

Dissecting a square into congruent polygons

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We study the dissection of a square into congruent convex polygons. Yuan *et al.* [Dissecting the square into five congruent parts, Discrete Math. **339** (2016) 288-298] asked whether, if the number of tiles is a prime number ≥ 3 , it is true that the tile must be a rectangle. We conjecture that the same conclusion still holds even if the number of tiles is an odd number ≥ 3 . Our conjecture has been confirmed for triangles in earlier works. We prove that the conjecture holds if either the tile is a convex q -gon with $q \geq 6$ or it is a right-angle trapezoid.

Keywords: Tiling, Eulerian graph, hypotenuse graph

1 Introduction

Let Ω be a polygon in \mathbb{R}^2 , and let $\{P_j; j = 1, \dots, N\}$ be a family of polygons. We call $\{P_j\}_{j=1}^N$ a *tiling* or *dissection* of Ω , if $\Omega = \bigcup_{j=1}^N P_j$ and the right hand side is a non-overlapping union, that is, the interiors of the tiles are pairwise disjoint. In particular, we are interested in the tiling

$$\Omega = \bigcup_{j=1}^N P_j, \quad (1)$$

where Ω is a square, and all $P_j, j \in \{1, \dots, N\}$, are congruent to a convex polygon P . In this case, we also say that P can tile Ω . (Two sets A and B are congruent if $A = g(B)$ where g is a composition of a rotation, possibly a reflection and a translation.)

In the 1980's, Ludwig Danzer conjectured that if $N = 5$ in (1), then P must be a rectangle (see [12]). Yuan *et al.* [12] proved that Danzer's conjecture is true, and asked whether, if the number of tiles is a prime number ≥ 3 , it is true that the tile must be a rectangle. Except $N = 5$, this question was answered confirmatively for $N = 3$ in [6].

In this paper, we formulate a stronger conjecture:

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Conjecture 1. *If a convex polygon P can tile a square and the number of tiles is an odd number ≥ 3 , then P must be a rectangle.*

A polygon is called a q -gon if it has q vertices and thus q sides. Instead of considering the number of tiles N case by case as in [6; 12], we study the problem on q case by case.

When $q = 3$, Conjecture 1 is confirmed by Thomas [11] and Monsky [7]. Actually, they proved the following surprising result: If a rectangle is tiled by N triangles with the same area, then N must be an even number.

Our first result is to show that Conjecture 1 is true for $q \geq 6$. Actually, we prove the following stronger result:

Theorem 1.1 *Let R be a rectangle and K be a convex q -gon with $q \geq 6$. Then K cannot tile R .*

The proof of the above theorem is motivated by Feng *et al.* [3].

In another paper [9], we show that Theorem 1.1 still holds for $q = 5$, with the help of a computer to verify hundreds of cases. So, for Conjecture 1, the only remaining case is $q = 4$, which seems to be very difficult. In this paper, we give a partial answer for this case. A *right-angle trapezoid* is a trapezoid with angles $\pi/2, \pi/2, \alpha, \pi - \alpha$ where $0 < \alpha < \pi/2$, where the two right angles are adjacent.

Theorem 1.2 *Let P be a right-angle trapezoid. If P can tile a square, then the number of tiles must be even.*

To prove Theorem 1.2, we introduce a hypotenuse graph G related to the tiling (1). We show that Conjecture 1 is true if every connected component of the graph G is Eulerian; indeed, this is the case when $\alpha \neq \pi/3$. If $\alpha = \pi/3$, we need to investigate carefully the forbidden configurations of tiles in the tiling (1), which is the most difficult part of the proof of Theorem 1.2.

There are some works on other dissection problems of a square into polygons, see for instance [2; 5; 8].

The paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we recall some results on Eulerian graphs. In Section 4, we define the hypotenuse graph and show that Conjecture 1 is true if the graph is Eulerian. In Sections 5 and 6, we show that Conjecture 1 is true when $\alpha \neq \pi/3$ and $\alpha = \pi/3$, respectively. In Section 7, we pose several questions.

2 A convex q -gon cannot tile a rectangle when $q \geq 6$

Let Ω be a rectangle in the plane. Let $q \geq 3$ and P be a convex q -gon. Suppose

$$\Omega = \cup_{j=1}^N P_j$$

is a tiling of Ω , where $P_j, j \in \{1, \dots, N\}$, are congruent to P .

Denote by \mathcal{V}_Ω the set consisting of the four vertices of Ω , and \mathcal{V}_j the vertex set of $P_j, j \in \{1, \dots, N\}$. Let $\mathcal{V} = \cup_{j=1}^N \mathcal{V}_j$. Clearly $\mathcal{V}_\Omega \subset \mathcal{V}$.

For $w \in \mathcal{V}$, let

$$\mathcal{I}(w) := \{j : w \in \mathcal{V}_j\};$$

namely, $\mathcal{I}(w)$ is the set of indices of tiles having w as a vertex. For $j \in \mathcal{I}(w)$, denote by $\theta_j(w)$ the angle of the vertex w in P_j . Then, for $w \in \mathcal{V}$, we have

$$\sum_{j \in \mathcal{I}(w)} \theta_j(w) = \begin{cases} \frac{\pi}{2} & \text{if } w \in \mathcal{V}_\Omega; \\ \pi & \text{if } w \text{ lies on an (open) side of a tile or of } \Omega; \\ 2\pi & \text{otherwise.} \end{cases} \quad (2)$$

Define

$$\mathcal{F} = \{w \in \mathcal{V} : \sum_{j \in \mathcal{I}(w)} \theta_j(w) = 2\pi\}, \quad \mathcal{H} = \{w \in \mathcal{V} : \sum_{j \in \mathcal{I}(w)} \theta_j(w) = \pi\}.$$

For a set A , let $|A|$ denote its cardinality.

