

Evacuating Robots from a Disk Using Face-to-Face Communication*

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Assume that two robots are located at the centre of a unit disk. Their goal is to *evacuate* from the disk through an *exit* at an unknown location on the boundary of the disk. At any time the robots can move anywhere they choose on the disk, independently of each other, with maximum speed 1. The robots can cooperate by exchanging information whenever they meet. We study algorithms for the two robots to minimize the *evacuation time*: the time when *both* robots reach the exit.

Czyzowicz et al. (2014) gave an algorithm defining trajectories for the two robots yielding evacuation time at most 5.740 and also proved that any algorithm has evacuation time at least $3 + \frac{\pi}{4} + \sqrt{2} \approx 5.199$. We improve both the upper and lower bound on the evacuation time of a unit disk. Namely, we present a new non-trivial algorithm whose evacuation time is at most 5.628 and show that any algorithm has evacuation time at least $3 + \frac{\pi}{6} + \sqrt{3} \approx 5.255$. To achieve the upper bound, we designed an algorithm which proposes a forced meeting between the two robots, even if the exit has not been found by either of them. We also show that such a strategy is provably optimal for a related problem of searching for an exit placed at the vertices of a regular hexagon.

Keywords: Disk, Evacuation, Face-to-Face Model

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1 Introduction

The goal of traditional search problems is to find an object that is located in a specific domain. This subject of research has a long history and there is a plethora of models investigated in the mathematical and theoretical computer science literature with emphasis on probabilistic search in Stone (1975), game theoretic applications in Alpern and Gal (2003), cops and robbers in Bonato and Nowakowski (2011), classical pursuit and evasion in Nahin (2012), search problems and group testing in Ahlswede and Wegener (1987), and many more.

In this paper, we investigate the problem of searching for a stationary point target called an *exit* at an unknown location using two robots. This type of collaborative search is advantageous in that it reduces the required search time by distributing the search effort between the two robots. In previous work on collaborative search, the goal has generally been to minimize the time taken by the *first* robot to find the object of the search. In contrast, in the work at hand, we are interested in minimizing the time when the *last robot* finds the exit. In particular, suppose two robots are in the interior of a region with a single exit. The robots need to evacuate the region but the location of the exit is unknown to them. The robots can cooperate to search for the exit, but it is not enough for one robot to find the exit, we require *both* robots to reach the exit as soon as possible.

We study the problem of two robots that start at the same time at the centre of a unit disk and attempt to reach an exit placed at an unknown location on the boundary of the disk. At any time the robots can move anywhere they choose within the disk. Indeed, they can take short-cuts by moving in the interior of the disk if desired. We assume that their maximum speed is 1. The robots can communicate with each other only if they are at the same point at the same time: we call this communication model *face-to-face communication*. Our goal is to schedule the trajectories of the robots so as to minimize the *evacuation time*, which is the time it takes both robots to reach the exit (for the worst case location of the exit).

Our main contributions pertain to improved upper and lower bounds for the evacuation problem on the disc. The main ideas are derived by the study of a related evacuation problem in which the exit lies in one of the vertices of a regular hexagon. For the latter problem, we introduce a novel and non-intuitive search strategy in which searchers are forced to detour and meet if the exit is not found early enough. Surprisingly, we show this trajectory to induce optimal evacuation cost for that problem. The implications are two-fold. First, the lower bound we obtain applies directly to the main evacuation problem on the disc. Second, we use the same ideas of a detour and forced meeting to improve the best evacuation cost known (at the time when the result was first announced; see Section 1.1 for newer developments).

1.1 Related work

The most related work to ours is Czyzowicz et al. (2014), where the evacuation problem for a set of robots all starting from the centre of a unit disk was introduced and studied. Two communication models are introduced in Czyzowicz et al. (2014). In the *wireless* model, the two robots can communicate at any time regardless of their locations. In particular, a robot that finds the exit can immediately communicate its location to the other robot. The other model is called the *non-wireless or local* model in Czyzowicz et al. (2014), and is the same as our face-to-face model: two robots can only communicate when they are face to face, that is, they are at the same point location at the same time. In Czyzowicz et al. (2014), for the case of 2 robots, an algorithm with evacuation time $1 + \frac{2\pi}{3} + \sqrt{3} \approx 4.826$

