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On path-cycle decompositions of triangle-free graphs

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In this work, we study conditions for the existence of length-constrained path-cycle decompositions, that is, partitions of the edge set of a graph into paths and cycles of a given minimum length. Our main contribution is the characterization of the class of all triangle-free graphs with odd distance at least 3 that admit a path-cycle decomposition with elements of length at least 4. As a consequence, it follows that Gallai's conjecture on path decomposition holds in a broad class of sparse graphs.

Keywords: path decomposition, cycle decomposition, length constraint, odd distance, triangle-free, Conjecture of Gallai

1 Introduction

All graphs considered here are simple, that is, without loops and multiple edges. As usual, for a graph G, we denote by $V(G)$ and $E(G)$ its vertex set and edge set, respectively. A collection of subgraphs $\mathcal{D}$ of G is called a *decomposition of* G if each edge of G is contained in exactly one subgraph of $\mathcal{D}$. If a decomposition $\mathcal{D}$ of G consists only of paths (resp. cycles), then we say that it is a *path decomposition* (resp. *cycle decomposition*) of G, and if it consists of paths and cycles, then we say that it is a *path-cycle decomposition*.

About fifty years ago, according to Lovász Lovász ((1968)), Gallai conjectured the following bound on the cardinality of a path decomposition of a connected graph:

Conjecture 1 (Gallai's Conjecture) *Every connected graph G on n vertices has a path decomposition of cardinality at most* $\lceil \frac{n}{2} \rceil$.

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Despite many efforts and attempts Donald ((1980)); Fan ((2005)); Harding and McGuinness ((2014)); Lovász ((1968)); Pyber ((1996)) to prove Gallai's conjecture, it remains unsolved. The earliest and major progress towards this conjecture was made by Lovász. In Lovász ((1968)), he proved that every graph on n vertices has a path-cycle decomposition of cardinality at most $\lfloor n/2 \rfloor$. Since any path-cycle decomposition of a graph has at least $p/2$ paths, where p is the number of odd vertices in the graph, Lovász's result implies that graphs with at most one even vertex satisfy Gallai's conjecture; an *even vertex* (resp. *odd vertex*) is a vertex with even degree (resp. odd degree). Dean and Kouider Dean and Kouider ((2000)) proved that every graph G has a path decomposition into at most $\frac{p}{2} + \left\lfloor \frac{2}{3} q \right\rfloor$ paths, where p (resp. q) is the number of odd (resp. even non-isolated) vertices in G. Later, Harding and McGuinness Harding and McGuinness ((2014)) proved that this bound can be greatly improved for graphs of large girth. They showed the bound $\frac{p}{2} + \left\lfloor \frac{g+1}{2g} q \right\rfloor$, where $g \geq 4$ denotes the girth of the graph G. As these works suggest, it seems to be particularly difficult to guarantee the validity of Gallai's conjecture on graphs with many even vertices. Indeed, the broadest subclass of Eulerian graphs, excluding the complete graphs on an odd number of vertices, for which the conjecture is known to be true is the one composed of graphs of maximum degree 4 (see Favaron and Kouider ((1988))).

Our work contributes in several directions. Nevertheless, it is interesting in its own right. Let $\mathcal{F}$ denote the set of paths and cycles of length (number of edges) at least 4. Our main result is the characterization of the class of triangle-free graphs where odd vertices are at distance at least three that do not admit a decomposition into copies of graphs from $\mathcal{F}$. We call those graphs *hanging-square graphs*; which among others, satisfy Gallai's conjecture and can be recognized in polynomial time. As a consequence, we can guarantee that Gallai's conjecture holds for graphs with a large number of even vertices and linear number of edges. In particular, we verify the conjecture for a subclass of planar graphs. In the following subsection, we formalize our results.

Contributions

The *odd distance* of a graph G, denoted by $d_o(G)$, is the minimum distance between any pair of odd vertices of G. A path-cycle decomposition $\mathcal{D}$ of G is called a *4-pc decomposition* if every element of $\mathcal{D}$ has length at least 4; in other words, a 4-pc decomposition is a decomposition into copies of graphs from $\mathcal{F}$. We recall that a graph is *triangle-free* if it does not have cycles of length 3.

We focus our studies on the following set of graphs:

$$\mathcal{G} := \{G : G \text{ is a connected triangle-free graph with } d_o(G) \geq 3\}.$$

We note that $\mathcal{G}$ corresponds to the family of graphs that can be obtained from any triangle-free Eulerian graph by removing a matching M (possibly empty) that satisfies the following property: for every pair $e, e' \in M$, the minimum distance between an end vertex of e and an end vertex of e' is at least 3.

The main contributions of this work are the next theorem and some of its consequences. (For the definitions of *hanging-square graph* and *skeleton*, see Section 2.)

Theorem 1 *A graph in $\mathcal{G}$ has a 4-pc decomposition if and only if it is not a hanging-square graph.*

As a corollary, we have the following statements.

Corollary 2 *Every graph in $\mathcal{G}$ on n vertices with at most $4\lceil n/2 \rceil$ edges has a path decomposition into at most $\lceil n/2 \rceil$ paths.*

From the proof of Corollary 2, one can also obtain that each path of the decomposition has length at least 3, which is best possible. Gallai's conjecture is open in the class of planar graphs. This is quite surprising if we consider that Hajós' conjecture, which states that every Eulerian graph on n vertices has a cycle decomposition of cardinality at most $\lfloor n/2 \rfloor$, has been positively settled for planar graphs Seyffarth ((1992)). Since a planar triangle-free graph on n vertices has at most $2n-4$ edges, the next result follows immediately from Corollary 2.

Corollary 3 *Every planar graph in $\mathcal{G}$ satisfies Gallai's conjecture.*

We believe that this work contributes with a substantial step towards showing that Gallai's conjecture holds for the class of planar graphs.

As we will see, the class of hanging square graphs can be defined recursively, and for that, we define first the subclass of such graphs that are acyclic, and name them skeletons. For them, the following holds.

Corollary 4 *Every tree in $\mathcal{G}$ either has a decomposition into paths of length at least 4 or is a skeleton.*

We observe that the skeletons can be recognized in polynomial time. Thus, in view of the above corollary, the *certificate* that a tree is a skeleton (its *building sequence*, as we will define) can be seen as a short certificate that it cannot be decomposed into paths of length at least 4. At the end of Section 5, we discuss a polynomial time strategy to recognize hanging-square graphs.

Organization of the paper

In Section 2, we define the class of skeletons and hanging-square graphs. Section 3 is devoted to the proof of Theorem 1. In Subsection 3.4 we show Corollary 2. Section 4 contains several properties regarding hanging-square graphs which are fundamental for the proofs of Theorem 1. Finally, Section 5 contains some concluding remarks.

2 Hanging-square graphs

Throughout this paper, we denote a path of length k (number of its edges) by a sequence of its vertices, as for example, $P = x_0x_1 \cdots x_k$ and say that it is a *k-path*.

In the following, we introduce a class of trees, called *skeletons*. For that, we consider k-paths $x_0 \cdots x_k$, with $k \in \{3,4,6\}$, and for each $k \in \{4,6\}$, we say that (the middle vertex) $x_{\frac{k}{2}}$ is the *joint* of P.

Definition 1 (Skeletons) *A* skeleton *is a tree T that admits a black-red coloring λ of $V(T)$, a sequence $T_0, T_1, \ldots, T_t$ of trees, and a sequence $P_1, \ldots, P_t$ of paths, each of which is a 4-path or a 6-path, such that:*

- *T_0 is a 3-path, $T_t = T$, and for $i \in [t]$ the tree T_i is obtained from T_{i-1} by adding P_i so that the joint of P_i is identified with a vertex of T_{i-1};*
- *for $T_0 = x_0x_1x_2x_3$, we have $\lambda(x_0) = \lambda(x_3) =$ black and $\lambda(x_1) = \lambda(x_2) =$ red;*
- *for each $P_i = x_0 \cdots x_k$, $i \in [t]$, it holds that $\lambda(x_0) = \lambda(x_k) =$ black and $\lambda(x_1) = \lambda(x_{k-1}) =$ red. If $k = 4$, then $\lambda(x_2) =$ red. If $k = 6$, then $\lambda(x_3) =$ black and $\lambda(x_2) = \lambda(x_4) =$ red.*

In Figure 1 we show an example of a skeleton T, obtained from a sequence $T_0, T_1, \ldots, T_7 = T$.

The following observation helps understanding skeletons.

Observation 1 *Let T be a skeleton and λ, λ' be colorings of $V(T)$ as in Definition 1. Then, $\lambda = \lambda'$, the set of odd vertices of T is $\{v : \lambda(v) = black\}$ and the set of even vertices of T is $\{v : \lambda(v) = red\}$.*

The sequence $P_0, P_1, \ldots, P_t$, where $P_0 = T_0$, is called a *building sequence* of T and each P_i a *building path*. We might denote the skeleton T by its building sequence $P_0, P_1, \ldots, P_t$. Every time we consider a skeleton T, we implicitly assume that it comes with a black-red coloring λ, as defined above. The following result is not needed in what follows, but it is a noteworthy property of skeletons.

Observation 2 *If T is a skeleton, then all building sequences of T have the same number of building paths.*

We refer to each 3-path of T colored black-red-red-black as a *brrb-path*. Those paths play a fundamental role throughout this work. In particular, the following holds: for every brrb-path P in T, there is a building sequence of T that starts at P (see Proposition 7).