Lemma 2.1 *Let*

$$F = \sum_{w \in \mathcal{F}} |\mathcal{I}(w)|, \quad H = \sum_{w \in \mathcal{H}} |\mathcal{I}(w)|, \quad \text{and} \quad \hbar = \sum_{w \in \mathcal{V}_\Omega} |\mathcal{I}(w)|.$$

Then

$$\frac{2|\mathcal{F}| + |\mathcal{H}| + 2}{F + H + \hbar} = \frac{q-2}{q}. \quad (3)$$

Proof: Notice that the sum of angles of P is $(q-2)\pi$. Since

$$\begin{aligned} \sum_{w \in \mathcal{F}} \sum_{j \in \mathcal{I}(w)} \theta_j(w) &= 2|\mathcal{F}|\pi, \\ \sum_{w \in \mathcal{H}} \sum_{j \in \mathcal{I}(w)} \theta_j(w) &= |\mathcal{H}|\pi, \\ \sum_{w \in \mathcal{V}_\Omega} \sum_{j \in \mathcal{I}(w)} \theta_j(w) &= 2\pi, \end{aligned}$$

we infer that

$$2|\mathcal{F}|\pi + |\mathcal{H}|\pi + 2\pi = N(q-2)\pi.$$

On the other hand we have

$$F + H + \hbar = \sum_{j=1}^N |\mathcal{V}_j| = N \cdot q.$$

Taking the ratio of the above two equations now yields the lemma. \square

Lemma 2.2 *Let*

$$\Delta = F + H + \hbar - 3|\mathcal{F}| - 2|\mathcal{H}| - |\mathcal{V}_\Omega|. \quad (4)$$

Then $\Delta \geq 0$ and

$$(q-6)|\mathcal{F}| + (q-4)|\mathcal{H}| + (q-2)\Delta + 2q = 8. \quad (5)$$

Proof: Since all angles of P are less than π , we have $|\mathcal{I}(w)| \geq 3$ for $w \in \mathcal{F}$, and $|\mathcal{I}(w)| \geq 2$ for $w \in \mathcal{H}$. It follows that $F \geq 3|\mathcal{F}|$, $H \geq 2|\mathcal{H}|$ and $\hbar \geq 4$, which imply that $\Delta \geq 0$.

By (3) we have

$$\begin{aligned} 2q|\mathcal{F}| + q|\mathcal{H}| + 2q &= (q-2)(F + H + \hbar) \\ &= (q-2)(3|\mathcal{F}| + 2|\mathcal{H}| + 4 + \Delta). \end{aligned}$$

Rearranging the terms of the above equation, we obtain (5). The lemma is proved. \square

Proof of Theorem 1.1. If $q \geq 6$, then the left hand side of (5) is no less than 12, which is absurd. \square

3 Eulerian graphs

In this section we recall some notions and results of graph theory. See [1].

Let $G = (V, \Gamma)$ be a directed graph, where V is the *vertex set* and Γ is the *edge set*. Each edge γ is associated to an ordered pair (u, v) in V , and we say γ is *incident out* of u and *incident into* v . We also call u and v the *origin* and *terminus* of γ , respectively. The number of edges incident out of a vertex u is the *outdegree* of u and is denoted by $\deg^+(u)$. The number of edges incident into a vertex u is the *indegree* of u and is denoted by $\deg^-(u)$. We remark that in the graph G we allow multi-edges from a vertex u to v .

A *directed walk* joining the vertex v_0 to the vertex v_k in G is an alternating sequence $v_0\gamma_1v_1\gamma_2v_2 \dots \gamma_kv_k$ with γ_i incident out of v_{i-1} and incident into v_i .

Similarly, for an undirected graph, we can define *trail*, *path* and *cycle*, see [1].

G is *connected* if for any $u, v \in V$, there is a path joining u and v . A connected graph G is called *Eulerian* if there is a closed trail containing all the edges of G .

Theorem 3.1 *Let $G = (V, \Gamma)$ be a connected directed graph. The following three statements are all equivalent:*

- (i) *For every $u \in V$, $\deg^+(u) = \deg^-(u)$.*
- (ii) *G is Eulerian.*
- (iii) *G is a union of edge-disjoint directed cycles.*

Remark 3.1 For an undirected graph, similar results hold, see for instance, [1]; we will need such results in Section 6.

To close this section, we give a definition. We say that a (directed or undirected) graph is *component-wise Eulerian*, if every connected component of the graph is Eulerian.

4 Tiling a square with congruent right-angle trapezoids

Let Ω be a square in \mathbb{R}^2 . Let P be a trapezoid with angles $(\alpha, \pi - \alpha, \pi/2, \pi/2)$, where $0 < \alpha < \pi/2$; see Figure 1. Let

$$\Omega = \cup_{j=1}^N P_j, \quad (6)$$

be a tiling of Ω , where P_j are all congruent to P . Let ϕ_j be the isometry such that $P_j = \phi_j(P)$, $j \in \{1, \dots, N\}$. The rest of the paper proves that N is an even number.

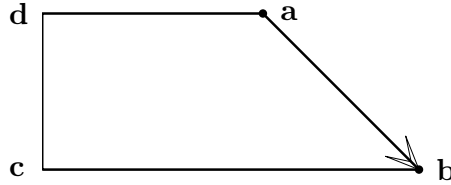


Fig. 1: The trapezoid P .

4.1 Hypotenuse graph

We denote the vertices of P by \mathbf{a} , \mathbf{b} , \mathbf{c} and \mathbf{d} ; see Figure 1. The line segment $\overline{[\mathbf{a}, \mathbf{b}]}$ is called the *hypotenuse* of P . We shall define a directed graph consisting of the (directed) hypotenuses of P_j , $j \in \{1, \dots, N\}$. More precisely, let

$$V = \bigcup_{j=1}^N \{\phi_j(\mathbf{a}), \phi_j(\mathbf{b})\}$$

be the vertex set. For each $j \in \{1, \dots, N\}$, we define a directed edge τ_j with origin $\phi_j(\mathbf{a})$ and terminus $\phi_j(\mathbf{b})$. To emphasize that the edge τ_j is motivated by the hypotenuse of P_j , we can even denote τ_j by the triple

$$\tau_j = (\phi_j(\mathbf{a}), \phi_j(\mathbf{b}), P_j) \quad (7)$$

Let

$$\Gamma = \{\tau_j; 1 \leq j \leq N\}$$

be the set of edges. We call (V, Γ) the *hypotenuse graph* of the tiling (6). (It may happen that two different edges have the same origin and terminus, which explains why we put P_j as the third entry of τ_j in (7).)

The goal of this section is to prove the following:

Theorem 4.1 *If the hypotenuse graph (V, Γ) of the tiling (6) is component-wise Eulerian, then N is even.*

For brevity, we use β to denote $\pi - \alpha$ hereafter.

Let $u \in V$. If θ is the angle of a tile P_j at the vertex u , then we say θ is an *angle around* u . If $\theta_1, \dots, \theta_k$ are the angles around u arranged in the clock-wise order, then we call $(\theta_1, \dots, \theta_k)$ the *angle pattern* at u .

