;v,1) = E_1(v,w_2)a(x;1,v) + E_2(v,w_2) + \frac{(v-1)D(v,x)}{v} + E_3(v,w_2)D(v/w_2,w_2x).$$

Solving this system of equations yields

$$a(x;1,v) = \frac{E_2(v,w_1) - E_2(v,w_2)}{E_1(v,w_2) - E_1(v,w_1)} + \frac{E_3(v,w_1)D(v/w_1, w_1x) - E_3(v,w_2)D(v/w_2, w_2x)}{E_1(v,w_2) - E_1(v,w_1)}$$

Involving expression (13) for $D(u,x)$,

$$w_1 + w_2 = \frac{v+1}{1-vx} \quad \text{and} \quad w_1 w_2 = \frac{v}{1-vx},$$

we get (16) after simplification. This completes the proof of the first statement.

The second statement follows from the fact that

$$1 + \sum_{n\geq 1} |\mathbf{I}_n(110,102)| x^n = a(x;1,1) + \frac{1}{1-x}$$

equals the right-hand side of (6), which ends the proof of the theorem. □

4.3 Pattern pair $(102,120)$

As for the pattern pair $(102,110)$, we consider the set $\mathcal{B}_{n,m,\ell}$ of $(102,120)$-avoiding inversion sequences of length n in which the largest entry is m with the left-most occurrence of m in position ℓ. Denote by $b_{n,m,\ell}$ the cardinality of $\mathcal{B}_{n,m,\ell}$. The first few values of the arrays $[b_{n,m,\ell}]_{0\leq m<\ell\leq n}$ are:

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 8 & 4 & 2 & 1 \\ 0 & 0 & 13 & 7 & 4 \\ 0 & 0 & 0 & 14 & 9 \\ 0 & 0 & 0 & 0 & 14 \end{bmatrix}.$$

The following recurrence relation for $b_{n,m,\ell}$ holds.

Lemma 4.8. *For $1 \leq m < \ell \leq n$, we have*

$$b_{n,m,\ell} = d_{\ell,m-1} + \sum_{j=1}^{m} \sum_{k=\ell}^{n-1} b_{n-1,j,k}. \tag{21}$$

Proof: For an element $e \in \mathcal{B}_{n,m,\ell}$, we distinguish two cases:

(a) If there are at least two m appearing as entries of e, then the left-most m is extraneous concerning avoidance of 102 and 120. Deletion of this m results in a member of $\mathcal{B}_{n,m,i}$ for some $\ell \leq i \leq n-1$. So there are $\sum_{i=\ell}^{n-1} b_{n-1,m,i}$ many members of $\mathcal{B}_{n,m,\ell}$ arise in this case.

(b) Otherwise, the letter m occurs only once within e. Suppose that j is the second largest letter of e, then we need further to consider two cases:

(b1) The left-most occurrence of j within e is in positions 1 through $\ell-1$. Since e is 102-avoiding, the subsequence formed by entries to the left of m must be weakly increasing. Moreover, as e is 120-avoiding, the entries appear after e are all equal j. So members of $\mathcal{B}_{n,m,\ell}$ arising in this case (with j as second largest entry) are in bijection with Dyck paths of length $\ell-1$ with last east step of height j, $0 \le j \le m-1$. Summing over all $0 \le j \le m-1$, there are in total $d_{\ell,m-1}$ many members of $\mathcal{B}_{n,m,\ell}$ arising in this case.

(b2) The left-most occurrence of j within e is in positions $\ell+1$ through n. In each case, the entry $e_\ell = m$ is extraneous since the first $\ell-1$ entries of e are governed by a j at some position after m. The deletion of e_ℓ from e then results in a member of $\mathcal{B}_{n,j,k}$ for some $1 \le j \le m-1$ and $\ell \le k \le n-1$. Summing over these indices gives $\sum_{j=1}^{m-1}\sum_{k=\ell}^{n-1} b_{n-1,j,k}$.

The above cases together gives (21), as desired. □

For any $n \ge 2$, let

$$b_n(u,v) := \sum_{\ell=2}^{n}\sum_{m=1}^{\ell-1} b_{n,m,\ell}u^m v^\ell.$$

Introduce the generating function $b(x;u,v)$ by

$$b(x;u,v) := \sum_{n\ge 2} b_n(u,v)x^n = uv^2x^2 + (2uv^2 + uv^3 + 2u^2v^3)x^3 + \cdots.$$

We could translate recurrence relation (21) into a functional equation for $B(x;u,v)$.

Lemma 4.9. *We have*

$$b(x;u,v) = \frac{u(D(u,vx) - vx - vxD(1,uvx))}{1-x} + \frac{vx}{(1-v)(1-uv)}b(x;uv,1) - \frac{vx}{(1-u)(1-v)}b(x;u,v) + \frac{uvx}{(1-u)(1-uv)}b(x;1,uv). \quad (22)$$

Proof: First we compute

$$\begin{aligned}
&\sum_{n\ge 2}x^n\sum_{\ell=2}^{n}v^\ell\sum_{m=1}^{\ell-1}d_{\ell,m-1}u^m \\
&= u\sum_{n\ge 2}x^n\sum_{\ell=2}^{n}v^\ell(d_\ell(u) - C_{\ell-1}u^{\ell-1}) \\
&= u\sum_{\ell\ge 2}v^\ell(d_\ell(u) - C_{\ell-1}u^{\ell-1})\sum_{n\ge \ell}x^n \\
&= \frac{u(D(u,vx) - vx - vxD(1,uvx))}{1-x}.
\end{aligned}$$

Next we calculate

$$\begin{aligned}
&\sum_{n\geq 2} x^n \sum_{\ell=2}^{n} v^\ell \sum_{m=1}^{\ell-1} u^m \sum_{j=1}^{m}\sum_{k=\ell}^{n-1} b_{n-1,j,k} \\
&= \sum_{n\geq 2} x^n \sum_{\ell=2}^{n} v^\ell \sum_{j=1}^{\ell-1}\sum_{k=\ell}^{n-1} b_{n-1,j,k} \sum_{m=j}^{\ell-1} u^m \\
&= \sum_{n\geq 2} x^n \sum_{j=1}^{n-1}\sum_{\ell=j+1}^{n} v^\ell \sum_{k=\ell} b_{n-1,j,k}\frac{u^j-u^\ell}{1-u} \\
&= \frac{1}{1-u}\sum_{n\geq 2} x^n \sum_{k=2}^{n-1}\sum_{j=1}^{k-1}\left(b_{n-1,j,k}u^j\frac{v^{j+1}-v^{k+1}}{1-v} - b_{n-1,j,k}\frac{(uv)^{j+1}-uv^{k+1}}{1-uv}\right) \\
&= \frac{1}{1-u}\sum_{n\geq 3} x^n \left(\frac{vb_{n-1}(uv,1)-vb_{n-1}(u,v)}{1-v} - \frac{uvb_{n-1}(uv,1)-uvb_{n-1}(1,uv)}{1-uv}\right) \\
&= \frac{vx}{(1-v)(1-uv)}b(x;uv,1) - \frac{vx}{(1-u)(1-v)}b(x;u,v) + \frac{uvx}{(1-u)(1-uv)}b(x;1,uv).
\end{aligned}$$