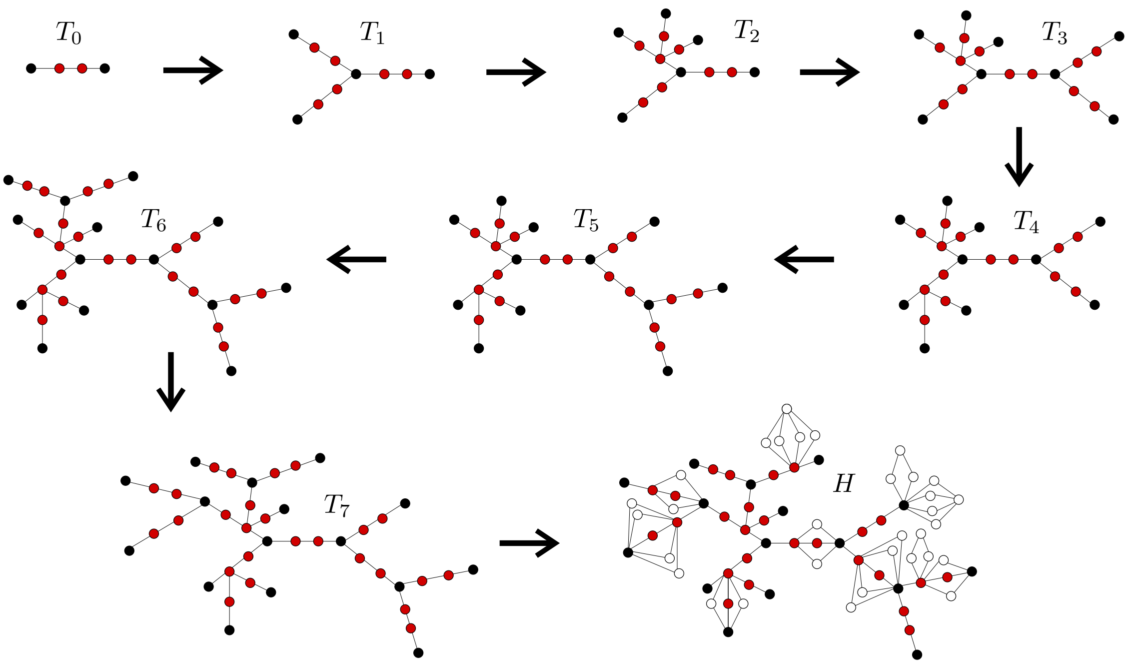

Fig. 1: Construction of a skeleton T obtained from the sequence $T_0, T_1, \ldots, T_7 = T$. It has a building sequence $P_0, P_1, \ldots, P_7$, where $P_0 = T_0$, P_2 and P_4 are 4-paths, and P_1,P_3,P_5 and P_7 are 6-paths. The graph H is an example of a hanging square graph with skeleton T.

For simplicity, we may refer to a cycle of length 4 as a *square*.

Definition 2 (Bunch of squares) *Let $k \geq 1$. A* bunch B of k squares *(or simply, a* bunch*) is a graph obtained from the union of k pairwise edge-disjoint squares, say $Q_1, \ldots, Q_k$, such that each of these squares contains two non-adjacent vertices a, b, and $V(Q_i) \cap V(Q_j) = \{a, b\}$ for $1 \leq i < j \leq k$.*

In other words, a bunch B of k squares is a complete bipartite graph $K_{2,2k}$, where one of the partition classes consists of a and b.

The common non-adjacent vertices a, b of a bunch are called *joints*. In the case that the bunch is a square (namely, k=1 in Definition 2) there are two pairs of joints. For consistency, we consider only one of them as joints. Thus, each bunch has exactly one pair of joints.

Let $\mathcal{B}$ be a set of bunches. We say that a bunch $B \in \mathcal{B}$ with joints a, b is *maximal* if there is no $B' \in \mathcal{B}$ distinct of B with joints a, b.

Definition 3 (Hanging-square graphs) *A graph H is a hanging-square graph if it is the union of a skeleton T_H and a set $\mathcal{B}$ of maximal bunches so that the following holds.*

(i) For each $B \in \mathcal{B}$, the vertices in $V(B) \cap V(T_H)$ are joints of B; accordingly, if $|V(B) \cap V(T_H)| = 1$, then we call B a 1-bunch*, and if $|V(B) \cap V(T_H)| = 2$, then we call B a* 2-bunch*. Moreover, if $B' \in \mathcal{B}$ and $V(B) \cap V(B') \neq \emptyset$, then $V(B) \cap V(B') \subseteq V(T_H)$.*

(ii) If B is a 2-bunch, then there exists a brrb-path $P_B = x_0x_1x_2x_3$ in T_H such that x_0, x_2 are the joints of B. Moreover, if $B' \in \mathcal{B}$ and $B \neq B'$, then $P_B \neq P_{B'}$. We say that the path P_B is occupied *at x_0, x_2 by B.*

(iii) If $P_B = x_0x_1x_2x_3$ is a brrb-path occupied at x_0, x_2 by a 2-bunch B, then $d_H(x_1) = 2$. Moreover, for $H' = H - E(B)$, we have either $d_{H'}(x_0) = 1$ or $d_{H'}(x_2) = 2$.

In particular, in Definition 3, item (ii) says that each brrb-path is occupied either at none, or at 2 vertices and, item (iii) says that x_1 is neither the joint of a 1-bunch, nor the joint of a building path of T_H, and that joints of 1-bunches and joints of building paths of T_H may correspond either to x_0 and x_3, or to x_2 and x_3 (namely, such joints cannot simultaneously correspond to x_0 and x_2).

We now prove that hanging-square graphs satisfy Gallai's conjecture. For that, we use a result of Fan Fan ((2005)). A *block* of a graph is a maximal connected subgraph without a cut vertex (i.e. a vertex whose deletion increases its number of connected components). Given a graph G, the *even graph* of G is the subgraph of G induced by the vertices of even degree in G.

Theorem 5 (Corollary of the Main Theorem of Fan ((2005))) *Let G be a graph. If each block of the even graph of G is a triangle-free graph of maximum degree at most 3, then G satisfies Gallai's conjecture.*

Proposition 6 *If H is a hanging-square graph, then H satisfies Gallai's conjecture.*

Proof: Let H be a hanging-square graph and T_H be its skeleton. If C and C' are squares such that $V(V) \cap V(C') \neq \emptyset$, $E(V) \cap E(C') = \emptyset$, and $C \cup C'$ is triangle-free, then one can easily find a decomposition of $C \cup C'$ into two paths. Hence, we can assume that $H - E(T_H)$ is a union of isolated squares and vertices.

Let H' be the even graph of H. On the one hand, if $v \in V(H') \cap V(T_H)$ and the degree of v is at least 2, then v is a cut vertex of H'. On the other hand, if $v \in V(H') - V(T_H)$, then v is a vertex of a square. Hence, each block of H' is a vertex, an edge or a square. Due to Theorem 5, graph H satisfies Gallai's conjecture. □

3 Proof of the main results

In this section we discuss the proof of Theorem 1 and of Corollary 2. Recall that a 4-pc decomposition is a decomposition into paths and cycles whose lengths are at least 4. The main result of this work is a characterization of the class of connected triangle-free graphs with odd distance at least 3 having no 4-pc decompositions. We prove that this class is exactly the class of all hanging-square graphs. The proof of Theorem 1 relies on the following two facts:

Fact 1. Each graph in $\mathcal{G}$ is a hanging-square graph, or has a 4-pc decomposition (Theorem 9).

Fact 2. Hanging-square graphs do not admit a 4-pc decomposition (Proposition 10).

The proof of Fact 1 needs the use of some technical results regarding the structure of the hanging-square graphs which are postponed to Section 4.

In Section 3.1, we introduce some notions and results that are helpful in the proofs. In particular, at the end of the subsection, we prove the first step to complete Theorem 9. Namely, we show that trees in $\mathcal{G}$ are skeletons, or have a 4-pc decomposition.

3.1 Properties of skeletons

To simplify notation, if E' is a a subset of the edge set of a graph G, we denote by $G - E'$ the graph obtained from G by first removing the edges from E' and then deleting all isolated vertices. Consistently, whenever we refer to the deletion of an edge set of a graph, we also consider that the resulting graph has no isolated vertices.

Observation 3 *Let T be a skeleton. Then, each red vertex in T has a neighbor that is black.*

Proof: Let $P_0, P_1, \ldots, P_t$ be a building sequence of T. We proceed by induction on t. For $t = 0$ the statement trivially holds. We assume that $t > 0$. Suppose P_t is a 4-path. The joint of P_t is red and is a vertex of the skeleton $P_0, P_1, \ldots, P_{t-1}$; thus, by induction hypothesis the joint has a neighbor that is black. If P_t is a 6-path, the proof follows by induction hypothesis and as a direct consequence of the coloring of P_t. □

Let P be a building path of a skeleton; that is, a 4-path or a 6-path. We write $P = P' \sqcup \tilde{P}$ if $\{P', \tilde{P}\}$ is a decomposition of P into two paths of equal length. Note that both, P' and $\tilde{P}$, have as an end vertex the joint of P.

Observation 4 *Let T be a skeleton. Then each edge of T belongs to a brrb-path P.*

Proof: Let λ be the coloring of T and $P_0, P_1, \ldots, P_t$ be a building sequence of T such that $P_i = P'_i \sqcup \tilde{P}_i$. Let $e \in E(T)$. If $e \in E(P_0)$, then $P = P_0$. If $e \in P_i$ for some i, and P_i is a 6-path, then $P \in \{P'_i, \tilde{P}_i\}$. If P_i is a 4-path, according to the proof of Observation 3, the joint v of P_i is adjacent to a vertex u (that belongs to the skeleton $P_0, P_1, \ldots, P_{i-1}$) such that $\lambda(u)$ is black. In this case, $P \in \{P'_i \cup vu, \tilde{P}_i \cup vu\}$. □

The following is a key property that will be used many times throughout this paper.