Lemma 4.1 *(V, Γ) is component-wise Eulerian if and only if for each $u \in V$, the angle pattern at u falls into*

$$(\alpha, \beta), (\alpha, \beta, \alpha, \beta), (\alpha, \alpha, \beta, \beta), (\alpha, \beta, \pi/2, \pi/2), (\alpha, \pi/2, \beta, \pi/2) \quad (8)$$

up to a rotation or a reversion.

Proof: Suppose (V, Γ) is component-wise Eulerian. For $u \in V$, an angle α at u determines an incoming edge, and an angle β at u determines an outgoing edge. So the angle pattern at u contains either one α and one β , or two α and two β . So the angle pattern at u falls into (8) up to a rotation.

Assume that all the angle patterns fall into (8). Then $\deg^-(u) = \deg^+(u)$, so (V, Γ) is component-wise Eulerian. \square

4.2 Pairing principle and feasible cycles

In the rest of this section, we will always assume that (V, Γ) is component-wise Eulerian. Let

$$V_1 = \{u \in V : \text{the angle pattern at } u \text{ is } (\alpha, \alpha, \beta, \beta) \text{ up to a rotation}\}.$$

For each $u \in V_1$, we denote the tiles around u corresponding to $(\alpha, \alpha, \beta, \beta)$ by

$$(L_{u,\alpha}, R_{u,\alpha}, R_{u,\beta}, L_{u,\beta}).$$

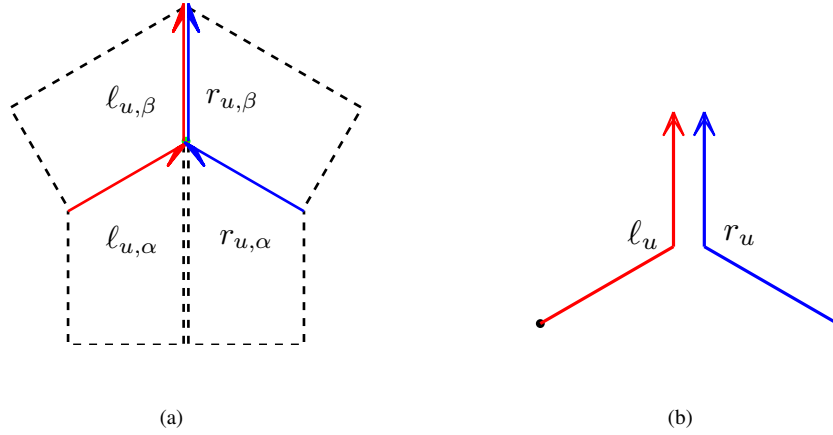


Fig. 2: Pairing rule for the angle pattern $(\alpha, \alpha, \beta, \beta)$. There are essentially four cases depending on the relative positions of α and β in a tile, here we illustrate only one case.

Then the α angle of $L_{u,\alpha}$ and the angle β in $L_{u,\beta}$ form an angle measuring π , and so do α in $R_{u,\alpha}$ and β in $R_{u,\beta}$. Denote the edges of Γ associated to $L_{u,\alpha}, R_{u,\alpha}, R_{u,\beta}, L_{u,\beta}$ by $l_{u,\alpha}, r_{u,\alpha}, r_{u,\beta}$ and $l_{u,\beta}$, respectively. See Figure 2(a).

We regard the path $l_{u,\alpha} + l_{u,\beta}$ as a single edge, and denote it by l_u ; similarly, define $r_u = r_{u,\alpha} + r_{u,\beta}$, see Figure 2(b). Replacing the old edges by these new edges, we obtain a new graph

$$\Gamma^* = (\Gamma \setminus \cup_{u \in V_1} \{l_{u,\alpha}, r_{u,\alpha}, l_{u,\beta}, r_{u,\beta}\}) \cup (\cup_{u \in V_1} \{l_u, r_u\}), \quad (9)$$

where the corresponding vertex set is $V^* = V \setminus V_1$.

A cycle in (V, Γ) is called *feasible* if it is also a cycle in (V^*, Γ^*) .

Clearly, (V^*, Γ^*) is still component-wise Eulerian since for each $u \in V^*$, the degrees of u in Γ and Γ^* are the same. Therefore, we have

Lemma 4.2 (V, Γ) is a union of edge-disjoint feasible cycles.

4.3 Structure of feasible cycles

For a sequence of edges $\gamma_1, \dots, \gamma_m$ such that the terminus of γ_k coincides with the origin of γ_{k+1} for all $k = 1, \dots, m-1$, we use $\gamma_1 + \dots + \gamma_m$ to denote the trail formed by these edges.

Let \mathcal{C} be a feasible cycle in the graph (V, Γ) and let us write it as

$$\mathcal{C} = \gamma_1 + \dots + \gamma_m.$$

Hereafter, we always use

$$K_i = f_i(P)$$

to denote the tile containing γ_i , where $f_i \in \{\phi_1, \dots, \phi_N\}$. We denote two vectors by

$$\vec{\gamma}_i = \overrightarrow{[f_i(\mathbf{a}), f_i(\mathbf{b})]} \quad \text{and} \quad \vec{\rho}_i = \overrightarrow{[f_i(\mathbf{d}), f_i(\mathbf{a})]}.$$

We say $\phi_i(P)$ is *positively oriented* if its vertices $\phi_i(\mathbf{a}), \phi_i(\mathbf{b}), \phi_i(\mathbf{c})$ and $\phi_i(\mathbf{d})$ form a clockwise sequence on the boundary of $\phi_i(P)$; otherwise we say $\phi_i(P)$ is *negatively oriented*.

For two edges γ and γ' in Γ , we write

$$\gamma \sim \gamma'$$

if $\vec{\gamma}$ is either parallel or perpendicular to $\vec{\gamma}'$. Indeed, \sim is an equivalence relation on Γ . The following observation is crucial in our discussion.