By the above two expressions, multiplying both sides of (21) by $x^n u^m v^\ell$ and summing over $1 \leq m < \ell \leq n$ gives the functional equation (22). □

We are going to solve the functional equation (22), which will result in the following expression for $b(x,1,1)$.

Theorem 4.10. *We have*

$$b(x;1,1) = \frac{x^2 + x^2\sqrt{1-4x}}{(x-1)((3x-1)\sqrt{1-4x} - 4x^2 + 5x - 1)}. \tag{23}$$

Equivalently,

$$|\mathbf{I}_n(120,102)| = 1 + \sum_{i=1}^{n-1}\binom{2i}{i-1}.$$

Proof: The proof is similar to that of Theorem 4.7. We apply the kernel method and set the coefficients of $b(x;u,v)$ on both sides of (22) equal:

$$1 = -\frac{vx}{(1-u)(1-v)}.$$

Solving this equation for u gives

$$u = u(x,v) = \frac{1-v+vx}{1-v} \tag{24}$$

Let us denote

$$\beta = \beta(x,v) = uv = \frac{v - v^2 + v^2 x}{1-v}.$$

Substituting the expression for u given by (24) into (22), and simplifying yields

$$b(x;1,\beta)=\frac{\beta-v}{\beta-\beta v}b(x;\beta,1)+\frac{(v-\beta)(1-\beta))}{v^2x(x-1)}D(\beta/v,vx) + \frac{(v-\beta)(1-\beta))}{v(1-x)}(D(1,\beta x)+1). \quad (25)$$

Observe that $\beta(x,w)=v$ whenever $w=w(x,v)$ is a root of the quadratic equation

$$(x-1)w^2+(1+v)w-v=0. \quad (26)$$

Replacing v by w in (25) gives

$$b(x;1,v)=\frac{v-w}{v-vw}b(x;v,1)+\frac{(w-v)(1-v))}{w^2x(x-1)}D(v/w,wx) + \frac{(w-v)(1-v))}{w(1-x)}(D(1,vx)+1). \quad (27)$$

substituting the two roots of the quadratic equation (26) into (27) yields a system of two equations, which after eliminating $b(x;1,v)$ and then setting $v=1$ gives the desired expression (23) for $b(x;1,1)$. This completes the proof of the theorem. □

5 Classes 4140(D,E,F): Bell numbers and Stirling numbers

Let Π_n be the set of all set partitions of $[n]$ and denote $\Pi_{n,k}$ the set of partitions in Π_n with k blocks. It is well known that

$$|\Pi_n|=B_n \quad \text{and} \quad |\Pi_{n,k}|=S(n,k),$$

where B_n is the n-th *Bell number* and $\{S(n,k)\}_{1\le k\le n}$ is the triangle of *Stirling numbers of the second kind*. The Stirling numbers $S(n,k)$ satisfy the following basic recurrence (cf. [30, Sec. 1.4]):

$$S(n,k)=kS(n-1,k)+S(n-1,k-1).$$

It was shown by Corteel et al. [15, Theorem 11] that

Theorem 5.1 (Corteel–Martinez–Savage–Weselcouch)**.** *The number of* 011*-avoiding inversion sequences in* $\mathbf{I}_n$ *with* k *zeros is* $S(n,k)$*.*

The proof of this result in [15] was by constructing a recursive bijection from $\mathbf{I}_{n,k}(011)$ to $\Pi_{n,k}$, where $\mathbf{I}_{n,k}(011):=\{e\in\mathbf{I}_n(011):\mathrm{zero}(e)=k\}$. We note that there exists another natural bijection $\eta:\mathbf{I}_{n,k}(011)\to\Pi_{n,k}$ observed in [13]. Given $e\in\mathbf{I}_{n,k}(011)$, the partition $\eta(e)$ can be constructed for $i=1,2,\ldots,n$, step by step, as follows:

- if $e_i=0$, then construct a new block $\{i\}$;
- otherwise, $e_i>0$ and insert i into the block containing e_i.

For example, if $e = (0,0,2,1,0,4) \in \mathbf{I}_{6,3}(011)$, then $\eta(e) = \{\{1,4,6\},\{2,3\},\{5\}\} \in \Pi_{6,3}$. The bijection η provides another proof of Theorem 5.1 .

A statistic whose distribution over a combinatorial object gives the Stirling numbers of the second kind is said to be *2nd Stirling* over such object. In this language, Theorem 5.1 is equivalent to the assertion that "zero" is 2nd Stirling over $\mathbf{I}_n(011)$. Since an inversion sequence contains the pattern 101 or 110 must also contain the pattern 011, we have

Corollary 5.2. *For* $n \geq 1$, $\mathbf{I}_n(011,101) = \mathbf{I}_n(011) = \mathbf{I}_n(011,110)$.

It has been shown in [12, Sec. 6] that $|\mathbf{I}_n(000,101)| = |\mathbf{I}_n(000,110)| = B_n$ and moreover, the statistic "rmin" is 2nd Stirling over $|\mathbf{I}_n(000,110)|$. Unfortunately, all the five classical statistics defined in the introduction are not 2nd Stirling over $\mathbf{I}_n(000,101)$. We pose the following challenging open problem for further research.

Problem 5.3. Find a natural statistic whose distribution over $\mathbf{I}_n(000,101)$ is 2nd Stirling.

For a sequence $e \in \mathbf{I}_n$, let $\mathrm{rep}(e) := n - \mathrm{dist}(e)$ be the number of repeated entries of e. Martinez and Savage [27, Sec. 2.17.3] showed that $|\mathbf{I}_n(010,101)| = B_n$ by constructing a bijection between $\mathbf{I}_n(010,101)$ and $\mathbf{I}_n(011)$. We have the following refinement.