Proposition 7 *Let T be a skeleton. For every brrb-path P in T, there is a building sequence of T that starts at P.*

Proof: We proceed by induction on $|E(T)|$. In case that T is a 3-path the result is trivial. Assume $|E(T)| > 3$ and let $P_0, P_1, \ldots, P_t$ be a building sequence of T. By induction hypothesis, the statement holds for the skeleton T' with building sequence $P_0, P_1, \ldots, P_{t-1}$. Therefore, it suffices to prove the statement in the case that P is a brrb-path of T such that $E(P) \cap E(P_t) \neq \emptyset$; otherwise, the result follows directly from the induction hypothesis.

We suppose first that $P_t = P'_t \sqcup \tilde{P}_t$ is a 6-path. Then, P is either P'_t or $\tilde{P}_t$. Without loss of generality, let $P = P'_t$. Let v denote the joint of P_t. Given that v is a black vertex in T', by Observation 4, T' contains a 3-path P' with end vertex v. By induction hypothesis, T' has a building sequence that starts at P', say $P', P'_1, \ldots, P'_{t-1}$. Hence, $P'_t, (P' \cup \tilde{P}_t), P'_1, \ldots, P'_{t-1}$ is a building sequence of T.

Secondly, we suppose that $P_t = P'_t \sqcup \tilde{P}_t$ is a 4-path and let v be its joint. Without loss of generality, we can assume that $P = P'_t \cup vu$, for some neighbor u of v in T'. As before, by Observation 4 and induction hypothesis, there exists a brrb-path P'' containing the edge vu, and furthermore P'' is the starting path of a building sequence of T', say $P'', P'_1, \ldots, P'_{t-1}$. Hence, $(P'_t \cup vu), ([P'' - vu] \cup \tilde{P}_t), P'_1, \ldots, P'_{t-1}$ is a building sequence of T. □

Finally, we prove that trees in $\mathcal{G}$ are skeletons or have a 4-pc decomposition. We first make the following observation.

Observation 5 *Let T be a skeleton and u be an odd degree vertex of T. By Observation 4 and Proposition 7, there exists a brrb-path P with end vertex u such that P is the starting path of a building sequence of T. Therefore, the graph $T \cup uv$, where v is a vertex not in $V(T)$, has a 4-pc decomposition, which is given by $P \cup uv$ and the set of paths of the aforementioned building sequence of T.*

Lemma 8 *Let T be a tree in $\mathcal{G}$. Then T has a 4-pc decomposition or is a skeleton.*

Proof: If T is a minimum counterexample to Lemma 8, then the following two properties hold.

Claim 1: T contains no leaf v such that the minimum distance from v to any other odd vertex of T is at least 4.

If such a leaf v exists, then the tree $T - vu$, where u is the only neighbor of v, has odd distance at least 3. By the minimality of T, we conclude that $T - vu$ has a 4-pc decomposition or is a skeleton. In the former case, since u is odd in $T - vu$, we can trivially extend the 4-pc decomposition of $T - vu$ to one of T, a contradiction. If $T - vu$ is a skeleton, by Observation 5, T has a 4-pc decomposition, again a contradiction.

Claim 2: T contains no path P of length at least 4 such that $T - E(P)$ is connected and both leaves of P have degree 1 in T.

If such a path P exists, then $T - E(P)$ is a tree of odd distance at least 3. If $T - E(P)$ has a 4-pc decomposition, clearly T also has such a decomposition, a contradiction. Thus, by the minimality of T, we conclude that $T - E(P)$ is a skeleton, in which case, by Lemma 17, we have that T has a 4-pc decomposition, a contradiction.

Claim 3: If P is a 3-path with odd end vertices, then $T - E(P)$ has a 4-pc decomposition and is connected.

Let $P = v_0v_1v_2v_3$. The graph $T - E(P)$ has at most 4 components, say H_0, H_1, H_2 and H_3, containing v_0, v_1, v_2, and v_3, respectively. Set $H_i = \emptyset$ if H_i does not exist. Notice that none of the

H_j can be a skeleton because the distance between odd vertices needs to be at least 3. To see this, notice that in H_i the vertex v_i has even degree. If H_i is a skeleton, then v_i has a neighbour u_i in H_i of odd degree (by Observation 3), but then u_i has distance at most 2 from v_0 or v_3. Hence, every component has a 4-pc decomposition. If there are at least two such components, then $P \cup H_i$ is a skeleton for each $i \in \{0, 1, 2, 3\}$; this follows because $P \cup H_i$ has fewer edges than T and if it had a 4-pc decomposition, then T would have a 4-pc decomposition as well, a contradiction. But this means that $P \cup H_0 \cup H_1 \cup H_2 \cup H_3$ is a skeleton. Thus, there must be only one component.

Let v_0 be a leaf of T and v_3 be an odd vertex of T at distance 3 from v_0 in T (by *Claim 1*, such vertices always exist). Let $P' = v_0v_1v_2v_3$ denote the 3-path of T with end vertices v_0, v_3 and let H_i with $i \in \{1, 2, 3\}$ the only component of $T - E(P')$ which is not empty, as in *Claim 3*. Due to Claim 3, H_i has a 4-pc decomposition.

Let $\mathcal{D}$ be a 4-pc decomposition of H_i such that the paths containing v_i are as long as possible. Because of maximality, every such path ends in odd vertices of T. If $i = 1$ (analogously for $i = 2$), then every path in $\mathcal{D}$ containing v_1 is of length 4 and has v_1 as its middle vertex, this is because T does not have a 4-pc decomposition. Let $x_0x_1x_2x_3x_4$ denote such a path. Due to Claim 3, x_0 and x_4 are leaves of T and the degree of its inner vertices distinct of v_1, namely of x_1 and x_3, is two. But then, we obtain that the path $x_0x_1v_1v_2v_3$ violates *Claim 2*. For the case $i = 3$, note that again due to that T does not have a 4-pc decomposition, every path in $\mathcal{D}$ that contains v_3 is of length 6 and has v_3 as its middle vertex. Let $x_0x_1x_2vy_2y_1y_0$ denote such a path. Due to Claim 3, the end vertices x_0 and y_0 are leaves of T and the degree of the inner vertices distinct from v_3, namely of x_1, x_2, y_1 and y_2, is two. Then, the path $P' \cup x_0x_1x_2v$ violates *Claim 2*. □

3.2 Minimum counterexample argument

In this subsection we study the properties of a minimum counterexample to Fact 1. Due to Lemma 8, we already know that a minimum counterexample has cycles. As we already mentioned, the proof of this theorem is based on the results that will be presented in Section 4. Recall that $\mathcal{G}$ denotes the set of all connected, triangle-free graphs with odd distance at least 3. From now on, we may denote the number of edges of a graph G by $\ell(G)$.

Theorem 9 *Let $G \in \mathcal{G}$. If G is not a hanging-square graph, then G has a 4-pc decomposition.*

Proof: For the purpose of contradiction, let $G \in \mathcal{G}$ be an edge-minimum graph that is not a hanging-square graph and does not have a 4-pc decomposition. Due to Lemma 8, G has a cycle. Since G does not have a 4-pc decomposition, for every cycle C of G, the graph $G - E(C)$ consists of at least one component, and by the minimality of G at least one component of $G - E(C)$ is a hanging-square graph. Recall that if $E' \subset E(G)$, we denote by $G - E'$ the graph obtained from G by first removing the edges from E' and then deleting all isolated vertices. We analyze two cases.

Case 1. There exists a cycle C in G such that $G - E(C)$ has exactly one component.

Let $H = G - E(C)$. By the observation above, H is a hanging-square graph. We claim that C has length 4. Indeed, if the length of C is at least 5, by Lemma 12 we have that $G = C \cup H$ has a 4-pc decomposition, a contradiction.

Suppose that there exists a square Q in H (such a square is part of a bunch or it is a bunch itself). We have that $G - E(Q)$ either has exactly one component or, is formed by two components, namely, C and

$H - E(Q)$. In the case that $G - E(Q)$ is formed by two components, by Lemma 13 we have that G has a 4-pc decomposition, again a contradiction. In the case that $G - E(Q)$ has one component, say H', given that G does not have a 4-pc decomposition, we have that H' is a hanging-square graph, but then by Lemma 18 there exists a 4-pc decomposition of G, a contradiction.

Thus, we conclude that H has no squares, and therefore $H = T_H$ (that is, H is a skeleton). If H is a 3-path, then $G = H \cup C$ is a hanging-square graph, a contradiction to our assumption. Let P be the last path in a building sequence of T_H. Since P has length 4 or 6 and G has no 4-pc decomposition, we have that also $G - E(P)$ has no 4-pc decomposition. By minimality of G, the graph $G - E(P)$ either has two connected components, namely $H - E(P)$ and C, or $G - E(P)$ is a hanging-square graph. Hence, by Lemma 14 we have that $G = H \cup C$ has a 4-pc decomposition, again a contradiction. Summarizing, we conclude that *Case 1* leads to a contradiction.

Case 2. The deletion of the edge set of any cycle gives at least two components.

Let C be a cycle of G of minimum length and let $H = G - E(C)$.

We first suppose that one of the components of H, say K, is not a hanging-square graph. By the minimality of G, the graph K has a 4-pc decomposition. Let $H' = G - E(K)$. Since G is a minimum counterexample, H' is a hanging-square graph. Moreover, since $H' - E(C)$ is connected, H has exactly two components, namely K and $H' - E(C)$, and since H' is a hanging-square graph, C is a square of H'. In addition, given that C disconnects G, we have $V(K) \cap V(H') \subset V_c$, where $V_c = V(C) \setminus V(T_{H'})$. Let us consider a 4-pc decomposition of K such that all its paths have length at most 7. Let D be an element of such a 4-pc decomposition that contains a vertex of C. If D is a cycle, then by Lemma 13 we have that the graph $H' \cup D$ has a 4-pc decomposition. If D is a path, then Lemma 15 provides a 4-pc decomposition of $H' \cup D$; both cases yield a contradiction.

Hence, we conclude that all components of H are hanging-square graphs.