is given for the wireless model; this is shown to be optimal. For the face-to-face model, they prove an upper bound of 5.740 and a lower bound of 5.199 on the evacuation time. Since the first announcement of our results in Czyzowicz et al. (2015a), Brandt et al. (2017) improved our upper bound from 5.628 to 5.625. Their contributions also pertain to a significant simplification of the search trajectories (that use detours but without forced meetings), along with a clever performance analysis. Very recently, Disser and Schmitt (2019) reported a further improvement of 5.6234 by employing and generalizing techniques introduced in Brandt et al. (2017). However, to the best of our knowledge, the lower bound we provide in this manuscript is the best currently known.

Since the introduction of the problem in Czyzowicz et al. (2014), a number of interesting variations emerged some of which are summarized in recent survey Czyzowicz et al. (2019b) (see also Flocchini et al. (2019) for a collection of broadly related problems). Some representative examples of related problems include an evacuation problem with speed bounds in the wireless model Lamprou et al. (2016), evacuation from triangles Czyzowicz et al. (2015b); Chuangpishit et al. (2017), evacuation from multiple rays Brandt et al. (2020), evacuation from graphs Borowiecki et al. (2016), search with terrain dependent speeds Czyzowicz et al. (2017b), wireless evacuation from multiple exits Czyzowicz et al. (2018a); Pattanayak et al. (2018), priority evacuation Czyzowicz et al. (2018b,c), evacuation with faulty robots Czyzowicz et al. (2017a); Pattanayak et al. (2019); Georgiou et al. (2019b) or with probabilistically faulty robots Bonato et al., evacuation with immobile agents Georgiou et al. (2016, 2019a), time energy tradeoffs for evacuation on the line Czyzowicz et al. (2019a); Kranakis et al. (2019), worst-case average-case tradeoffs for evacuation on the disc Chuangpishit et al. (2018), and searching graphs Angelopoulos et al. (2019).

In a different direction, Baeza-Yates *et al* posed the question of minimizing the worst-case trajectory of a single robot searching for a target point at an unknown location in the plane Baeza Yates et al. (1993). This was generalized to multiple robots in López-Ortiz and Sweet (2001), and more recently has been studied in Emek et al. (2014); Lenzen et al. (2014). However, in these papers, the robots cannot communicate, and moreover, the objective is for the first robot to find the target. Two seminal and influential papers (that appeared almost at the same time) on probabilistic search that concern minimizing the *expected time* for the robot to find the target are Beck (1964) and Bellman (1963). As for two surveys on search theory the interested reader is referred to Benkoski et al. (1991) and Dobbie (1968). The latter survey also contains an interesting classification of search problems by search objectives, distribution of effort, point target (stationary, large, moving), two-sided search, and other criteria. The evacuation problem considered in our paper is related to searching on a line, in that we are searching on the boundary of a disk but with the additional ability to make short-cuts in order to enable the robots to meet sooner and thus evacuate faster. Our problem is also related to the *rendezvous problem* and the problem of *gathering* Alpern and Beck (1999); Flocchini et al. (2005). Indeed our problem can be seen as a version of a rendezvous problem for three robots, where one of them remains stationary.

1.2 Preliminaries and notation

We assume that two robots R_1 and R_2 are initially at the centre of a disk with radius 1, and that there is an exit at some location X on the boundary of the disk. The robots do not know X , but do know each other's algorithms. The robots move at a speed subject to a maximum speed, say 1. They cannot communicate except if they are at the same location at the same time. The *evacuation problem* is to define trajectories for the two robots that minimize the *evacuation time*. All our results (upper and lower bounds) pertain to deterministic algorithms that work against a (worst case) deterministic

adversary.

For two points A and B on the unit circle, the length of an arc AB is denoted by \widehat{AB} , while the length of the corresponding chord (line segment) will be denoted by \overline{AB} (arcs on the circle are always read clockwise, i.e., arc AB together with arc BA cover the whole circle). By $\angle ABC$ we denote the angle at B in the triangle ABC . Finally by \overrightarrow{AB} we denote the vector with initial point A and terminal point B , i.e. vector $B - A$ when B, A are understood as points in the Cartesian plane.