Lemma 4.3 *Let $\mathcal{C} = \gamma_1 + \dots + \gamma_m$ be a feasible cycle in Γ . Then for all $i = 1, \dots, m$, by identifying K_{m+1} with K_1 , we have that*

(i) *If K_i and K_{i+1} have different orientations, then $\gamma_i \sim \rho_{i+1}$ and $\rho_i \sim \gamma_{i+1}$.*

(ii) *If K_i and K_{i+1} have the same orientation, then $\gamma_i \sim \gamma_{i+1}$, and $\rho_i \sim \rho_{i+1}$.*

Proof: Let v be the terminus of γ_i as well as the origin of γ_{i+1} . By Lemma 4.1, the angle pattern at v must be one of

$$(\alpha, \beta), (\alpha, \beta, \alpha, \beta), (\alpha, \beta, \pi/2, \pi/2), (\alpha, \alpha, \beta, \beta), (\alpha, \pi/2, \beta, \pi/2), \quad (10)$$

up to a rotation or a reversion. In the first three cases, the angle of K_i at v and the angle of K_{i+1} at v form an angle measuring π , see Figure 3; in the fourth case, this is also true since \mathcal{C} is feasible. In the final case, K_i and K_{i+1} are separated by two right angles, see Figure 4.

In Figures 3 and 4, we illustrate all the possible ways to place K_i and K_{i+1} , and there are 8 of them. From the figures, one easily sees that the lemma holds. \square

Let E and F be two points in \mathbb{R}^2 . We will identify the vector \overrightarrow{EF} to a complex number. We use $\arg z$ to denote the principle argument of a complex number z .

Theorem 4.2 *If $\mathcal{C} = \gamma_1 + \dots + \gamma_m$ is a feasible cycle in Γ , then m is even.*

Proof: To facilitate the discussion, we set a coordinate system as follows: If all K_j are negatively oriented, then we set the coordinate system as in Figure 5(a); otherwise, we assume K_1 has positive orientation without loss of generality, and set the coordinate system as in Figure 5(b). In the following we use $A \oplus B$ instead of $A + B$ to emphasize that if $a, a' \in A$ and $b, b' \in B$, then $a + b = a' + b'$ holds only when $a = a'$ and $b = b'$. We claim that

$$\arg \vec{\gamma}_i \in \{0, \pi/2, \pi, 3\pi/2\} \oplus \{0, \beta\}. \quad (11)$$

If the orientations of K_1, \dots, K_m are the same, by Lemma 4.3, we have $\gamma_j \sim \gamma_1$ for all $j = 1, 2, \dots, m$. Since $\arg \vec{\gamma}_1 = 0$ or π , we have $\arg \vec{\gamma}_i \in \{0, \pi/2, \pi, 3\pi/2\}$.

If the orientations of K_1, \dots, K_m are not the same, we claim that

(i) If K_i has negative orientation, then $\arg \vec{\gamma}_i \equiv \alpha \pmod{\pi/2}$, $\arg \vec{\rho}_i \equiv 0 \pmod{\pi/2}$;

(ii) If K_i has positive orientation, then $\arg \vec{\gamma}_i \equiv 0 \pmod{\pi/2}$, $\arg \vec{\rho}_i \equiv \alpha \pmod{\pi/2}$.

For $i = 1$, K_1 has positive orientation by our convention, and $\arg \vec{\gamma}_1 = \pi$ and $\arg \vec{\rho}_1 = \alpha + \pi$ by our choice of the coordinate system, so the scenario of item (ii) occurs. Now the claim can be easily proved by Lemma 4.3. Our claim is proved.

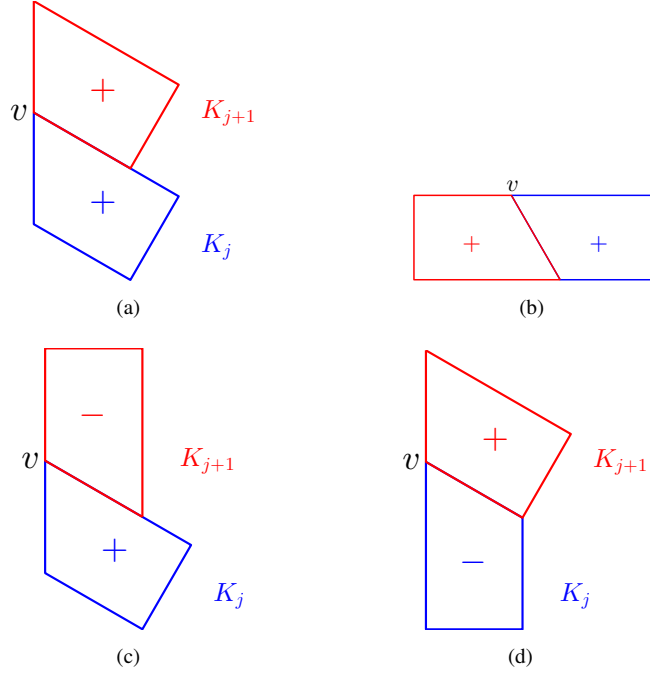


Fig. 3: Lemma 4.3: The four cases that $K_j \cap K_{j+1}$ is a line segment. Here ‘+’ means the tile is positively oriented and ‘-’ means the orientation is negative. The corresponding angle pattern is one of the first four cases of (10).

Let $\omega = e^{i\alpha}$. Recall that $|\gamma|$ denotes the length of γ . By applying a dilation to the tiling, we may assume $|\gamma_i| = 1$, then by the above claim, we have

$$\vec{\gamma}_i \in \{1, -1, i, -i, \omega, -\omega, i\omega, -i\omega\}.$$

Set

$$a = |\{i; \vec{\gamma}_i = 1\}| - |\{i; \vec{\gamma}_i = -1\}|, \quad b = |\{i; \vec{\gamma}_i = i\}| - |\{i; \vec{\gamma}_i = -i\}|,$$

$$c = |\{i; \vec{\gamma}_i = \omega\}| - |\{i; \vec{\gamma}_i = -\omega\}|, \quad d = |\{i; \vec{\gamma}_i = i\omega\}| - |\{i; \vec{\gamma}_i = -i\omega\}|.$$

\mathcal{C} is closed implies that $a + bi + c\omega + di\omega = 0$, so either $a = b = c = d = 0$, or

$$\left| \frac{a + bi}{c + di} \right| = |-\omega| = 1.$$

It follows that $a^2 + b^2 = c^2 + d^2$. Then $(a + b + c + d)^2$ is even, from which it follows that so is $a + b + c + d$. Therefore, the number of edges in \mathcal{C} is even. \square

Now, Theorem 4.1 is an immediate consequence of Lemma 4.2 and Theorem 4.2.