Theorem 5.4. *The statistic* "rep" *is 2nd Stirling over* $\mathbf{I}_n(010,101)$.

Proof: The mapping $e \mapsto e'$ constructed in [27], where $e'_i = 0$ if $e_i \in \{e_1, e_2, \ldots, e_{i-1}\}$ and otherwise $e'_i = e_i$, is a bijection from $\mathbf{I}_n(010,101)$ and $\mathbf{I}_n(011)$ which preserves the statistic "dist". Since each 011-avoiding inversion sequence has distinct positive entries, the statistics "rep" and "zero" coincide on $\mathbf{I}_n(011)$. The result then follows from Theorem 5.1. □

Theorem 5.5. *There exists a bijection* $\phi : \mathbf{I}_n(010,101) \to \mathbf{I}_n(010,100)$ *which preserves the triple* $(\mathrm{dist}, \mathrm{satu}, \mathrm{zero})$. *Consequently, the statistic* "rep" *is 2nd Stirling over* $\mathbf{I}_n(010,100)$.

Proof: Note that $e \in \mathbf{I}_n$ is $(010,101)$-avoiding if and only if whenever $e_i = e_j$ for some $i < j$, then $e_k = e_i$ for every $i \leq k \leq j$. For $e \in \mathbf{I}_n(010,101)$, define $\phi(e) = e'$ where $e'_i = \max\{e_1, e_2, \ldots, e_{i-2}\}$ if $e_{i-1} = e_i < \max\{e_1, e_2, \ldots, e_{i-2}\}$ and otherwise $e'_i = e_i$. For instance, if

$$e = (0,0,0,0,4,3,2,2,2,5,1,1) \in \mathbf{I}_{12}(010,101),$$

then

$$\phi(e) = (0,0,0,0,4,3,2,4,4,5,1,5) \in \mathbf{I}_{12}(010,100).$$

It is routine to check that ϕ is a bijection between $\mathbf{I}_n(010,101)$ and $\mathbf{I}_n(010,100)$ that preserves the triple $(\mathrm{dist}, \mathrm{satu}, \mathrm{zero})$. The second statement then follows from the bijection ϕ and Theorem 5.4. □

6 Class 5798B: A bijection to weighted ordered trees

The integer sequence

$$\{1, 1, 2, 6, 21, 80, 322, 1347, 5798, 25512, \ldots\}$$

with algebraic generating function

$$A(t) = 1 + \frac{tA(t)}{1 - tA(t)^2}$$

appearing as A106228 in the OEIS [28] enumerates many interesting combinatorial objects, including

- $(4123, 4132, 4213)$-avoiding permutations, proved by Albert, Homberger, Pantone, Shar and Vatter [1];
- $(101, 102)$-avoiding inversion sequences, conjectured by Martinez and Savage [27] and verified algebraically by Cao, Jin and Lin [12];
- weighted ordered trees, where each *interior vertex* (non-root, non-leaf) is weighted by a positive integer less than or equal to its outdegree (see [28, A106228]).

In this section, we will prove that this integer sequence also counts $(101, 021)$-avoiding inversion sequences. This is achieved via a new constructed "type"-preserving bijection from ordered trees of n edges to Dyck paths of length n.

Let $\mathbf{t} = (t_1, t_2, \dots, t_k)$ be a sequence of positive integers with $t_1 + t_2 + \cdots + t_k = n$. A Dyck path $D = h_1 h_2 \cdots h_n$ of length n is said to have *type* $\mathbf{t}$ if

$$h_i \neq h_{i+1} \Longleftrightarrow i \in \{t_1, t_1 + t_2, \dots, t_1 + t_2 + \cdots + t_{k-1}\}.$$

In other words, this Dyck path begins with t_1 east steps followed by at least one north step, and then continues with t_2 east steps followed by at least one north step, and then so on. For example, the Dyck path in the right-bottom side of Fig. 2 has type $(3, 1, 2, 1)$.

We next introduce the type for ordered trees. For the basic definitions and terminology concerning ordered trees (or plane trees), see [31, Sec. 1.5]. In an ordered tree T, the outdegree of a vertex $v \in V(T)$, denoted $\deg_v$, is the number of descendants of v. We will order all vertices of T by using the *depth-first order* (also known as *preorder*). For example, the preorder of the vertices of the ordered tree in the right-top side of Fig. 2 are a, b, c, d, e, f, g, h. If v_1 is the root of T and $v_2, v_3, \dots, v_k$ are the interior vertices of T in preorder, then we say T has type $(\deg_{v_1}, \deg_{v_2}, \dots, \deg_{v_k})$. For instance, the tree in our running example has type $(\deg_a, \deg_c, \deg_d, \deg_g) = (3, 1, 2, 1)$. Let $\mathfrak{T}_n$ be the set of all ordered trees with n edges. It is well known that $|\mathfrak{T}_n| = C_n$ and there are already several bijections between $\mathfrak{T}_n$ and $\mathfrak{D}_n$; see the monograph [31] of Stanley on Catalan number. The main result of this section is a construction of a bijection from $\mathfrak{T}_n$ to $\mathfrak{D}_n$, which seems new to the best of our knowledge.

Theorem 6.1. *There exists a type-preserving bijection Φ from $\mathfrak{T}_n$ to $\mathfrak{D}_n$.*

Recall that in Section 4.1, the geometric representation $\mathcal{H}$ is a bijection between $\mathbf{I}_n(021)$ and $\mathcal{A}_n$. This h reduces to a bijection between $\mathbf{I}_n(101, 021)$ and colored Dyck paths in $\mathcal{A}_n$ such that red east steps of the same height are connected. The latter objects are readily seen to be in one-to-one correspondence with Dyck paths in $\mathfrak{D}_n$ such that each first appearance of east steps of the same height is weighted by a positive integer less than or equal to the number of east steps of that height. The later objects are in bijection with weighted ordered trees where each interior vertex is weighted by a positive integer less than or equal to its outdegree because of Theorem 6.1. This proves the following enumerative result.