Let $\{H_i\}_{i \in [k]}$ denote the set of all components of H. We first assume that for some $i \in [k]$, the graph H_i has a square Q. By assumption (Case 2), the deletion of $E(Q)$ yields at least 2 components, thus, $V(H_i) \cap V(C) \subset V_q$, where $V_q = V(Q) \setminus V(T_{H_i})$ and H_i does not contain further squares.

Let P_0 be a brrb-path such that $|V(P_0) \cap V(Q)|$ is maximum; if Q is a 2-bunch we let P_0 be the brrb-path occupied by Q, otherwise, we just let P_0 be any of the brrb-paths that contain the joint of Q. Suppose that T_{H_i} has a building sequence starting at P_0 and ending at P such that $P_0 \neq P$. Since $V(H_i) \cap V(C) \subset V_q$, the graph $G - E(P)$ is connected and, because of our assumption, it is not a hanging-square graph. By the minimality of G, the graph $G - E(P)$ has a 4-pc decomposition. But this yields a 4-pc decomposition of G, a contradiction. Therefore, H_i is the union of P_0 and a square Q and $G - E(Q)$ has exactly two components, P_0 and $\tilde{H}$ (with C a cycle in $\tilde{H}$). If $\tilde{H}$ is not a hanging-square graph, then we proceed as in the previous argument (with $K = \tilde{H}$) to obtain a 4-pc decomposition of G and thus, a contradiction. Therefore, we can assume that $\tilde{H}$ is a hanging-square graph and C is a square of $\tilde{H}$. Using the same argument as before, $\tilde{H}$ is the union of a 3-path $\tilde{P}$ and a square C. By Lemma 16, we have that there is a 4-pc decomposition of $H_i \cup \tilde{H}$ and therefore of G, a contradiction.

Finally, we are left with the case that H_i is a skeleton for each $i \in [k]$. Suppose that for some $i \in [k]$, H_i is not a 3-path. Let P be the last path of a building sequence of H_i. If $(H_i \cup C) - E(P)$ is connected, then $G - E(P)$ is connected and is not a hanging-square graph. Thus, $G - E(P)$ has a 4-pc decomposition, and so does G, a contradiction. Then, $(H_i \cup C) - E(P)$ is not connected for all such paths. This implies that $V(C) \cap V(H_i) \subset V(P) \setminus \{v\}$, where v is the joint of P. Now, let $v' \in V(C) \cap V(H_i)$ and $P_0, P_1, \dots, P_t$

be a building sequence of H_i such that $v' \in P_0$. Since $v' \notin V(P_t) \setminus \{v''\}$, where v'' is the joint of P_t, we have that $(H_i \cup C) - E(P_t)$ is connected. This contradicts the previous assertion.

Thus, for every $i \in [k]$, H_i is a 3-path. Note that the degree in G of a vertex in C is in $\{2, 3, 4\}$. Let $x_1, x_2, \ldots x_t$ be the subsequence of vertices of (the sequence that defines) C that have degrees in $\{3, 4\}$. Assume first that every component intersects C in exactly one vertex, and denote by H_i the 3-path that contains vertex x_i. Furthermore, denote by $C(i, j)$ a path in C with end vertices x_i, x_j. We need one more notation: we denote by L_i and S_i the longest and the shortest, respectively, subpaths in H_i with end vertex x_i. We decompose G into paths $P'_i, i \in [t]$, where $P'_i = L_i \cup C(i, i+1) \cup S_{i+1}$ for all $i \in [t-1]$ and $P_t = L_t \cup C(t, 1) \cup S_1$. From next observation follows that the length of each path $P_i, i \in [t]$, is at least 4.

Observation 6 *If v is a vertex of degree 3 in C, then its neighbors in C have degree 2, since otherwise there exists an odd vertex within distance 2 of v.*

It remains to analyze the case that there is a $brrb$-path $\tilde{H}$ that intersects C in at least 2 vertices. If C contains two adjacent vertices of $\tilde{H}$, then there exists in G a cycle shorter than C, a contradiction. If C contains an inner and an end vertex of $\tilde{H}$, then $\tilde{H} \cup C$ is a hanging-square graph, because in this case the length of C is necessarily 4 (otherwise there would be a shorter cycle). Let v be a vertex of C that has degree 2 in $\tilde{H} \cup C$ (since C has length 4, there are two such vertices), by Observation 6 the vertex v has degree 2 in G as well. Then, $G = \tilde{H} \cup C$, a contradiction. If C contains both ends of $\tilde{H}$, then necessarily C has length 6 and again by Observation 6 we have that $G = \tilde{H} \cup C$, a contradiction, as this implies that G has a 4-pc decomposition. This completes the proof that G is a tree. □

3.3 Hanging-square graphs do not have 4-pc decompositions

In this section we complete the proof of Theorem 1.

Proposition 10 *If H is a hanging-square graph, then H does not have a 4-pc decomposition.*

Proof: Let H be a minimum counterexample to the statement. Let T_H denote the skeleton of H and $\mathcal{B}_i$ denote the set of (maximal) i-bunches of squares of H for $i \in \{1, 2\}$ (see Definition 3). Let $\mathcal{D}$ be a 4-pc decomposition of H. Clearly, $|E(H)| > 3$.

We note that if $\mathcal{D}$ contains a cycle C, then $\mathcal{D} \setminus \{C\}$ is a 4-pc decomposition of $H - E(C)$, which is a hanging-square graph, a contradiction to the minimality of H. Thus, we assume that $\mathcal{D}$ consists only of paths.

Recall that by Proposition 7, for each brrb-path P of T_H, there exists a building sequence of T_H that starts at P. We first prove two claims:

Claim 1 *For every $e \in E(T_H)$, either $H - e$ is connected or one of the components of $H - e$ has at most 2 edges.*

Proof of Claim 1. By Observation 4, there exists a brrb-path P which contains e. By contradiction, suppose that the components H', H'' of $H - e$ satisfy

$$|E(H')| \geq 3 \text{ and } |E(H'')| \geq 3. \quad (1)$$

We show that this situation yields a contradiction. We have that $H' \cup P$ and $H'' \cup P$ are hanging-square graphs with fewer than $|E(H)|$ edges. Then, by the minimality of H, neither $H' \cup P$, nor $H'' \cup P$ has a

4-pc decomposition. Let $P(e) \in \mathcal{D}$ denote the path that covers e; recall that $\mathcal{D}$ is a 4-pc decomposition of H which consists only of paths. If $H' \cup P$ contains at least 4 edges from $P(e)$, then the restriction of $\mathcal{D}$ to $H' \cup P$ is a 4-pc decomposition of $H' \cup P$, a contradiction. Then, we can assume that each of the hanging-square graphs $H' \cup P$ and $H'' \cup P$ have at most 3 edges from $P(e)$. Let $P = x_0x_1x_2x_3$ and suppose that $e = x_0x_1$, H' contains x_0 and H'' contains x_1, x_2, x_3. If no edge of $P(e)$ belongs to $E(H')$, then $H'' \cup P$ contains all edges of $P(e)$, a contradiction. Therefore, at least one edge from $P(e)$ belongs to $E(H')$. It implies that the restriction of $\mathcal{D}$ to H' consists of a set of paths of length at least 4, say $\mathcal{P}'$, and a path Q of length at least 1 with end vertex x_0. Therefore $\mathcal{P}' \cup \{Q \cup P\}$ is a 4-pc decomposition of $H' \cup P$, a contradiction to the minimality of H. So, we now assume that $e = x_1x_2$, H' contains x_0, x_1 and H'' contains x_2, x_3. We note that the condition $e = x_1x_2$ implies that P is not occupied at x_0, x_2. Without loss of generality, suppose that H' has at least 2 edges from $P(e)$. Then, the restriction of $\mathcal{D}$ to H' consists of a set of paths of length at least 4, say $\mathcal{P}'$, and a path Q of length at least 2 with end vertex x_1. Therefore $\mathcal{P}' \cup \{Q \cup \{x_1x_2, x_2x_3\}\}$ is a 4-pc decomposition of $H' \cup P$, again a contradiction to the minimality of H. $\square$

Claim 2 *If $v \in V(T_H)$ is the joint of a building path or of a 1-bunch, then every building path of H and every 1-bunch of H has joint v and every 2-bunch has v as a joint.*

Proof of Claim 2. If there are distinct $v, u \in V(T_H)$ which are joints of building paths or of 1-bunches, then, by definition of hanging-square graphs, there exists an edge $e \in E(T_H)$ so that $H - e$ has two components, each with at least 3 edges, contradicting Claim 1. Similarly, if v is the joint of all building paths and 1-bunches and a 2-bunch of H does not have v as a joint, then again due to the definition of hanging-square graphs, there exists an edge e such that $H - e$ has two components each with at least 3 edges, a contradiction to Claim 1. $\square$

Let $P = x_0x_1x_2x_3$ be a brrb-path of T_H. If no vertex of P is the joint of a building path or of a 1-bunch, then a longest path in H has length at most 4 and hence, $\mathcal{D}$ consists of paths of length 4. But this implies that $|E(H)|$ is even, a contradiction.

Suppose x_2 is the joint of a building path or of a 1-bunch. Let $Q \in \mathcal{D}$ be the path that contains the edge x_2x_3. We have that $|E(Q)| = 4$, since the longest path in H that contains x_2x_3 has length 4, due to Claim 2. Further, Q contains either edges of a 1-bunch or of a 2-bunch, and in either case the deletion of $E(Q)$ generates a hanging-square graph, a contradiction to the minimality of H.

Finally, suppose x_0 is the joint of a building path or of a 1-bunch. Since x_0 is an odd degree vertex and due to Claim 2, there exists a path $Q \in \mathcal{D}$ that ends at x_0, but the longest such path in H has length 3, a contradiction. $\square$

3.4 Proof of Corollary 2

Let P be a path and $x, y \in V(P)$. We denote by $P(x, y)$ the subpath in P with end vertices x, y. Let C be a cycle and $x, y \in V(P)$, with $x \neq y$. We denote by $C(x, y)$ one of the two subpaths in C with end vertices x, y.