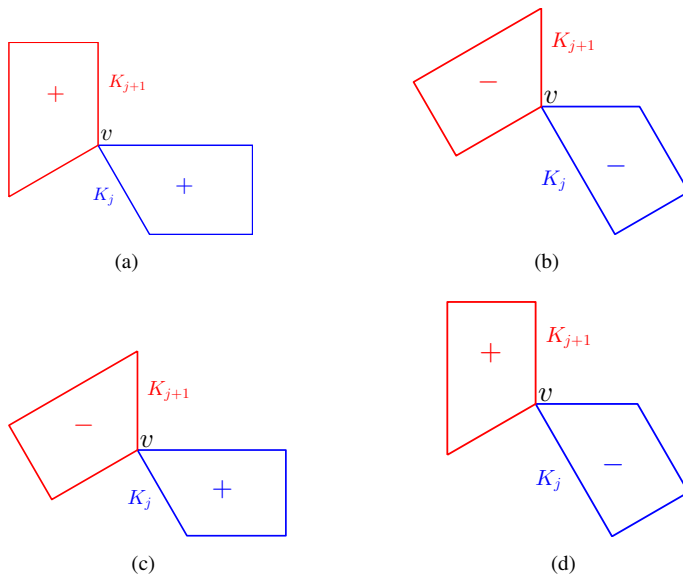


Fig. 4: Lemma 4.3: The four cases that $K_j \cap K_{j+1}$ is a single point. The corresponding angle pattern is the last case of (10).

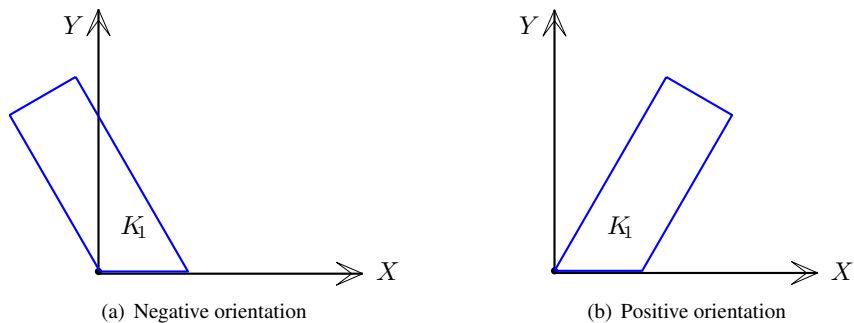


Fig. 5: Setting the coordinate system.

5 The proof of Theorem 1.2 when $\alpha \neq \pi/3$

Let $\Omega = \bigcup_{j=1}^N P_j$ be a tiling, where each P_j is congruent with a right angle trapezoid P with an angle $0 < \alpha < \pi/2$. Recall that (V, Γ) is the hypotenuse graph of the tiling (6).

5.1 The case $\alpha \notin \{\pi/4, \pi/3\}$

Let $u \in V$ and let $(\beta_1, \dots, \beta_k)$ be the angle pattern at u . Then $\beta_1 + \dots + \beta_k = \pi/2, \pi$ or 2π , and we call $\beta_1 + \dots + \beta_k$ a V -decomposition at u . Since u is taken from V , at least one angle around u is α or β .

Proof of Theorem 1.2 when $\alpha \notin \{\pi/3, \pi/4\}$. Suppose the hypotenuse graph (V, Γ) is not component-wise Eulerian. Then there exists $u \in V$ such that $\deg^+(u) < \deg^-(u)$. Suppose the V -decomposition at u is

$$a\alpha + b\beta + c\pi/2, \quad \text{where } 0 \leq a < b \text{ and } c \geq 0.$$

From

$$2\pi \geq a\alpha + b\beta + c\pi/2 > a(\alpha + \beta) = a\pi,$$

we conclude that $a < 2$.

If $a = 1$, then we have $(b-1)\beta + c\pi/2 = 0$ or π , which is impossible.

If $a = 0$, then $b\beta + c\pi/2 = \pi$ or 2π , which implies the V -decomposition at u is either $3\beta = 2\pi$ or $2\beta + \pi/2 = 2\pi$. In the former case $\alpha = \pi/3$ and in the latter case $\alpha = \pi/4$.

So (V, Γ) must be component-wise Eulerian, and N is even by Theorem 4.1. \square

5.2 The case $\alpha = \pi/4$

In this case, instead of using the hypotenuse graph (V, Γ) , we will use an undirected graph. Let (V, Γ_0) be an undirected graph, which is obtained by regarding every edge $\gamma \in \Gamma$ as an undirected edge. Clearly for every $u \in V$, the degree of u is even. Consequently, Γ_0 is component-wise Eulerian, and it is an edge-disjoint union of cycles.

Theorem 5.1 *Any cycle of (V, Γ_0) consists of an even number of edges. Consequently, $N = |\Gamma_0|$ is an even number.*

Proof: Let $\mathcal{C} = \gamma_1 + \dots + \gamma_m$ be a cycle in Γ . We choose a direction of the cycle, and regard all the edges involved as a directed edge, and then as a vector, and also as a complex number. Clearly,

$$\arg \vec{\gamma}_i \in \left\{ \frac{k\pi}{4}; 0 \leq k \leq 7 \right\}.$$

Therefore, one can show that m is even by a direct calculation, or by the same argument as in Theorem 4.2. \square

Consequently, Theorem 1.2 holds when $\alpha = \pi/4$.

6 The proof of Theorem 1.2 when $\alpha = \pi/3$

Let Ω be a square, P be a right angle trapezoid with an angle $\alpha = \pi/3$. Let

$$\Omega = \bigcup_{j=1}^N P_j \quad (12)$$

be a tiling of Ω , where each P_j is congruent with P . From now on, we assume that N is an odd number, and we are going to deduce a contradiction. For a polygon P , we shall use ∂P to denote its boundary.

Let \mathcal{V} be the union of the vertex sets of all P_j , and Λ be the set consisting of all sides of all P_j . Recall that (V, Γ) is the hypotenuse graph of the tiling (12).

Definition 6.1 For $u, v \in \mathcal{V}$, we call the line segment $\overline{[u, v]}$ a *basic segment* if for any $e \in \Lambda$, either $e \subset \overline{[u, v]}$, or $e \cap \overline{[u, v]}$ is either a point or empty.

If $\overline{[u, v]}$ is a basic segment and it is not a proper subset of any other basic segment, then we call $\overline{[u, v]}$ a *maximal segment*.

For a basic segment $\overline{[u, v]}$, the line containing the segment divides the plane into two parts. If we assume u as the origin and v as the terminus, then we call the left hand side half plane the *upper part*, and the other half plane the *lower part*.