Corollary 6.2. *Inversion sequences avoiding the pair $(101, 021)$ are counted by the integer sequence A106228. Equivalently, $|\mathbf{I}_n(101, 102)| = |\mathbf{I}_n(101, 021)|$ for $n \geq 1$.*

The construction of Φ will be recursive based on the number of interior vertices of rooted trees. For an ordered tree $T \in \mathfrak{T}_n$ with v as the latest (in preorder) interior vertex, define $\mathrm{cap}(T)$ to be the number of vertices of T after v (in preorder), called the *capacity of* T. While for a Dyck path (or just a word of integers) $D = h_1 h_2 \cdots h_n$, define $\mathrm{cap}(D) = n - h_n$ the *capacity of* D. We now construct $\Phi(T)$ recursively as follows:

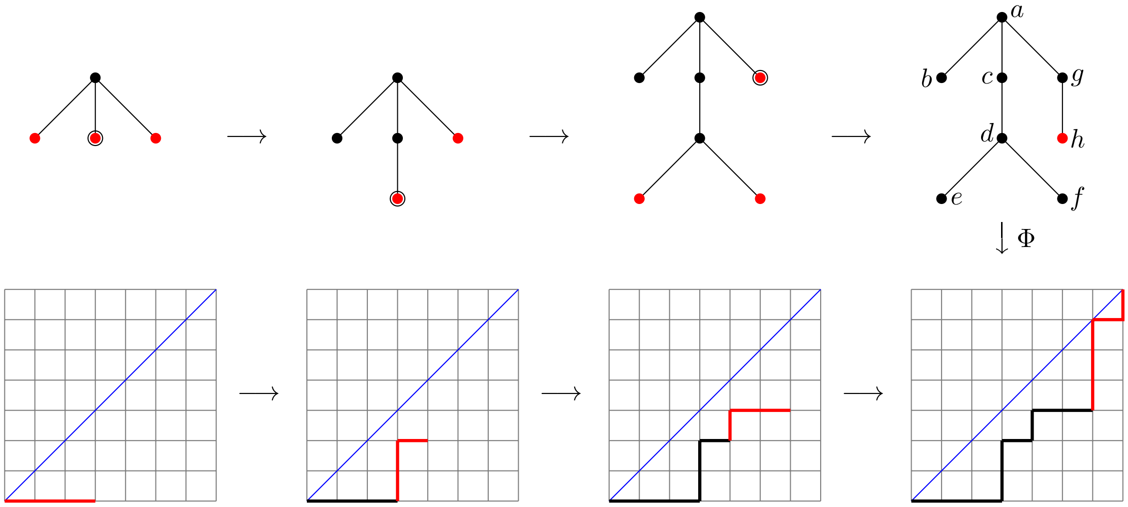

Fig. 2: An example of the construction of the bijection Φ

- if T has no interior vertex, then define $\Phi(T) = 00\cdots0$, the Dyck path consisting of n east steps of height 0;
- otherwise, suppose T has at least one interior vertex and v is the latest (in preorder) interior vertex; let T' be the ordered tree obtained by deleting all descendants of v in T and let $\Phi(T') = h'_1h'_2\cdots h'_\ell$, where $\ell = n - \deg_v$; let v' be the latest interior vertex of T' (if T' has no interior vertex, then set v' to be the root of T') and suppose that v is the h-th leaf after v' (in preorder) in T' with $1 \le h \le \mathrm{cap}(T')$; define
$$\Phi(T) = h'_1h'_2\cdots h'_\ell h_{\ell+1}h_{\ell+2}\cdots h_n,$$
where $h_{\ell+1} = h_{\ell+2} = \cdots = h_n = h'_\ell + h$.

See Fig. 2 for an example of the construction of $\Phi(T)$.

The following crucial property of Φ can be checked routinely by induction on the number of interior vertices of T.

Lemma 6.3. *The mapping Φ is capacity preserving, namely,* $\mathrm{cap}(T) = \mathrm{cap}(D)$ *with $D = \Phi(T)$ considered as a word of integers.*

It follows from the above lemma that the resulting word $\Phi(T)$ is a Dyck path and so the mapping Φ is well-defined. It can be checked from the recursive construction that Φ is a type-preserving injection and thus is a bijection due to the cardinality reason.

7 Class 11624 and indecomposable set partitions

A set partition of $[n]$ is a *indecomposable partition* (or *unsplitable partition*) if the only subset of its blocks that partitions an initial segment of $[n]$ is the full set of all blocks. For our purpose, we arrange the blocks of a partition in decreasing order of their smallest entry, as in $\{7\}\{4\}\{3,5\}\{1,2,6\}$. Since the last three

$n\backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	2	3	1		
4	6	9	6	1	
5	22	31	28	10	1

Tab. 3: The first values of $T_{n,k}$

blocks form a partition of $[6]$, this partition is not indecomposable. The indecomposable partitions can be used to index a free generating set of symmetric functions in noncommutative variables (see [14]).

The indecomposable partitions are counted by the integer sequence [28, A074664], which has several other combinatorial interpretations, including irreducible set partitions and non-nesting permutations. In this section, we prove that this integer sequence also counts $(101, 110)$-avoiding inversion sequences. In order to achieve this, we need to introduce a refinement of this integer sequence.

The triangle $\{T_{n,k}\}_{1\leq k\leq n}$, appearing as A086211 in OEIS [28], are defined by

$$T_{n,k} = T_{n-1,k-1} + kT_{n-1,k} + \sum_{j=k+1}^{n-1} T_{n-1,j} \tag{28}$$

with $T_{1,i} = \delta_{1,i}$. The first few values of $T_{n,k}$ are listed in Table 3. As was proven by Callan [11], $T_{n,k}$ counts the indecomposable partitions of $[n+1]$ such that $n+1$ lies in the k-th block. For example, $T_{3,2} = 3$ and all the indecomposable partitions of $[4]$ such that 4 lies in the second block are $\{2\}\{1,3,4\}$, $\{2,3\}\{1,4\}$ and $\{3\}\{1,2,4\}$. The following result provides another interpretation for $T_{n,k}$.

Theorem 7.1. *For $1 \leq k \leq n$, we have*

$$|\{e \in \mathbf{I}_n(101, 110) : \mathrm{zero}(e) = k\}| = T_{n,k}.$$

In particular, the cardinality of $\mathbf{I}_n(101, 110)$ equals the number of indecomposable partitions of $[n+1]$.

Proof: Let

$$Z_{n,k} = \{e \in \mathbf{I}_n(101, 110) : \mathrm{zero}(e) = k\} \quad \text{and} \quad z_{n,k} = |Z_{n,k}|.$$

It will be sufficient to show that

$$z_{n,k} = z_{n-1,k-1} + kz_{n-1,k} + \sum_{j=k+1}^{n-1} z_{n-1,j} \tag{29}$$

with initial condition $z_{1,i} = \delta_{1,i}$.