We need the following lemma to complete the proof of Corollary 2.

Lemma 11 *Let C be a cycle and D be a path or a cycle. Suppose C and D are of length at least 4 and at most 7. If $V(C) \cap V(D) \neq \emptyset$, $E(C) \cap E(D) = \emptyset$ and $C \cup D$ is a triangle-free graph which, in the case that C is a cycle of length 4 and D is a path, does not satisfy the following (see Figure 2):*

(i) $|V(C) \cap V(D)| = 1$ *and* $V(C) \cap V(D)$ *is an end vertex of* D*,*

(ii) $|V(C) \cap V(D)| = 2$*, and* $V(C) \cap V(D)$ *contains an end vertex* x *of* D *and a vertex at distance two of* x *in* D*,*

then $C \cup D$ *has a decomposition into paths of length at least 4.*

Fig. 2: The two graphs of Lemma 11. Dashed lines represent paths.

Proof: Note that if D and C are cycles of length 4, then one can easily find a decomposition into two paths of length 4. We now show that the case that D is a cycle follows from the case that D is a path. Suppose that D is a cycle of length at least 5. We claim that $|V(C) \cap V(D)| \leq 4$. Otherwise, there is a path $v_1v_2v_3$ in C such that $v_i \in V(C) \cap V(D)$ for each $i \in \{1, 2, 3\}$. Since $C \cup D$ is triangle-free, each path $D(v_1, v_2)$ and $D(v_2, v_3)$ is of length at least 3, and due to that the length of D is at most 7, we have that v_1v_3 is an edge of D, and then, $\{v_1, v_2, v_3\}$ is a triangle of $C \cup D$, a contradiction. Therefore, there is a vertex x in D of degree two in $C \cup D$. Let $e = xy$ be an edge in D. Then the path $D' := D - e$ is of length at least 4. If the graph $C \cup D'$ satisfies condition (i) or (ii), then one can easily find the desired decomposition of $C \cup D$. If $C \cup D'$ does not satisfy (i) and (ii), under the assumption of the validity of the lemma for the cycle path case, we show that we can find the desired decomposition of $C \cup D$. Note that vertices y and x are the only odd vertices in $C \cup (D - e)$ and x has degree 1 in $C \cup (D - e)$. Let $\mathcal{D}$ be a decomposition of $C \cup D'$ into paths of length at least 4. If a path in $\mathcal{D}$ ending in y does not use the vertex x, then we can easily extend $\mathcal{D}$ to the desired decomposition of $C \cup D$. Now suppose that each path in $\mathcal{D}$ ending in y uses the vertex x and let P be such path. We have that $P \cup e$ is a cycle of length at least 5 and that $C \cup D' - E(P)$ is a cycle of length 8 or 9. Moreover, $\mathcal{D} \setminus P$ is a decomposition of $C \cup D' - E(P)$ into paths of length 4 or 5. Let P' be a path in $\mathcal{D} \setminus P$ that contains vertex y. Then, via the assumption there is a decomposition $\mathcal{D}'$ into paths of length at least 4 of the graph $P \cup e \cup P'$. Hence, taking the union of $\mathcal{D}'$ and the path in $\mathcal{D} \setminus \{P, P'\}$ we have a decomposition of $C \cup D$ into paths of length at least 4.

We assume now that D is a path. In the case that $|V(C) \cap V(D)| = 1$, it is not hard to see that, unless $C \cup D$ satisfies condition (i), we can obtain the desired path decomposition. Set $V(C) \cap V(D) = \{u\}$. One of the paths of the desired path decomposition of $C \cup D$ is obtained by the union of a longest subpath P of D with end vertex u and a subpath P' of C with u as an end vertex: if the length of P is at least 3, then take P' consisting of only one edge; and if not, consider P' such that $\ell(P') = \ell(P) = 2$; in both cases, since $\ell(P) < \ell(D)$ because $P \cup D$ does not satisfy (i), the complement of $P \cup P'$ is a path of length at least 4. Let $V(C) \cap V(D) = \{u, v\}$, let x, y be the end vertices of D such that $D(x, u)$ does not contain v and let P_1, P_2, P_3 be the three subpaths of $C \cup D$ with end vertices u, v. Without loss of generality, suppose $\ell(D(x, u)) \geq \ell(D(y, v))$ and $\ell(P_1) \leq \ell(P_2) \leq \ell(P_3)$. One of the paths of the desired path decomposition of $C \cup D$ is obtained by the union of $D(x, u)$, P_1 and a subpath P' of P_3 with end vertex v; the other path is its complement. Let us check that it is possible to choose a suitable P'. If $\ell(P_2) \geq 3$

and $\ell(P_3) > 3$, then P' can be chosen so that $\ell(P') = 3$, and the result follows. If $\ell(P_2) = \ell(P_3) = 3$, then $\ell(D(x,u)) \geq 1$, and hence $\ell(D(x,u) \cup P_1) \geq 2$, and the result holds if we choose P' so that $\ell(P') = 2$. If $\ell(P_2) = 2$ and $\ell(P_3) \geq 3$, then $\ell(P_1) = 2$ and again the result holds if we choose P' so that $\ell(P') = 2$. Finally, if $\ell(P_2) = \ell(P_3) = 2$, then $\ell(P_1) = 2$ and $\ell(D(x,u)) \geq \ell(D(y,v)) \geq 1$ since $C \cup D$ does not satisfy (ii), and the result follows if we choose P' consisting of one edge.

For the cases that $|V(C) \cap V(D)| = i$ with $i \geq 3$, let $V(C) \cap V(D) = \{v_0, \ldots, v_{i-1}\}$ and x, y be the end vertices of D such that $D := x \cdots v_0 \cdots v_2 \cdots \cdots v_{i-1} \cdots y$.

If there is a vertex $w \in V(C)$ of degree 2 in $C \cup D$ adjacent to v_0 (symmetrically for v_{i-1}) then, the graphs $wv_0 \cup D(v_0, y)$ and $(C - wv_0) \cup D(x, v_0)$ form a path decomposition of $C \cup D$. These paths are of length at least 4 unless $i = 3$ and $\ell(D(v_0, y)) = 2$ or $\ell(C) = 4$ and $x = v_0$. If $\ell(D(v_0, y)) = 2$ (resp. $\ell(C) = 4$ and $x = v_0$), then, since $C \cup D$ is triangle-free there is a vertex $w' \in V(C)$ such that $C(v_0, w') \cup D(v_0, y)$ and $(C - C(v_0, w')) \cup D(x, v_0)$ (resp. $D(w', v_0) \cup C(v_0, v_2) \cup D(v_2, y)$ and $(C - C(v_0, v_2)) \cup D(w', v_2)$) form a desired path decomposition. Thus, we can assume that

$$\text{there is no vertex in } V(C) \text{ of degree 2 in } C \cup D \text{ adjacent to } v_0 \text{ or to } v_{i-1}. \tag{2}$$

Suppose that v_0 and v_1 are neighbors in C (analogously, v_{i-1} and v_{i-2} are neighbors in C). Then the path $D(v_0, v_1)$ is of length at least 3 because $C \cup D$ is triangle-free. Let w be the neighbour of v_1 in $D(v_0, v_1)$. Thus, we have that $wv_1 \cup (C - v_0v_1) \cup D(v_0, x)$ and its complement with respect to $C \cup D$ is a desired path decomposition, and we can assume that

$$v_0 \text{ and } v_1 \text{ (resp. } v_{i-1} \text{ and } v_{i-2}\text{) are not neighbors in } C \tag{3}$$

Suppose now that there are $j \in \{0, \ldots, i-2\}$ and $v \in V(C \cup D)$ of degree two in $C \cup D$ such that the following two conditions hold (see Figure 3(a)):

- v_0 and v_{j+1} are neighbors in C (resp. v_{i-1} and v_j are neighbors in C)
- v is an inner vertex of $D(v_j, v_{j+1})$ or it is adjacent to v_j (resp. v_{j+1}) in the path $C(v_j, v_{j+1})$, where $C(v_j, v_{j+1})$ is the path of C connecting v_j and v_{j+1} which does not contain v_0 (resp. v_{i-1}).

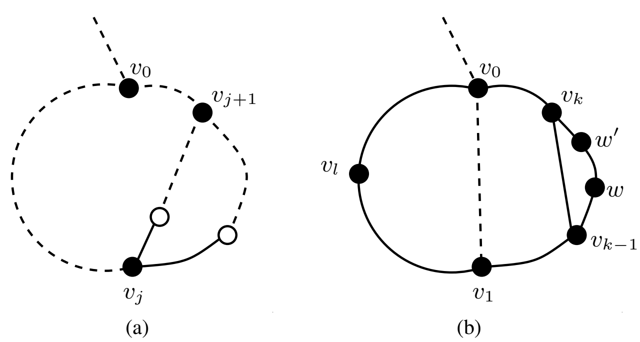

Fig. 3: In (a), vertex v is one of the white vertices. Dashed lines represent paths.