Denote by ∂P_j the boundary of P_j . Clearly $\bigcup_{j=1}^N \partial P_j$ is a non-overlapping union of maximal segments.

By applying a dilation, we may assume the lengths of the four sides of the tile P to be

$$x + 1, 2, x, \sqrt{3}.$$

Lemma 6.1 *There exist $r, s \in \mathbb{Q}$ with $s > 0$ such that $x = r + s\sqrt{3}$.*

Proof: Let $L_j, 1 \leq j \leq 4$, be the four sides of Ω . Clearly L_j are maximal segments. We identify L_1 and L_3 , and identify L_2 with L_4 , so that $L_1 = L_3$ and $L_2 = L_4$ have both upper part and lower part. Let M denote the collection of maximal segments of the tiling $\Omega = \bigcup_{j=1}^N P_j$ after this identification.

Let $\overline{[u, v]}$ be a maximal line segment. Let L be the line containing $\overline{[u, v]}$. Since $\overline{[u, v]}$ is tiled by some sides of tiles on the upper part of L , there exist $a_1, b_1, c_1, d_1 \in \mathbb{N}$ such that

$$|\overline{[u, v]}| = a_1 x + b_1(x + 1) + c_1 \sqrt{3} + 2d_1.$$

A similar relation exists at the lower part of L . Hence there exist $a, b, c, d \in \mathbb{Z}$ such that

$$ax + b(x + 1) + c\sqrt{3} + 2d = 0. \quad (13)$$

If $a + b \neq 0$, setting $r = -\frac{b+2d}{a+b}$, $s = -\frac{c}{a+b}$, then $x = r + s\sqrt{3}$ and $r, s \in \mathbb{Q}$. The lemma holds in this case.

If $a + b = 0$ for every $\overline{[u, v]} \in M$, then $b + 2d + c\sqrt{3} = 0$, so $c = 0$. Let $X_{\overline{[u, v]}}$ be the collection of tiles whose side of length $\sqrt{3}$ is a subset of $\overline{[u, v]}$, then $|X_{\overline{[u, v]}}|$ is an even number, since each part of L contains half of these tiles. Since

$$\{P_1, \dots, P_N\} = \bigcup_{\overline{[u, v]} \in M} X_{\overline{[u, v]}}$$

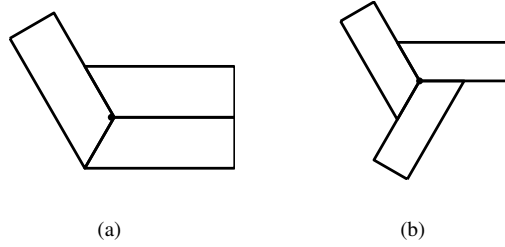


Fig. 6: Up to symmetry, there are two configurations for the angle pattern (β, β, β) .

is a partition, we conclude that N is even, which is a contradiction. The first assertion is proved.

To prove the second assertion, we use an area argument. Denote the areas of Ω and P by S_Ω and S_P , respectively. Obviously $S_\Omega = NS_P$ and

$$S_P = \frac{1}{2}(2x+1)\sqrt{3} = \frac{2r+1}{2}\sqrt{3} + 3s.$$

Let ℓ be the side length of Ω . Then $\ell = A + B\sqrt{3}$ where $A, B \in \mathbb{Q}$. So

$$S_\Omega = A^2 + 3B^2 + 2AB\sqrt{3}.$$

Hence $A^2 + 3B^2 = 3Ns$, which implies $s \geq 0$. Finally, $s \neq 0$ since $\ell > 0$. The second assertion is proved. \square

As a direct consequence of $s > 0$, we have

Corollary 6.2 *The set $\{ax + b(x+1) + c\sqrt{3}; a, b, c \in \mathbb{N}\}$ contains no positive even numbers. Therefore, if the upper part of a basic segment is tiled by sides of length 2 only, then so is the lower part.*

Lemma 6.3 *There is a vertex $v \in V$ such that the angle pattern at v is (β, β, β) .*

Proof: Since N is odd, the hypotenuse graph of the trapezoid tiling is not component-wise Eulerian. Therefore, since the total number of angles measuring α and the total number of angles measuring β are equal, there exists a vertex $u \in V$ such that $\deg^-(u) > \deg^+(u)$, so in the angle pattern at u , the number of angles measuring β is larger than those measuring α . This can only happen when the angle pattern is (β, β, β) . (See Figure 6.) The lemma is proved. \square

Before proceeding to the proof of Theorem 1.2 when $\alpha = \pi/3$, we give some definitions.

Definition 6.2 Let $u, v \in \mathcal{V}$. We call $\overline{[u, v]}$ a *half maximal segment* if it is a basic segment and there exists $u' \in \mathcal{V}$ such that $\overline{[u', v]}$ is a maximal segment containing $\overline{[u, v]}$.

By definition, a maximal segment itself is a half maximal segment.

Let $\overline{[u, v]}$ be a half maximal segment. Let K_1, \dots, K_k be the tiles in the upper part of $\overline{[u, v]}$, from left to right, such that one side of K_j is contained in $\overline{[u, v]}$. We denote the lengths of these sides by a_j , and call (a_1, \dots, a_k) the *upper side sequence* of $\overline{[u, v]}$. Similarly, we can define the *lower side sequence*.

Let (a_1, \dots, a_k) and $(b_1, \dots, b_{k'})$ be the upper and lower side sequence of $\overline{[u, v]}$, respectively. Let $T(j)$ denote the tile contributing the side a_j , and $S(j)$ the tile contributing the side b_j . But for clarity, we will use $T(j, a_j)$ and $S(j, b_j)$ instead of $T(j)$ and $S(j)$. We call $T(j, a_j)$ an *upper tile*, and $S(j, b_j)$ a *lower tile*.

Definition 6.3 Let $\overline{[u, v]}$ be a half maximal segment with upper and lower side sequences (a_1, \dots, a_k) and $(b_1, \dots, b_{k'})$, respectively. If $a_1 = 2, b_1 \neq 2$, or $a_1 \neq 2, b_1 = 2$, then we call $\overline{[u, v]}$ a *special segment*.

By Corollary 6.2, we see that if $\overline{[u, v]}$ is a special segment, then neither (a_1, \dots, a_k) nor $(b_1, \dots, b_{k'})$ is $(2, \dots, 2)$.