Let $Z_{n,k,\ell}$ be the set of sequences $e \in Z_{n,k}$ such that within e there are exactly ℓ ones. For $e \in \mathbf{I}_n$, let us introduce the operation $\sigma_{-1}(e) = (e'_2, e'_3, \ldots, e'_n)$, where $e'_i = e_i - \chi(e_i > 0)$. It is clear that $\sigma_{-1}(e) \in Z_{n-1,k+\ell-1}$. In fact, if $\ell = 0$, then σ_{-1} is a bijection between $Z_{n,k,0}$ and $Z_{n-1,k-1}$ and so $|Z_{n,k,0}| = z_{n,-1,k-1}$. However, if $\ell = 1$, each element of $Z_{n-1,k}$ is the image under σ_{-1} of k elements of $Z_{n,k,1}$; and if $\ell \geq 2$, then each element of $Z_{n-1,k+\ell-1}$ is the image under σ_{-1} of an unique element from $Z_{n,k,\ell}$. To see this, for $e \in Z_{n,k,\ell}$, let $\tilde{e}$ be the subsequence of e consisting of all of the zeros and ones in e. Since e avoids both 101 and 110, when $\ell \geq 2$, the ℓ ones in $\tilde{e}$ must be consecutive and in the last ℓ positions of $\tilde{e}$. But if $\ell = 1$, then there are k ways to place the entry one in $\tilde{e}$. It follows that

$$|Z_{n,k,1}| = kz_{n-1,k} \quad \text{and} \quad |Z_{n,k,\ell}| = z_{n-1,k+\ell-1} \text{ for } \ell \geq 2.$$

Taking all the above cases into account gives

$$z_{n,k} = \sum_{\ell \geq 0} |Z_{n,k,\ell}| = z_{n-1,k-1} + kz_{n-1,k} + \sum_{\ell \geq 2} z_{n-1,k+\ell-1},$$

which proves (29). This completes the proof of the theorem. □

8 Wilf-equivalences

Two sets of patterns Π and Π' are said to be *Wilf-equivalent* on $\mathbf{I}_n$ if for each positive integer n,

$$|\mathbf{I}_n(\Pi)| = |\mathbf{I}_n(\Pi')|.$$

We denote the Wilf-equivalence of Π and Π' by $\Pi \sim \Pi'$. Our results in previous sections together with the works in [9, 27, 12] have already established most of the Wilf-equivalences for inversion sequences avoiding pairs of patterns of length 3; see Tables 1 and 2. The computer data indicates that there are exactly 48 Wilf-equivalence classes, of which 4 classes are remain unproved. The purpose of this section is to prove these remaining 4 classes, which would completely classify all the Wilf-equivalences for inversion sequences avoiding pairs of patterns of length 3.

We will apply a bijection $\varphi : \mathbf{I}_n(210) \to \mathbf{I}_n(201)$ of Corteel et al. [15, Thm. 5]. For the sake of convenience, we review the construction of φ as follows. It was proven in [15, Obs. 3] that 210-avoiding inversion sequences are precisely those that can be partitioned into two weakly increasing subsequences. Given $e \in \mathbf{I}_n(210)$, suppose that $e_{a_1} \leq e_{a_2} \leq \cdots \leq e_{a_t}$ is the sequence of weak left-to-right maxima of e and $e_{b_1} \leq e_{b_2} \leq \cdots \leq e_{b_{n-t}}$ is the subsequence of remaining elements of e. Define $\varphi(e) = (f_1, f_2, \ldots, f_n)$, where

- $f_{a_i} = e_{a_i}$ for $i = 1, \ldots, t$ and
- for each $j = 1, 2, \ldots, n-t$, extract an element of the multiset $B = \{e_{b_1}, e_{b_2}, \ldots, e_{b_{n-t}}\}$ and assign it to $f_{b_1}, f_{b_2}, \ldots, f_{b_{n-t}}$ according to the rule:

$$f_{b_j} = \max\{k \,|\, k \in B - \{f_{b_1}, f_{b_2}, \ldots, f_{b_{j-1}}\} \text{ and } k < \max(e_1, \ldots, e_{b_{j-1}})\}.$$

As an example, we have

$$e = (0, 0, 0, 3, 1, 4, 2, 2) \mapsto \varphi(e) = (0, 0, 0, 3, 2, 4, 2, 1), \tag{30}$$

where the weak left-to-right maxima of e are colored blue.

Theorem 8.1. *The following three Wilf-equivalences hold:*

$$(011, 201) \sim (011, 210), \quad (000, 201) \sim (000, 210) \quad \textit{and} \quad (100, 021) \sim (110, 021).$$

Proof: An inversion sequence is 011-avoiding if its positive entries are distinct, and is 000-avoiding if no entry occurs more than twice. Notice that the bijection φ between $\mathbf{I}_n(210)$ and $\mathbf{I}_n(201)$ is just rearrangement of the entries of inversion sequences. Thus, φ preserves both the patterns 011 and 000. It then follows that φ restricts to a bijection between $\mathbf{I}_n(011, 210)$ (resp. $\mathbf{I}_n(000, 210)$) and $\mathbf{I}_n(011, 201)$ (resp. $\mathbf{I}_n(000, 201)$). This proves the first two Wilf-equivalences.

For the third Wilf-equivalence, define the mapping $e \mapsto e'$ where $e \in \mathbf{I}_n(100, 021)$, by

- $e' = e$, if the entries of e are weakly increasing;
- otherwise, e contains exactly one zero entry, say $e_k = 0$, after the first positive entry. Then let e' be the inversion sequence obtained from e by changing all positive entries before e_k to zeros, except all these that are the first appearances of each positive value.

For instance, we have

$$e = (0, 0, 1, 1, 2, 2, 2, 0, 3, 3, 5) \mapsto e' = (0, 0, 1, 0, 2, 0, 0, 0, 3, 3, 5).$$

It is routine to check that this mapping establishes a one-to-one correspondence between $\mathbf{I}_n(100, 021)$ and $\mathbf{I}_n(110, 021)$. □

Example (30) indicates that φ does not restrict to a bijection between $\mathbf{I}_n(010, 201)$ and $\mathbf{I}_n(010, 210)$, because it does not preserve the pattern 010. However, we still have the following equinumerosity.

Theorem 8.2. *There exists a bijection $\psi : \mathbf{I}_n(201) \to \mathbf{I}_n(210)$ which preserves the pattern 010 and the triple of statistics* $(\mathrm{zero}, \mathrm{dist}, \mathrm{satu})$. *In particular,* $(010, 201) \sim (010, 210)$.