1.3 Outline, results of the paper and new improvements

In Czyzowicz et al. (2014) an algorithm is given defining a trajectory for two robots in the face-to-face communication model with evacuation time 5.740 and it is also proved that any such algorithm has evacuation time at least $3 + \frac{\pi}{4} + \sqrt{2} > 5.199$.

Our main contribution in this paper is to improve both the upper and lower bounds on the evacuation time. Namely, we give a new algorithm whose evacuation time is at most 5.628 (see Section 2) and also prove that any algorithm has evacuation time at least $3 + \frac{\pi}{6} + \sqrt{3} > 5.255$ (see Section 3). To prove our lower bound on the disk, we first give tight bounds for the problem of evacuating a regular hexagon where the exit is placed at an unknown vertex. We observe that, surprisingly, in our optimal evacuation algorithm for the hexagon, the two robots are forced to meet after visiting a subset of vertices, even if an exit has not been found at that time. We use the idea of such a forced meeting in the design of our disk evacuation algorithm, inducing this way an improvement of the upper bound from 5.740 to 5.628. It is still unknown whether such a forced meeting is necessary. However, since the first announcement of the upper bound, Brandt et al. (2017) proposed an elegant analysis of a simplified trajectory that avoids forced meetings, still improving the upper bound from 5.628 to 5.625. The authors in the latter paper even proposed a refinement of their technique involving multiple detours as the robots are searching for the exit. Their ideas were later materialized in Disser and Schmitt (2019) that reported a further improvement of 5.6234.

2 Evacuation Algorithms

In this section we give two new evacuation algorithms for two robots in the face-to-face model that take evacuation time approximately 5.644 and 5.628 respectively. We begin by presenting Algorithm \mathcal{A} proposed by Czyzowicz et al. (2014) which has been shown to have evacuation time 5.740. Our goal is to understand the worst possible configuration for this algorithm, and to subsequently modify it accordingly so as to improve its performance.

All the algorithms we present follow the same general structure: The two robots R_1 and R_2 start by moving together to an arbitrary point A on the boundary of the disk. Then R_1 explores the arc $A'A$, where A' is the antipodal point of A , by moving along some trajectory defined by the algorithm. At the same time, R_2 explores the arc AA' , following a trajectory that is the reflection of R_1 's trajectory. If either of the robots finds the exit, it immediately uses the *Meeting Protocol* defined below to meet the other robot (note that the other robot has not yet found the exit and hence keeps exploring). After meeting, the two robots travel together on the shortest path to the exit, thereby completing the evacuation. At all times, the two robots travel at unit speed. Without loss of generality, we assume for our analysis that R_1 finds the exit and then *catches* R_2 .

Meeting Protocol for R_1 : If at any time t_0 , R_1 finds the exit at point X , it computes the shortest additional time t such that R_2 , after traveling distance $t_0 + t$, is located at point M satisfying $\overline{XM} = t$. Robot R_1 moves along the segment XM . At time $t_0 + t$ the two robots meet at M and traverse directly back to the exit at X incurring total time cost $t_0 + 2t$.

2.1 Evacuation Algorithm \mathcal{A} of Czyzowicz et al. (2014)

We proceed by describing the trajectories of the two robots in Algorithm \mathcal{A} . As mentioned above, both robots start from the centre O of the disk and move together to an arbitrary position A on the boundary of the disk. R_2 then moves clockwise along the boundary of the disk up to distance π , see the left-hand side of Figure 1, and robot R_1 moves counter clockwise on the trajectory that is a reflection of R_2 's trajectory with respect to the line passing through O and A . When R_1 finds the exit, it invokes the *meeting protocol* in order to meet R_2 , after which the evacuation is completed.

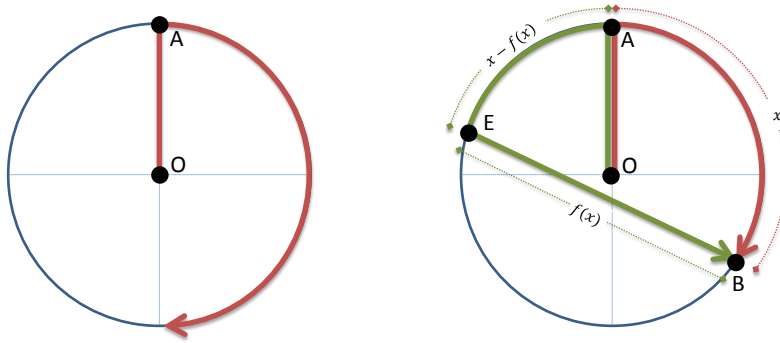


Fig. 1: Evacuation Algorithm \mathcal{A} with exit position E . The trajectory of robot R_2 is depicted on the left. The movement paths of robots R_1, R_2 are shown on the right, until the moment they meet at point B on the circle.