Now we regard the points in V as complex numbers. We define the *head information* of a special segment $\overline{[u, v]}$ to be

$$(u, \mathbf{x}, \delta, \theta),$$

where $\mathbf{x} = \frac{v-u}{|v-u|}$; $\delta = \text{upper}$ and θ is the angle of $T(1, a_1)$ at u if $a_1 = 2$, and $\delta = \text{lower}$ and θ is the angle of $S(1, b_1)$ at u if $b_1 = 2$.

Let $\omega = \exp(2\pi i/3)$. For a given vector $\mathbf{x} \neq 0$, we define a partial order on \mathbb{C} as follows: We say $u \prec v$ if $u \neq v$ and $v - u = a\mathbf{x} + b\omega\mathbf{x}$ with $a, b \geq 0$.

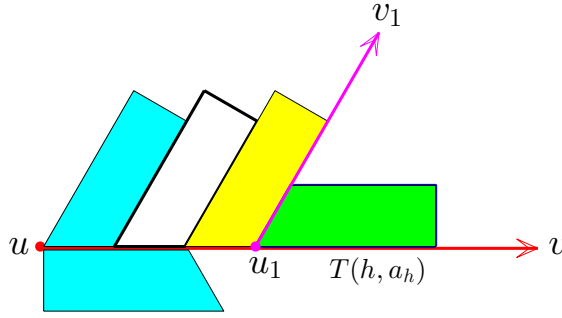


Fig. 7: $\overline{[u, v]}$ is a special segment with head information $(u, \mathbf{x}, \text{upper}, \alpha)$. It produces a new special segment $\overline{[u_1, v_1]}$, where $u_1 = u + 2(h-1)\mathbf{x}$.

Lemma 6.4 If $\overline{[u, v]}$ is a special segment with head information $(u, \mathbf{x}, \text{upper}, \alpha)$, then there exists a special segment $\overline{[u_1, v_1]}$ with head information $(u_1, \omega\mathbf{x}, \text{lower}, \alpha)$ and $u \prec u_1$.

Similarly, if $\overline{[u, v]}$ is a special segment with head information $(u, \omega\mathbf{x}, \text{lower}, \alpha)$, then there exists a special segment $\overline{[u_1, v_1]}$ with head information $(u_1, \mathbf{x}, \text{upper}, \alpha)$ and $u \prec u_1$.

Proof: Let (a_1, \dots, a_k) and $(b_1, \dots, b_{k'})$ be the upper and lower side sequence of $\overline{[u, v]}$, respectively.

Suppose the head information of $\overline{[u, v]}$ is $(u, \mathbf{x}, \text{upper}, \alpha)$. Then $a_1 = 2$ and $b_1 \neq 2$. Let h be the minimal integer such that $a_h \neq 2$, then $h \geq 2$. (The existence of h is guaranteed by Corollary 6.2.) Let

$$u_1 = u + 2(h-1)\mathbf{x}.$$

By Corollary 6.2, $u_1 \prec v$. Since $T(h-1, a_{h-1})$ contributes an angle β at u_1 , the tile $T(h, a_h)$ must contribute an angle α at u , and the orientation of $T(h, a_h)$ is positive. See Figure 7. Hence the pattern

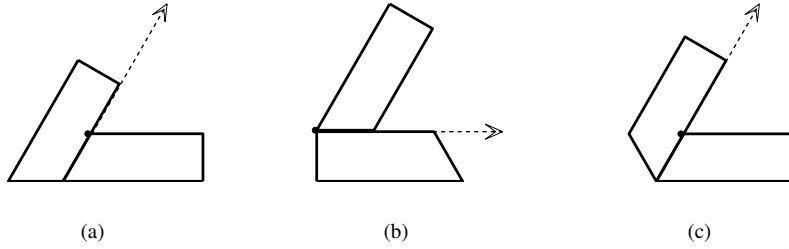


Fig. 8: Examples of forbidden patterns. There are many others.

of Figure 8 (a) occurs, and there is a special segment with head information $(u_1, \omega\mathbf{x}, \text{lower}, \alpha)$. The first assertion is proved.

The second assertion can be proved in the same manner as the first one. \square

Corollary 6.5 *Special segments with head information $(u, \mathbf{x}, \delta, \alpha)$ do not exist.*

Proof: If $\overline{[u, v]}$ is a special segment with head information $(u, \mathbf{x}, \delta, \alpha)$, then by Lemma 6.4, there exists a sequence of special segments $\overline{[u_k, v_k]}$, $k \geq 1$, such that $u_k \prec u_{k+1}$ for all k . This implies that \mathcal{V} is an infinite set since it contains all u_k , which is absurd. \square

By Corollary 6.5, the patterns in Figure 8 cannot occur in the tiling (12).

Lemma 6.6 *Let $\overline{[u, v]}$ be a special segment with head information in one of the following forms:*

$$(u, \mathbf{x}, \text{upper}, \beta), (u, \omega\mathbf{x}, \text{lower}, \beta). \quad (14)$$

Then there exists a special segment $\overline{[u', v']}$ with head information in (14) (after replacing u by u') and $u \prec u'$.

Proof: Let (a_1, \dots, a_k) and $(b_1, \dots, b_{k'})$ be the upper and lower side sequences of $\overline{[u, v]}$, respectively.

First, let us assume that the head information of $\overline{[u, v]}$ is $(u, \mathbf{x}, \text{upper}, \beta)$. Then $a_1 = 2, b_1 \neq 2$. Let h be the minimal integer such that $a_h \neq 2$, then $h \geq 2$. (Again the existence of h is guaranteed by Corollary 6.2.) Let

$$u_1 = u + 2(h-1)\mathbf{x}.$$

(We remark that u_1 is a kind of turning point.) Notice that $u \prec u_1$.

First, we argue that $T(h-1, a_{h-1})$ provides an angle α at u_1 . Otherwise, $T(h-1, a_{h-1})$ provides an angle β at u_1 implies that $T(h, a_h)$ provides an angle α at u_1 . Then the tiles $T(h-1, a_{h-1})$ and $T(h, a_h)$ form the forbidden pattern in Figure 8 (a). Our assertion is proved. Now $a_h = x$ or $x+1$ since $a_h \neq \sqrt{3}$.