In this case, there exists a desired path-decomposition of $C \cup D$. Because of assumption (3), $j \neq 0$ (resp. $j \neq i-2$). Let C' denote the path in C, disjoint from $C(v_j, v_{j+1})$, joining v_0 and v_j (resp. v_{j+1} and v_{i-1}). If v is an inner vertex of $D(v_j, v_{j+1})$ then, one can easily verify that the graph $D(v, v_{j+1}) \cup C(v_j, v_{j+1}) \cup C' \cup D(v_0, x)$ (resp. $D(v, v_j) \cup C(v_j, v_{j+1}) \cup C' \cup D(v_{i-1}, y)$) and its complement with respect to $C \cup D$ are paths of length at least four. Similarly, if v is adjacent to v_j (resp. v_{j+1}) as stated in the second condition, the graph $(C(v_j, v_{j+1}) - v_j v) \cup D(v_j, v_{j+1}) \cup C' \cup D(v_0, x)$ (resp. $(C(v_j, v_{j+1}) - v_j v) \cup D(v_j, v_{j+1}) \cup C' \cup D(v_0, x)$) and its complement with respect to $C \cup D$ form the desired path decomposition. Hence, we can assume that

$$\text{such pair } j,\, v \text{ does not exist.} \tag{4}$$

We now finish the proof. Let C_1 and C_2 denote the two paths joining v_0 and v_1 in C. Due to assumption (3), there are $l, k \neq 1$ in $\{2, \ldots, i-1\}$ such that v_l is an inner vertex of C_1 and v_k is an inner vertex of C_2. Because of assumption (2), we can assume that $v_l v_0$ and $v_0 v_k$ are edges of C. Hence, due to assumption (4) and the fact that $C \cup D$ is triangle-free, we have $|l-k| > 1$, and $D(v_k, v_{k-1})$, $D(v_l, v_{l-1})$ are edges. Without loss of generality, we can assume $l < k$. If v_{k-1} is an inner vertex of $C_2(v_1, v_k)$, then there are two more inner vertices w, w', since $C \cup D$ is triangle free and because of assumption (4) one of these vertices, the neighbour of v_{k-1} in C (say w), is in $|V(C) \cap V(D)|$ (see Figure 3(b)). Indeed, since $\ell(C) \leq 7$, the path C_2 is $v_0 v_k w' w v_{k-1} v_1$. If v_k were v_{i-1}, then because of assumption (4), $D(v_0, v_1)$ is an edge and since $\ell(C) \leq 7$, the vertices v_0, v_1, v_l would create a triangle, a contradiction. In fact, $D(v_0, v_1)$ is a path of length at least 2. Thus v_{i-1} is w or w'. In either case, because of assumptions (2) and (4), both w, w' are in $V(C) \cap V(D)$, and the vertex v_k is connected to w or to w' throughout edges of D, which is not possible without creating a triangle. Now, assume that v_{k-1} is an inner vertex of $C_1 - v_l v_0$. As before, if v_k were v_{i-1}, then because of assumption (4), $D(v_0, v_1)$ would be an edge. Since $C \cup D$ is triangle free and because of assumption (2) there is a vertex v_p with $p \neq 0, 1$ neighbor of v_k in C, and either $p < l$ or $l < p$. If $p < l$, then, since $C \cup D$ is triangle free $\ell(C) + \ell(D) \geq 15$, a contradiction. Assume now that $l < p$. Because of assumption (4) and since $C \cup D$ is triangle-free, there exists a vertex $v_t \neq v_1$ such that $v_t \in V(C) \cap V(D)$ is an inner vertex of $D(v_1, v_l)$ and $v_t v_l$ is an edge. But this implies that $\ell(C) + \ell(D) \geq 15$, again a contradiction. Finally, if $v_k \neq v_{i-1}$, then v_{i-1} is an inner vertex of $C_1(v_l, v_{k-1})$, $C_1(v_1, v_{k-1})$, or $C_2(v_1, v_k)$. It can be checked that any case leads to a contradiction. □

Proof Proof of Corollary 2: Let G be a graph in $\mathcal{G}$ on n vertices with $|E(G)| \leq 4\lceil n/2 \rceil$. If G is a hanging-square graph, then we use Proposition 6 to conclude the result. If G is a cycle, then we can decompose G into two paths and the statement holds. Otherwise, by Theorem 1, G admits a 4-pc decomposition. Let us consider a 4-pc decomposition $\mathcal{D}$ of G which is maximal with respect to the number of paths. Clearly, $|\mathcal{D}| \leq \lceil n/2 \rceil$. Note that since $\mathcal{D}$ is maximal with respect to the number of paths, every element in $\mathcal{D}$ has length at most 7. If $\mathcal{D}$ contains paths only, then the corollary holds. If not, suppose that $\mathcal{D}$ contains a cycle. Because of Lemma 11, every cycle in $\mathcal{D}$ is of length 4 and if D is an element of $\mathcal{D}$ such that $V(C) \cap V(D) \neq \emptyset$ and $E(C) \cap E(D)=\emptyset$, then D is a path and the graph $C \cup D$ is one of the two graphs described in Lemma 11 (see Figure 2). Moreover, since G is connected and is not a cycle, such D exists for each cycle in $\mathcal{D}$. Let $C_1, C_2, \ldots, C_k$ be the cycles in $\mathcal{D}$ and $D_1, D_2, \ldots, D_k$ be paths (not necessarily distinct) such that $V(C_i) \cap V(D_i) \neq \emptyset$ and $E(C_i) \cap E(D_i)=\emptyset$ for each $i \in \{1, \ldots, k\}$. Because of the structure of the graphs $C_i \cup D_i$ (Figure 2), for each i there is at most one $j \neq i$ such that

$D_i = D_j$. Thus, in order to complete the statement, it suffices to show that for each such graph $C_i \cup D_i$ or $C_i \cup D_i \cup C_j$ with $D_i = D_j$ there is a decomposition into 2 or 3 paths, respectively, which is a fairly trivial task. □

4 Properties of hanging-square graphs

This section addresses properties of hanging-square graphs. These properties were used in the proof of the main results, but some of them are interesting in their own right. Recall that if H is a hanging-square graph, then T_H is its skeleton (see Definitions 1 and 3).

Lemma 12 *Let H be a hanging-square graph and C be a cycle of length at least 5 such that $E(H) \cap E(C) = \emptyset$ and $V(T_H) \cap V(C) \neq \emptyset$. If $H \cup C \in \mathcal{G}$, then $H \cup C$ has a 4-pc decomposition.*

Proof: Let λ be the coloring of T_H and $v \in V(T_H) \cap V(C)$. If $\lambda(v)$ is black, then by Observation 4 there exists a brrb-path P such that v is an end vertex of P. Moreover, by Proposition 7, P is the starting path of some building sequence of T_H. Hence, it suffices to prove that $P \cup C$ has a 4-pc decomposition. Let v' be a neighbor of v in C. Then, $v' \notin V(P)$, because $d_o(P \cup C) \geq 3$ and $P \cup C$ is triangle-free, and thus, $P \cup vv'$, $C - vv'$ are paths of length at least 4 that decompose $P \cup C$. We now suppose that $\lambda(v)$ is red. Again by Observation 4 and Proposition 7, there exists a brrb-path P containing v that is the starting path of some building sequence of T_H. In this case, we have that v is an inner vertex of P. We can assume that C does not intersect the ends of P, otherwise we use the previous case to complete the proof. We decompose $P \cup C$ into two paths of length at least 4 in the following way. Let vx be the edge of P such that $\lambda(x)$ is black and let $vv'u$ be a path of length 2 in C. Given that C does not intersect the ends of P and that $P \cup C$ is triangle-free, both $(P - xv) \cup vv'u$ and $(C - vv'u) \cup xv$ are paths of length at least 4. □

Lemma 13 *Let H be a hanging-square graph, Q be a square of H, and C be a cycle of length at least 4 such that $E(H) \cap E(C) = \emptyset$, $V_q \cap V(C) \neq \emptyset$, where $V_q = V(Q) \setminus V(T_H)$, and $H \cup C$ is not a hanging-square graph. If $H \cup C \in \mathcal{G}$, then $H \cup C$ has a 4-pc decomposition.*

Proof: Let λ be the coloring of T_H. We split the proof into two cases.

Case $V(Q) \cap V(T_H) = \{v\}$. Let $Q(v, v')$ be a shortest path in Q connecting v and $v' \in V_q \cap V(C)$. As before, by Observation 4 and Proposition 7, if $\lambda(v)$ is black (resp. red), then there exists a brrb-path P that has v as an end (resp. inner) vertex and P is the starting path of some building sequence of T_H. Hence, it suffices to show that we can find a 4-pc decomposition of $P \cup Q \cup C$.

Firstly, we suppose that $V(C) \cap V(P) = \emptyset$. It is easy to see that, we can decompose $P \cup Q \cup C$ into two paths of length at least 4: form the first path P' by starting at an arbitrary end of P, continue by picking up the edges of $Q - Q(v, v')$ and add one more edge incident with v' in C. This is a path of length at least 4, and it is easy to check that also the remaining edges form a path of length at least 4.

Secondly, we suppose that $V(C) \cap V(P) \neq \emptyset$. If $\ell(C) \geq 5$, then the result holds by Lemma 12. We assume that C has length 4. Let $P = x_0x_1x_2x_3$ (we can assume that either $v = x_0$, or $v = x_2$). Moreover, $|V_q \cap V(C)| \leq 2$. If $|V_q \cap V(C)| = \{u, w\}$, with $u \neq w$, then necessarily u and w are adjacent to v and $|V(C) \cap V(P)| = 1$. Let $V(C) \cap V(P) = \{z\}$. If $\lambda(v)$ is black, then $z = x_2$. If $\lambda(v)$ is red, then

$z = x_0$. In the first case we decompose $P \cup Q \cup C$ into two paths, with one of them defined by $vuzwu'$, where $u' \in V_q$. In the second case we decompose $P \cup Q \cup C$ into two paths, with one of them defined by x_0ux_2wu', where $u' \in V_q$. We now study the case that $|V_q \cap V(C)| = \{u\}$. If u is adjacent to v, then $v \notin V(C) \cap V(P)$. Moreover, $|V(C) \cap V(P)| = 1$. Let $V(C) \cap V(P) = \{z\}$. For each choice of v in $\{x_0, x_2\}$ we have that $P \cup Q \cup C$ can be decomposed into a cycle and a path, with the cycle defined by $uv \cup C(u, z) \cup P(z, v)$, where $uv \in E(Q)$ and $C(u, z)$ is a shortest path in C (with end vertices u and z).