Proof: The idea underlying the construction of ψ is to replace iteratively, occurrences of the pattern 210 in an element of $\mathbf{I}_n(201) \setminus \mathbf{I}_n(210)$ with copies of the pattern 201, preserving the pattern 010.

We call a subsequence $e_a e_b e_c$ ($a < b < c$) of an inversion sequence e that is order isomorphic to 210 (resp. 201) a k-occurrence of 210 (resp. 201) if $e_a = k$. Since ψ is simply the identity on $\mathbf{I}_n(201) \cap \mathbf{I}_n(210)$, we only need to define the map ψ from $\mathbf{I}_n(201) \setminus \mathbf{I}_n(210)$ to $\mathbf{I}_n(210) \setminus \mathbf{I}_n(201)$.

We need to introduce a fundamental operation on an inversion sequence e. Suppose S is a subsequence of e and the set of distinct entries in S is $\{v_1, v_2, \ldots, v_\ell\}$ with $v_1 < v_2 < \cdots < v_\ell$ for some $\ell \geq 2$. Introduce the **cyclic exchange** on e with respect to S according to the following two cases:

- if $v_1 = 0$, then rearrange the entries of S so that all entries of S appear in weakly increasing order;
- otherwise, change all entries of S with value v_t to v_{t+1} for $1 \leq t \leq \ell - 1$ and change all entries of S with value v_ℓ to v_1.

Denote by $\mathcal{C}(e, S)$ the resulting sequence (not necessary an inversion sequence in general). For instance, if $e = (0, 0, 0, 3, 2, 2, 4, 1, 3, 0, 1, 1)$ (resp. $(0, 0, 0, 3, 1, 4, 3, 1, 3, 1, 2, 2)$) with the subsequence S in blue, then

$$\mathcal{C}(e, S) = (0, 0, 0, 3, 0, 1, 4, 1, 3, 1, 2, 2) \quad \text{(resp. } (0, 0, 0, 3, 2, 4, 1, 2, 3, 2, 3, 3)).$$

Since whenever 0 is a letter of S, the sequence $\mathcal{C}(e, S)$ is just rearrangement of e, the operation $\mathcal{C}$ preserves the number of zero entries. It is clear that the operation $\mathcal{C}$ also preserves the number of distinct entries of inversion sequences.

Let $e \in \mathbf{I}_n(201) \setminus \mathbf{I}_n(210)$. Then, e contains the pattern 210 but avoids 201. We apply the following algorithm on e, where t is a temporary variable:

(1) (Start) suppose the distinct entries greater than one appearing in e are $2 \le k_1 < k_2 < \cdots < k_m \le n-1$; set $e^{(0)} = e$, $t \leftarrow 1$ and go to step (2);

(2) go to step (3) if $e^{(t-1)}$ has at least one k_t-occurrence of 210, otherwise, set $e^{(t)} = e^{(t-1)}$, $t \leftarrow t+1$ and do step (2) again;

(3) let $S^{(t)}$ be the subsequence of $e^{(t-1)}$ consisting of entries after the left-most occurrence of k_t whose values are smaller than k_t; set $e^{(t)} = \mathcal{C}(e^{(t-1)}, S^{(t)})$, $t \leftarrow t+1$ and go to step (2).

Finally, this algorithm terminates when $t = m+1$ and we define $\psi(e) = e^{(m)}$. For example, if $e = (0,0,1,2,3,2,2,4,3,4,8,7,5,4,3,0)$ is a sequence in $\mathbf{I}_{15}(201) \setminus \mathbf{I}_{15}(210)$, then

$$\begin{aligned} e = (0,0,1,2,3,2,2,4,3,4,8,7,5,4,3,0) &\to (0,0,1,2,3,0,2,4,3,4,8,7,5,4,3,2) \\ \to (0,0,1,2,3,0,1,4,2,4,8,7,5,4,2,3) &\to (0,0,1,2,3,0,1,4,2,4,8,7,5,2,3,4) \\ \to (0,0,1,2,3,0,1,4,2,4,8,7,2,3,4,5) &\to (0,0,1,2,3,0,1,4,2,4,8,2,3,4,5,7) = \psi(e). \end{aligned}$$

We need to show that ψ is well-defined, i.e., $\psi(e) \in \mathbf{I}_n(210) \setminus \mathbf{I}_n(201)$. It suffices to prove that if $e^{(t-1)}$ has no k_i-occurrence of 210 for any $1 \le i \le t-1$ and contains k_t-occurrence of 210, then $e^{(t)}$ has no k_i-occurrence of 210 for any $1 \le i \le t$ and contains k_t-occurrence of 201. To see this, suppose that the distinct entries of $S^{(t)}$ are $v_1 < v_2 < \cdots < v_\ell$, then v_ℓ must be k_{t-1}. Otherwise, $v_\ell < k_{t-1}$ and all k_{t-1}'s in $e^{(t-1)}$ will appear to the left of the left-most k_t, which implies that $e^{(t-1)}$ has k_{t-1}-occurrence of 210, a contradiction. Next, we claim that all v_ℓ's in $S^{(t)}$ appear before all v_i's for $1 \le i \le \ell-1$. For otherwise, k_t and $k_{t-1} = v_\ell$ will play the roles of 2 and 1 in a pattern 201 of $e^{(t-1)}$, which forces e to contain the pattern 201 because $e^{(t-1)}$ is obtained from e by modifying the entries smaller than k_{t-1}. Thus, we come to the crucial conclusion that

($\star$) all entries of $S^{(t)}$ must appear from left to right in the order: $v_\ell = k_{t-1}, v_1, v_2, \ldots, v_{\ell-1}$.

So after applying the cyclic exchange on $e^{(t-1)}$ with respect to $S^{(t)}$, the entries after the left-most k_t and smaller than k_t appear in weakly increasing order, which removes all k_t-occurrences of 210 and replaces them with k_t-occurrences of 201 in $e^{(t)}$. This proves that ψ is well-defined.

In view of the property ($\star$), each cyclic exchange operation involved in step (3) is reversible and so ψ is bijective. We have already known that the cyclic exchange operation $\mathcal{C}$ preserves the pair of statistics $(\mathrm{zero}, \mathrm{dist})$, and so does ψ. Moreover, since in step (3) of the algorithm of constructing ψ, we just modify the entries after some k_t with values smaller than k_t, ψ also preserves the statistic 'satu'. Finally, we claim that if $e^{(t)} = \mathcal{C}(e^{(t-1)}, S^{(t)})$, then $e^{(t-1)}$ contains 010 if and only if $e^{(t)}$ contains 010. If aba ($a < b$) is a 010-subsequence of $e^{(t-1)}$, then this aba can not be a subsequence of $S^{(t)}$ because of property ($\star$). We distinguish several cases:

1) all letters of aba are not in $S^{(t)}$, then aba is a 010 pattern of $e^{(t)}$.