Case 1. If $a_h = x$, then $T(h, a_h)$ provides an angle β at u_1 , and there is a half maximal segment $\overline{[u_1, v_1]}$ with direction $\mathbf{y} = \omega\mathbf{x}$. Moreover, it follows that $\overline{[u_1, v_1]}$ is a special segment with head information $(u_1, \omega\mathbf{x}, \text{lower}, \beta)$, as we desired. See Figure 9.

Case 2. If $a_h = x+1$, then $T(h, a_h)$ provides an angle α at u_1 . Let $\overline{[u_1, v_1]}$ be the half maximal segment with direction $\omega\mathbf{x}$.

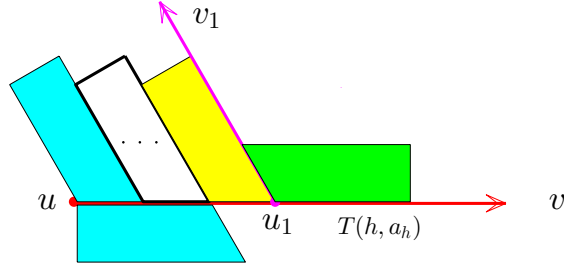


Fig. 9: Case 1 of the proof of Lemma 6.6. The yellow tile and the green tile form a special segment with head information $(u_1, \omega \mathbf{x}, \text{lower}, \beta)$.

Let (c_1, \dots, c_q) and $(d_1, \dots, d_{q'})$ be the upper and lower side sequence of $\overline{[u_1, v_1]}$, respectively. Then $d_1 = 2$. We assert that $c_1 = 2$, for otherwise, the forbidden pattern in Figure 8 (c) will occur. Let p be the maximal integer such that $c_1 = \dots = c_p = d_1 = \dots = d_p = 2$. Denote

$$u_2 = u_1 + 2p(\omega \mathbf{x}).$$

Case 2.1. If $p < q$, then at least one of c_{p+1} and d_{p+1} is not 2.

If $c_{p+1} \neq 2$, since $\overline{[u_1, u_2]}$ is not a half maximal segment, the angle pattern at u_2 must be $(\alpha, \alpha, \beta, \beta)$. Let T be the upper tile at u_2 contributing an angle α , then T must be negatively oriented, so T and the last upper tile $T(p, c_p)$ form a forbidden pattern. (See Figure 10 (a))

If $d_{p+1} \neq 2$, similarly, we get a forbidden pattern at the lower part of $\overline{[u_1, v_1]}$.

Case 2.2. If $p = q$, then every c_j and d_j is 2, and $p = q = q'$. So, $v_1 = u_2$, and the angle pattern at v_1 must be (β, β, β) , and the configuration in Figure 6 (a) or its reflection occurs. Therefore, there exists a special segment $\overline{[u_2, v_2]}$ with head information $(u_2, \omega^2 \mathbf{x}, \text{lower}, \beta)$ or $(u_2, \mathbf{x}, \text{upper}, \beta)$, see Figure 10 (b).

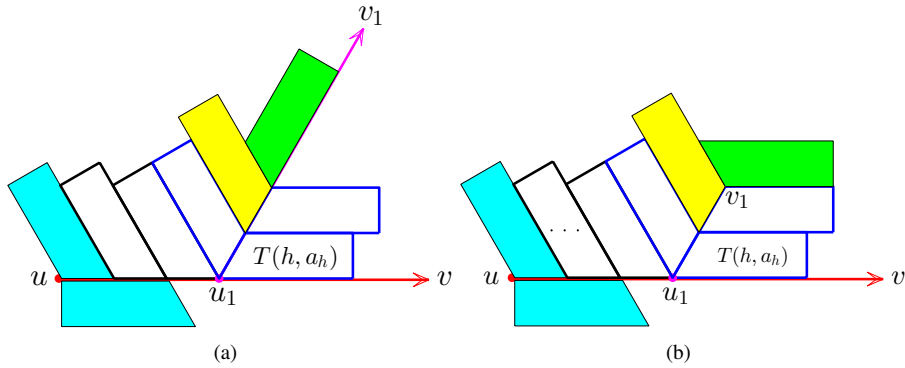


Fig. 10: Case 2 of the proof of Lemma 6.6. (a) Case 2.1: The yellow tile and the green tile form a forbidden pattern. (b) Case 2.2: The yellow tile and the green tile produce a new special segment.

The case that the head information of $\overline{[u, v]}$ is $(u, \omega \mathbf{x}, \text{lower}, \beta)$ can be dealt with in the same manner

as above. The lemma is proved. \square

Proof of Theorem 1.2 when $\alpha = \pi/3$. By Lemma 6.3, there exists a special segment with head information $(u, \mathbf{x}, \delta, \beta)$, see Figure 6. Then, by Lemma 6.6, there exists a sequence of special segments $[u_k, v_k]$, $k \geq 1$, such that $u_k \prec u_{k+1}$ for all k . But this contradicts the fact that \mathcal{V} is a finite set. Therefore, the assumption that N is odd is wrong. \square

7 Some questions

We close this paper with some questions.

Question 1. *What kind of quadrilaterals can tile a square?* We believe that if a quadrilateral can tile a square, then it must contain at least two right angles. See Figure 11.

Question 2. *Can we replace the square by a rectangle in Conjecture 1?* It is seen that the answer is yes for $q \neq 4$. For Theorem 1.2, this is also true except the case that $\alpha = \pi/3$. (The only place in which we use that Ω is a square rather than a rectangle is to prove $s > 0$ in Lemma 6.1.)

Question 3. *How does a right-angle trapezoid tile a square?* Let P be a right-angle trapezoid and (1) be a tiling of Ω by P . We believe that every connected component of the hypotenuse graph is a cycle consisting of two edges. In other words, the tiles must be paired by their hypotenuse. See Figure 11 (a).

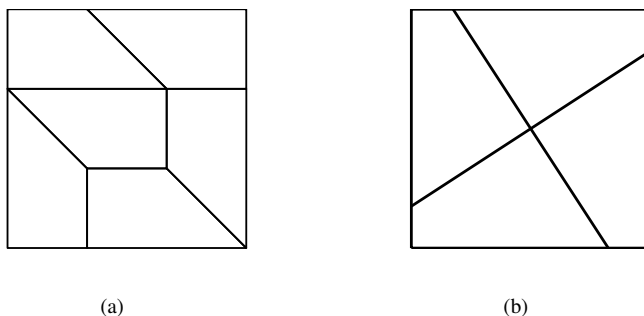


Fig. 11: Two classes of quadrilaterals which can tile a square.

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