We are left with the case that u is not adjacent to v. It is clear that $|V(C) \cap V(P)| \leq 2$. Moreover, we have that $v \notin V(C) \cap V(P)$, otherwise $H \cup C$ would be a hanging-square graph. We first suppose that $|V(C) \cap V(P)| = 2$. Let $Q(u, v)$ be a path in Q with end vertices u, v, and $C(x_1, x_3)$ be a path in C with end vertices x_1, x_3. We have that for either choice of v, $V(C) \cap V(P) = \{x_1, x_3\}$, and we can decompose $P \cup Q \cup C$ into a cycle and a path, where the cycle is defined by $Q(u, v) \cup ux_3 \cup C(x_1, x_3) \cup vx_1$, in the case that $v = x_0$, and by $x_1u \cup Q(u, v) \cup vx_3 \cup C(x_1, x_3)$, in the case that $v = x_2$. Finally, let $V(C) \cap V(P) = \{z\}$. For either choice of v, we can decompose $P \cup Q \cup C$ into a cycle and a path, where the cycle is defined by $Q(u, v) \cup C(u, z) \cup P(v, z)$, and $C(u, z)$ is a shortest path in C (with end vertices u, z).

Case $V(Q) \cap V(T_H) = \{v, u\}$, $v \neq u$. By Definition 3 and Observation 4, there exists a brrb-path P with v as an end vertex, u an innerl vertex and such that v and u are non-adjacent (in other words, P is occupied at v, u). By Proposition 7, P is the starting path of some building sequence of T_H. As before, it suffices to show that $P \cup Q \cup C$ has a 4-pc decomposition. Suppose that $V(C) \cap V(P) \neq \emptyset$. If the length of C is at least 5, then we use Lemma 12 to conclude the statement. Thus, the length of C is 4, and it is easy to see that $|V(C) \cap V(P)| > 1$ is not possible. Then the result follows from the previous case by interchanging the roles of C and Q. Hence, we can assume that $V(C) \cap V(P) = \emptyset$. If $V_q \cap V(C) = \{v'\}$, then $P \cup vv' \cup vu'$, $(C - v'u') \cup (Q - vv')$, where u' is a neighbor of v' in C, is a 4-pc decomposition of $P \cup Q \cup C$. Assume $V_q \cap V(C) = \{v', \tilde{v}\}$, $v' \neq \tilde{v}$. Let x denote the black neighbor of u in P and $C(v', \tilde{v})$ be a path in C with end vertices v', $\tilde{v}$. We can decompose $P \cup Q \cup C$ into a path given by $xu \cup uv' \cup C(v', \tilde{v}) \cup \tilde{v}v$ of length at least 4 and a cycle of length 4. □

Lemma 14 *Let T be a skeleton and P be the last path in some building sequence of T. Moreover, let C be a square such that $E(T) \cap E(C) = \emptyset$, $V(T) \cap V(C) \neq \emptyset$ and $T \cup C \in \mathcal{G}$ is not a hanging-square graph. In the following two cases $T \cup C$ has a 4-pc decomposition.*

1. *$[T - E(P)] \cup C$ is a hanging-square graph.*
2. *$[T - E(P)] \cup C$ consists of two connected components, namely $T - E(P)$ and C.*

Proof: Let λ be the coloring of T and v be the joint of P. We analyze cases *1* and *2* separately.
Case 1. Assume that $[T - E(P)] \cup C$ is a hanging-square graph. We can assume that there exists a brrb-path $P_0 = x_0x_1x_2x_3$ such that $E(P_0) \cap E(P) = \emptyset$ and P is the last path of a building sequence starting at P_0. Moreover, we can assume that if C is a 1-bunch with joint u, then $V(P_0) \cap V(C) = \{u\}$ and $u \in \{x_0, x_2\}$. Otherwise, P_0 is occupied at x_0, x_2 by C in $[T - E(P)] \cup C$.

Suppose that $[V(P) \setminus \{v\}] \cap V(C) = \emptyset$. Since $T \cup C$ is not a hanging-square graph, we have that C is a 2-bunch and that either P is a 4-path with joint x_1, or there exists another building path $P' \neq P$ of T with joint $v' \in \{x_0, x_2\}$ such that $v \neq v'$ and $v \in \{x_0, x_2\}$. In the first case we can easily see that $P_0 \cup P \cup C$ can be decomposed into two paths of length at least 4. In the second case, it is routine

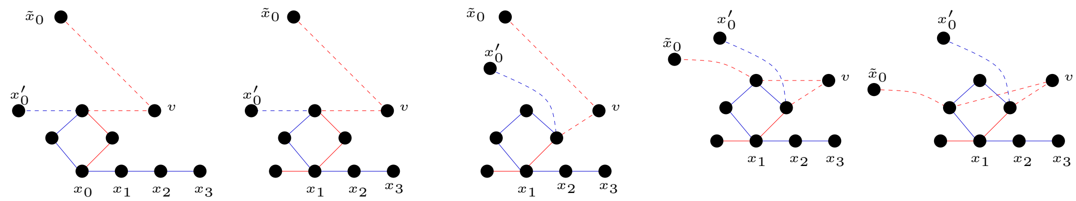

Fig. 4: Proof of Lemma 14, Case 1. In all figures, possibly $v \in \{x_1, x_2, x_3\}$. Recall that $P_0 = x_0x_1x_2x_3$ and $P = P' \sqcup \tilde{P}$. In the figures, P' has ends x'_0, v and $\tilde{P}$ has ends $\tilde{x}_0$, v. Moreover, the length of P' and $\tilde{P}$ is either 2 or 3 depending on whether P is a 4-path or a 6-path.

to check that $P_0 \cup P \cup C \cup P'$ can be decomposed into 3 paths of length at least 4. We now suppose that $[V(P) \setminus \{v\}] \cap V(C) \neq \emptyset$. Let us suppose that C is a 1-bunch. If $v = u$, then $T \cup C$ would be a hanging square graph, a contradiction. All the remaining cases are worked out in Figure 4. Hence, it remains to analyze the case that P_0 is occupied at x_0, x_2 by C. But in this case, P would intersect $V_c = V(C) \setminus V(P_0)$, which yields a contradiction to $d_o(T \cup C) \geq 3$. Hence, the result follows.

Case 2. Assume that $[T - E(P)] \cup C$ consists of exactly two connected components, namely $T - E(P)$ and C. We have that all inner vertices of P but its joint v have degree 2 in T, the ends of P have degree 1 in T and $V(C) \cap V(T) \subset V(P) \setminus \{v\}$. If P is a 6-path (resp. 4-path), then $\lambda(v)$ is black (resp. red) and there exists a brrb-path P_0 that is the starting path for some building sequence of $T - E(P)$ and such that v is an end (resp. inner) vertex of P_0. In both cases, it suffices to show that $P_0 \cup P \cup C$ has a 4-*pc* decomposition. Let $P = P' \sqcup \tilde{P}$. If C intersects only one of the paths $P' - v$, $\tilde{P} - v$, then it is easy to see that $T \cup C$ is a hanging-square graph. Hence, C intersects both $P' - v$ and $\tilde{P} - v$. Let $V(C) \cap V(P') = x$, $V(C) \cap V(\tilde{P}) = y$, x' (resp. $\tilde{x}$) be the end vertex of P' (resp. $\tilde{P}$) distinct from v.

We suppose now that P is a 6-path. Without loss of generality we can assume that $\ell(P'(x', x)) \leq \ell(\tilde{P}(\tilde{x}, y))$. Let $C(x, y)$ denote the shortest path in C connecting x and y. Then, the following two paths form a 4-pc decomposition of $P_0 \cup P \cup C$,

$$P'(x', x) \cup C(x, y) \cup \tilde{P}(y, v) \cup P_0 \quad \text{and} \quad \tilde{P}(\tilde{x}, y) \cup [C - C(x, y)] \cup P'(x, v).$$

The case that P is a 4-path is shown in Figure 5. □

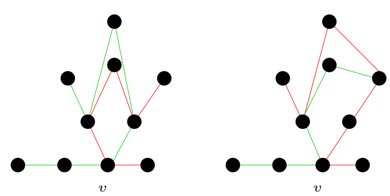

Fig. 5: All configurations in which P is a 4-path and *Case 2* of Lemma 14 occurs.

Lemma 15 *Let H be a hanging-square graph, Q be a square of H and $V_q = V(Q) \setminus V(T_H)$. Let P' be a path such that $4 \leq \ell(P') \leq 7$, $E(P') \cap E(H) = \emptyset$ and $V(P') \cap V(H) \subset V_q$. If $H \cup P'$ is triangle-free, then $H \cup P'$ has a 4-pc decomposition.*

Proof: First of all, by Observation 4, Proposition 7 and symmetry, we can assume that there is a brrb-path $P = x_0x_1x_2x_3$ that is the starting path of some building sequence of T_H such that if Q is a 1-bunch (resp. 2-bunch) with joint u, then $V(P) \cap V(Q) = \{u\} \in \{x_0, x_1\}$ (resp. P is occupied at x_0, x_2 by Q and $u \in \{x_0, x_2\}$). Let $P' = u_1 \cdots u_l$, $Q = y_0 \cdots y_3$ with $y_0 = u$ and $Q(x, y)$ denote a shortest path in Q connecting x and y.

As before, to prove that $H \cup P'$ has a 4-pc decomposition, it suffices to prove that $P \cup Q \cup P'$ has a 4-pc decomposition. We study cases according to the three possible values of $|V(P') \cap V_q|$.