2) both a's are entries of $S^{(t)}$, then

   2a) if $a = 0$, then subsequence $0k_t0$ is a 010 pattern of $e^{(t)}$, where the left 0 is the initial zero of $e^{(t)}$ and the right 0 is a zero entry (must exist) after k_t;

   2b) otherwise $a \ne 0$, then both a's will be changed into two entries with the same value smaller than b (since $b > k_t$), which together with b forms a 010 pattern of $e^{(t)}$.

3) only the right a is an entry of $S^{(t)}$, then still aba is a 010-subsequence of $e^{(t)}$.

4) ba are subsequence of $S^{(t)}$, then $ak_t a$ is a 010-subsequence of $e^{(t)}$.

In all cases, $e^{(t)}$ contains the pattern 010, which proves the 'only if' side of the claim. The 'if' side of the claim can be proven by similar discussions, which will be omitted. This completes the proof of the theorem. □

8.1 Revisiting $(011, 201)$ and $(011, 210)$: an unbalanced Wilf-equivalence conjecture

Following Burstein and Pantone [10], a Wilf-equivalence is called *unbalanced* if the two sets of patterns do not contain the same number of patterns of each length. The first two unbalanced Wilf-equivalences for classical patterns in permutations were proved in [10]. For inversion sequences, several instances of unbalanced Wilf-equivalences have been proved by Martinez and Savage [27] such as

$$|\mathbf{I}_n(021)| = S_n = |\mathbf{I}_n(210, 201, 101, 100)| \quad \text{and} \quad |\mathbf{I}_n(011)| = B_n = |\mathbf{I}_n(000, 110)|.$$

Regarding the pattern pairs $(011, 201)$ and $(011, 210)$, we make the following unbalanced Wilf-equivalence conjecture.

Conjecture 8.3. The following Wilf-equivalences hold:

$$(011, 201) \sim (011, 210) \sim (110, 210, 120, 010) \sim (100, 210, 120, 010).$$

For Conjecture 8.3, the first equivalence has been confirmed in Theorem 8.1, while the third equivalence was established in [27, Thm. 62], as

$$\mathbf{I}_n(110, 210, 120, 010) = \mathbf{I}_n(-, >, \geq) \quad \text{and} \quad \mathbf{I}_n(100, 210, 120, 010) = \mathbf{I}_n(\neq, \geq, \geq).$$

The open problem is the second equivalence. Though unable to prove Conjecture 8.3, we find functional equations that can be applied to compute $|\mathbf{I}_n(011, 201)|$ and $|\mathbf{I}_n(\neq, \geq, \geq)|$ for larger n, in the rest of this section.

Observe that 011-avoiding inversion sequences are these whose positive entries are distinct. For $1 \leq m < \ell \leq n$, let us consider the number $c_{n,m,\ell}$ of $(011, 201)$-avoiding inversion sequences of length n with $e_\ell = m$ as the unique largest entry. Let

$$c_{n,m}(v) = \sum_{\ell=m+1}^{n} c_{n,m,\ell} v^{\ell-m-1} \quad \text{and} \quad c_n(u, v) = \sum_{m=0}^{n-1} c_{n,m}(v) u^m.$$

Define the generating function $C(x; u, v)$ by

$$C(x; u, v) := \sum_{n \geq 1} c_n(u, v) x^n = x + (1 + u)x^2 + (1 + u + uv + 2u^2)x^3 + \cdots.$$

Then $C(x; u, v)$ satisfies the following functional equation.

Proposition 8.4. *For $1 \leq m < \ell \leq n$, we have the recursion*

$$c_{n,m,\ell} = \sum_{j=0}^{m-1} \sum_{i=j+1}^{\ell} c_{n-1,j,i} \tag{31}$$

with $c_{n,0,\ell} = \delta_{1,\ell}$. Equivalently, there holds the functional equation

$$\left(1 - \frac{ux}{v(1-v)} - \frac{u}{v}\right) C(x;u,v) = \frac{x}{1-x} - \frac{ux}{1-v} C(vx;u/v,1) + \frac{u(1-x)}{v} C(x;u,0). \tag{32}$$

Proof: Let $\mathcal{C}_{n,m,\ell}$ be the set of sequences $e \in \mathbf{I}_n(011, 201)$ with $e_\ell = m$ as the unique largest entry. For $e \in \mathcal{C}_{n,m,\ell}$, since e is 201-avoiding, the entries after e_ℓ are weakly decreasing. Thus, the deletion of e_ℓ from e results in an inversion sequence in $\mathcal{C}_{n-1,j,i}$ for some $0 \leq j \leq m-1$ and $j+1 \leq i \leq \ell$. Conversely, from each inversion sequence $e \in \mathcal{C}_{n-1,j,i}$ for $0 \leq j \leq m-1$ and $j+1 \leq i \leq \ell$, we can construct one sequence in $\mathcal{C}_{n,m,\ell}$ by adding a new entry m to the ℓ-th entry of e. This proves the recurrence relation (31) for $c_{n,m,\ell}$.

For $1 \leq m < \ell \leq n$, it follows from (31) that

$$c_{n,m,\ell} - c_{n,m-1,\ell} = \sum_{i=m}^{\ell} c_{n-1,m-1,i}.$$

Multiplying both sides by $v^{\ell-m-1}$ and summing over $m < \ell \leq n$ gives

$$\begin{aligned}
&c_{n,m}(v) - v^{-1}(c_{n,m-1}(v) - c_{n,m-1}(0)) \\
&= \sum_{\ell=m+1}^{n} v^{\ell-m-1} \sum_{i=m}^{\ell} c_{n-1,m-1,i} \\
&= -v^{-1} c_{n-1,m-1,m} + \sum_{i=m}^{n} c_{n-1,m-1,i} \sum_{\ell=i}^{n} v^{\ell-m-1} \\
&= \frac{1}{1-v}(v^{-1} c_{n-1,m-1}(v) - v^{n-m} c_{n-1,m-1}(1)) - v^{-1} c_{n-1,m-1}(0)
\end{aligned}$$

for $m \geq 1$ with $c_{n,0}(v) = 1$. Multiplying both sides by u^m and summing over $1 \leq m < n$ results in

$$\left(1 - \frac{u}{v}\right) c_n(u,v) + \frac{u}{v} c_n(u,0) = 1 + \frac{u}{1-v}(v^{-1} c_{n-1}(u,v) - v^{n-1} c_{n-1}(u/v,1)) - \frac{u}{v} c_{n-1}(u,0).$$