Let $V(P') \cap V_q = \{y_1, y_2, y_3\}$ and $y_i = u_{\pi(i)}$ for each i. In this case, Q must be a 1-bunch. By symmetry, either $\pi(1) < \pi(2) < \pi(3)$ or $\pi(2) < \pi(1) < \pi(3)$. In the first case, the path $P(x_3, u) \cup (Q - uy_1) \cup P'(y_1, u_1)$ and its complement with respect to $P \cup Q \cup P'$ and in the second case, the path $P(x_3, u) \cup uy_3 \cup P'(y_3, y_1) \cup y_1y_2 \cup P'(y_2, u_1)$ and its complement also with respect to $P \cup Q \cup P'$ form a 4-pc decompositions of $P \cup Q \cup P'$.

Let $V(P') \cap V_q = \{w\}$. If Q is a 2-bunch set $u = x_0$. Without loss of generality, we assume that $\ell(P'(u_1, w)) \leq \ell(P'(w, u_l))$. If $u \neq x_0$, then $P(x_3, u) \cup (Q - Q(u, w)) \cup P'(u_1, w)$ and $P(x_0, u) \cup Q(u, w) \cup P'(w, u_l)$ form a 4-*pc* decomposition of $P \cup Q \cup P'$, and if $u = x_0$, then $P(x_3, u) \cup Q(u, w) \cup P'(u_1, w)$ and $(Q - Q(u, w)) \cup P'(w, u_l)$ form a 4-*pc* decomposition of $P \cup Q \cup P'$.

Let $V(P') \cap V_q = \{u_i, u_j\}$ with $i < j$. If Q is a 2-bunch set $u = x_2$. Without loss of generality we assume that $\ell(P'(u_j, u_l)) \leq \ell(P'(u_i, u_1))$. If $u \neq x_0$ or $u = x_0$ then, $P(x_3, u) \cup (Q - Q(u, u_j)) \cup P'(u_j, u_l)$ and its complement with respect to $P \cup Q \cup P'$ form a 4-*pc* decomposition of $P \cup Q \cup P'$. □

Lemma 16 *Let H, H' be hanging-square graphs composed (only) of squares Q, Q' and brrb-paths P, P', respectively. Let $V_q = V(Q) \setminus V(T_H)$ and $V_{q'} = V(Q') \setminus V(T_{H'})$. Assume that $E(H) \cap E(H') = \emptyset$ and $V(H) \cap V(H') \subset V_q \cap V_{q'} \neq \emptyset$. If $H \cup H' \in \mathcal{G}$, then it has a 4-pc decomposition.*

Proof: Note that $|V(Q) \cap V(Q')| \in \{1, 2\}$, otherwise $Q \cup Q'$ would contain a triangle. Let $P = x_0 \cdots x_3$, $P' = x'_0 \cdots x'_3$ and $Q(x, y)$ (resp. $Q'(x, y)$) be a longest path in Q (resp. Q') connecting x and y. In addition, observe that if Q and Q' were 2-bunches, then there would exist a path of length 2 connecting x_0 and x'_0, a contradiction to $d_o(H \cup H') \geq 3$. Hence, at least one of Q, Q' is a 1-bunch.

We first suppose that $V(Q) \cap V(Q') = \{v\}$. Suppose that Q, Q' are 1-bunches and let u, u' be their joints, respectively. Without loss of generality $u \in \{x_0, x_1\}$ and $u' \in \{x'_0, x'_1\}$. In this case, a 4-pc decomposition of $H \cup H'$ is formed of two paths, with one of them, the following: $P(x_0, u) \cup Q(u, v) \cup Q'(u', v) \cup P'(x'_0, u')$. We now assume that Q' is a 2-bunch with joints x'_0, x'_2 and Q is the 1-bunch previously described. We have that the following paths belongs to the desired 4-pc decomposition of $H \cup H'$ into two paths: $Q'(x'_0, v) \cup Q(v, u) \cup P(u, x_3)$.

We can now assume that $V(Q) \cap V(Q') = \{v, w\}$ with $v \neq w$. As before, suppose that Q, Q' are 1-bunches with joints $u \in \{x_0, x_1\}$ and $u' \in \{x'_0, x'_1\}$. Observe that v, w are at distance one of u, u'. In this case, we see that one of the two paths in a 4-pc decomposition of $H \cup H'$ is $P(x_0, u) \cup Q(u, v) \cup [Q' - Q'(u', v)] \cup P'(u', x'_3)$. In the case that Q' is a 2-bunch with joints x'_0, x'_2, there exists a 4-pc decomposition into two paths, with one of the paths as follows: $P(x_0, u) \cup Q(u, w) \cup [Q' - Q'(w, x'_2)] \cup P'(x'_0, x'_2)$. □

Lemma 17 *Let T be a skeleton and P be a path of length at least 4 such that $T \cup P$ is a tree, $E(T) \cap E(P) = \emptyset$ and $V(T) \cap V(P) = \{v\}$, where v is not a leaf of P. If $T \cup P$ is not a skeleton and $d_o(T \cup P) \geq 3$, then it has a 4-pc decomposition.*

Proof: Let λ be the coloring of T. By Observation 4 and Proposition 7, there exists a brrb-path P' such that P' is the starting path of some building sequence of T, and if $\lambda(v)$ is black (resp. red), then v is an end vertex of P' (resp. an inner vertex of P'). Since P' is a starting path, it suffices to prove that $P' \cup P$ has a 4-pc decomposition. We consider the partition P_1, P_2 of P, where P_1 and P_2 have v as an end vertex. In the case that $\lambda(v)$ is black, we have that v is an odd vertex in T, and since $d_o(T \cup P) \geq 3$, we have that P_i has length at least 3, for each $i = 1, 2$. If $\ell(P_1) = \ell(P_2) = 3$, then $T \cup P$ is a skeleton, a case which we do not need to analyze. Therefore, without loss of generality we can assume that $\ell(P_1) \geq 4$, and then, $P' \cup P$ can be decomposed into P_1 and $P' \cup P_2$. Otherwise, $\lambda(v)$ is red, and then v is an even vertex in T and is at distance one (in T) of an odd vertex. And since $d_o(T \cup P) \geq 3$, we have that P_i has length at least 2, for each $i = 1, 2$. If $\ell(P_1) = \ell(P_2) = 2$, then $T \cup P$ is a skeleton, a case which we do not need to analyze. Hence, without loss of generality we can assume that $\ell(P_1) \geq 3$, and then, denoting by v' the end vertex of P' at distance one of v, we have that $P' \cup P$ can be decomposed into $P_1 \cup vv'$ and $(P' - vv') \cup P_2$. □

Lemma 18 *Let $G \in \mathcal{G}$ be a graph which is not a hanging square graph. If Q_1 and Q_2 are two edge-disjoint squares in G such that $G - E(Q_1)$ and $G - E(Q_2)$ are hanging-square graphs, then G has a 4-pc decomposition.*

Proof: Let $H = G - E(Q_2)$ and $V_1 = V(Q_1) \setminus V(T_H)$. If $V_1 \cap V(Q_2) \neq \emptyset$, then the result follows by Lemma 13. Therefore, we assume that $V_1 \cap V(Q_2) = \emptyset$. Since G is not a hanging-square graph, we can assume that there exists a brrb-path P in H such that $V(P) \cap V(Q_2) \neq \emptyset$ and $V(P) \cap V(Q_1) \neq \emptyset$. By Proposition 7, P is the starting path of some building sequence $P, P_1, \ldots, P_t$ of T_H. Without loss of generality, we can assume that $P = x_0x_1x_2x_3$ is occupied at x_0, x_2 by Q_1, in H. Then, either

(i) P is occupied at x_1, x_3 by Q_2, in $G - E(Q_1)$, or

(ii) Q_2 is a 1-bunch with joint x_1, or

(iii) Q_2 is a 1-bunch with joint x_0 (resp. x_2) and there exists D, a building graph in $\{P_1, \ldots, P_t\}$ or a 1-bunch, with joint x_2 (resp. x_0) .

In cases (i) and (ii) it is easy to prove that $P \cup Q_1 \cup Q_2$ has a decomposition into a path of length 5 and a path of length 6. In case (iii), it can be shown that $P \cup Q_1 \cup Q_2 \cup D$ decomposes into 3 paths of length at least 4. Further details of the proof are left to the reader. □

5 Concluding remarks

It would be interesting to broaden further the class of graphs for which properties on path (or path-cycle) decompositions can be well-characterized. In this direction, the study of the class of triangle-free graphs with odd distance at least 2 is a challenging problem. It is not so likely that a nice characterization (as in the case of odd distance at least 3) can be found, but it would be interesting to study path decomposition properties of this class of graphs.

Finally, we note that the class of hanging-square graphs can be recognized in polynomial time. To see this, let us say that a square in a graph G is *good* if it has either 2 vertices or 3 vertices of degree 2 in G. Given a triangle-free graph G with odd distance at least 3, we look for a square and check whether it is good. If yes, we look for a maximal bunch, say B, that contains it, and keep the information on its joints. Then, we delete the edges of B and repeat the process considering the resulting graph, while it is connected. If the resulting graph is not connected or if it contains cycles but none of them is a good square, then we can conclude that the original graph is not a hanging-square graph. If the resulting graph is a tree, say T, then we have to check whether it is a skeleton. It is not difficult to see that the latter step can be done in polynomial time, and that if T is a skeleton, then we can find in polynomial time a building sequence $\mathcal{S} = P_0, P_1, \ldots, P_t$ of T. If T is not a skeleton, then the original graph is not a hanging-square graph. If T is a skeleton, we have to check whether all maximal bunches, whose edges were deleted previously, have its joints intersecting properly (according to Definition 3) the paths P_i of $\mathcal{S}$.

We only sketched the ideas behind a polynomial-time recognition algorithm for the hanging-square graphs, as this algorithmic aspect is not the focus of this paper, but we wanted to discuss its consequence. We note that, in view of Theorem 1, the problem of deciding whether a graph in $\mathcal{G}$ admits a 4-pc decomposition can be solved in polynomial time (the certificate that it belongs to the class $\mathbf{coNP}$ being precisely the certificate that it is a hanging-square graph).

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