Multiplying both sides by x^n and summing over $n \geq 2$ yields (32) after simplification. □

Next we will use the generating tree technique (see [7]) to count $(\neq, \geq, \geq)$-avoiding inversion sequences. For each $e \in \mathbf{I}_n(\neq, \geq, \geq)$, let $m_1(e)$ and $m_2(e)$ be the greatest element and the second greatest element in $\{e_i : i \in [n]\} \cup \{-1\}$, respectively. Introduce the parameters (p, q) of e by

$$p = n - m_1(e) \quad \text{and} \quad q = m_1(e) - m_2(e).$$

For example, the parameters of $(0,0,2,1)$ is $(2,1)$ and the parameters of $(0,0,0,0)$ is $(4,1)$. We have the following rewriting rule for $(\neq, \geq, \geq)$-avoiding inversion sequences.

Lemma 8.5. *Let $e \in \mathbf{I}_n(\neq, \geq, \geq)$ be an inversion sequence with parameters (p, q). Exactly $p+q$ inversion sequences in $\mathbf{I}_{n+1}(\neq, \geq, \geq)$ when removing their last entries will become e, and their parameters are respectively:*

$$(p+1, q), (p+1, q-1), \ldots, (p+1, 1)$$
$$(p, 1), (p-1, 2), \ldots, (1, p).$$

Proof: It is clear that the sequence $f := (e_1, e_2, \ldots, e_n, b)$ is in $\mathbf{I}_{n+1}(\neq, \geq, \geq)$ if and only if $m_2(e) < b \leq n$. We distinguish the following three cases:

- If $m_2(e) < b < m_1(e)$, then $m_1(f) = m_1(e)$ and $m_2(f) = b$. These contribute the parameters $(p+1, q-1), (p+1, q-2), \ldots, (p+1, 1)$.
- If $b = m_1(e)$, then $m_1(f) = m_1(e)$ and $m_2(f) = m_2(e)$. So this case contributes the parameters $(p+1, q)$.
- If $m_1(e) < b \leq n$, then $m_1(f) = b$ and $m_2(f) = m_1(e)$. These contribute the parameters $(p, 1), (p-1, 2), \ldots, (1, p)$.

The above three cases together give the rewriting rule for $(\neq, \geq, \geq)$-avoiding inversion sequences. □

Using the above lemma, we can construct a *generating tree* (actually an infinite rooted tree) for $(\neq, \geq, \geq)$-avoiding inversion sequences by representing each element as its parameters as follows: the root is $(1, 1)$ and the children of a vertex labelled (p, q) are those generated according to the rewriting rule in Lemma 8.5. For an inversion sequence $e \in \mathbf{I}_n$, let $|e| = n$ denote the size of e. Introduce $E_{p,q}(x) = \sum_e x^{|e|}$, where the sum runs over all $(\geq, \geq, -)$-inversion sequences with parameters (p, q). Define the formal power series

$$E(u, v) = E(x; u, v) := \sum_{p,q \geq 1} E_{p,q}(x) u^p v^q = uvx + (uv + u^2 v)x^2 + \cdots.$$

We can turn this generating tree into a functional equation as follows.

Proposition 8.6. *We have the following functional equation for $E(u, v)$:*

$$\left(1 + \frac{uvx}{1-v}\right) E(u, v) = uvx + \left(\frac{uvx}{1-v} + \frac{uvx}{u-v}\right) E(u, 1) - \frac{uvx}{u-v} E(v, 1). \tag{33}$$

Proof: Since in the generating tree for $(\neq, \geq, \geq)$-avoiding inversion sequences, each vertex other than the root $(1, 1)$ can be generated by a unique parent, we have

$$\begin{aligned} E(u, v) &= uvx + x \sum_{p,q \geq 1} E_{p,q}(x) \left(u^{p+1} \sum_{i=1}^{q} v^i + \sum_{i=1}^{p} u^{p+1-i} v^i \right) \\ &= uvx + x \sum_{p,q \geq 1} E_{p,q}(x) \left(\frac{uv(u^p - u^p v^q)}{1-v} + \frac{uv(u^p - v^p)}{u-v} \right) \\ &= uvx + \frac{uvx}{1-v}(E(u, 1) - E(u, v)) + \frac{uvx}{u-v}(E(u, 1) - E(v, 1)), \end{aligned}$$

which is equivalent to (33). □

Can the two functional equations (33) and (32) be solved to prove Conjecture 8.3?

9 Concluding remarks

Since the publications of Corteel, Martinez, Savage and Weselcouch [15] and Mansour and Shattuck [27] on enumeration of inversion sequences avoiding a single pattern of length 3, patterns in inversion sequences have attracted considerable attentions. Many interesting enumeration results and surprising connections with special integer sequences and functions have already been found [2, 3, 5, 9, 12, 18, 22, 20, 23, 24, 27, 25, 35], some of which are still open. As one of the most attractive conjectures, Beaton, Bouvel, Guerrini and Rinaldi [5, Conj. 23] suspected that

$$|\{e \in \mathbf{I}_n(110) : \mathrm{zero}(e) = k\}| = |\{\pi \in \mathfrak{S}_n(\underline{23}14) : \mathrm{rmin}(\pi) = k\}|$$

for $1 \leq k \leq n$, where $\mathrm{rmin}(\pi)$ denotes the number of right-to-left minima of π.

In this paper, we finish the classification of all the Wilf-equivalences for inversion sequences avoiding pairs of length-3 patterns and establish their further connections to some special OEIS sequences and classical combinatorial objects. Can the generating functions, recurrence relations or any general expressions be obtained for the avoidance classes for the pattern pairs that have marked with "open" (new in the OEIS [28]) in column 3 in Table 2?

For classical patterns in inversion sequences, one further direction to continue would be to consider a triple (or multi-tuple) of length-3 patterns or a single pattern of length 4. In particular, regarding the integer sequence A279561 in OEIS [28], the following conjectured enumeration has been discovered by Jun Ma and the second named author.

Conjecture 9.1 (Lin and Ma 2019)**.** For $n \geq 1$, we have

$$|\mathbf{I}_n(0012)| = 1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}.$$

In other words, there holds the unbalanced Wilf-equivalence

$$(0012) \sim (021, 120).$$

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