The meeting-protocol trajectory of R_1 in Algorithm \mathcal{A} is depicted in the right-hand side of Figure 1, with exit point E and meeting point B . Clearly, for the two robots to meet, we must have $\overline{AB} = \overline{EA} + \overline{EB}$. Next we want to analyze the performance of the algorithm, with respect to $x := \overline{AB}$, i.e. the length x of the arc that R_2 travels, before it is met by R_1 . We also set $f(x) := \overline{EB}$. It follows that $\overline{EA} = x - f(x)$, and therefore⁽ⁱ⁾

$$f(x) = z, \text{ where } z \text{ is a solution of the equation } z = 2 \sin\left(x - \frac{z}{2}\right). \quad (1)$$

In other words, $f(x)$ ⁽ⁱⁱ⁾ is the length of the segment EB that R_1 needs to travel in the interior of the disk after locating the exit at E , to meet R_2 at point B .

Then the cost of Algorithm \mathcal{A} , given that the two robots meet at time x after they together reached the boundary of the disk at A , is $1 + x + f(x)$. Given that the distance $x - f(x)$ traveled by R_1 until finding the exit is between 0 and π , it directly follows that x can take any value between 0 and π as well. Hence,

⁽ⁱ⁾ We are using the elementary fact that in a unit circle, an arc of length α corresponds to a chord of length $2 \sin(\alpha/2)$

⁽ⁱⁱ⁾ Uniqueness of the root of the equation defining $f(x)$ is an easy exercise.

the worst case performance of Algorithm \mathcal{A} is determined by

$$\max_{x \in [0, \pi]} \{x + f(x)\}$$

The next lemma, along with more details of its proof, follows from Czyzowicz et al. (2014).

Lemma 2.1. *Expression $F(x) := x + f(x)$ attains its supremum at $x_0 \approx 2.85344$ (which is $\approx 0.908279\pi$). In particular, $F(x)$ is strictly increasing when $x \in [0, x_0]$ and strictly decreasing when $x \in [x_0, \pi]$.*

Proof: The behavior of $F(x)$, as x ranges in $[0, \pi]$, is shown in Figure 2. ■

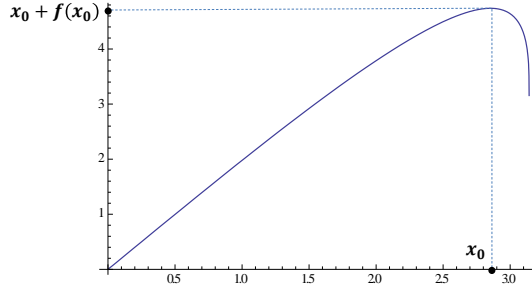


Fig. 2: The performance of Algorithm \mathcal{A} as a function of the meeting points of the robots.

By Lemma 2.1, the evacuation time of Algorithm \mathcal{A} is $1 + x_0 + f(x_0) < 5.740$. The worst case position of the exit is attained for $x_0 - f(x_0) \approx 0.308\pi$.

2.2 New evacuation algorithm $\mathcal{B}(\chi, \phi)$

We now show how to improve Algorithm \mathcal{A} and obtain an evacuation time of at most 5.644. The main idea for the improvement is to change the trajectory of the robots when the distance traveled on the boundary of the disk approaches the critical value x_0 of Lemma 2.1. Informally, robot R_2 could meet R_1 earlier if it makes a linear detour inside the interior of the disk towards R_1 a little before traversing distance x_0 .

We describe a generic family of algorithms that realizes this idea. The specific trajectory of each algorithm is determined by two parameters χ and ϕ where $\chi \in [\pi/2, x_0]$ and $\phi \in [0, f(\chi)/2]$, whose optimal values will be determined later. For ease of exposition, we assume that R_1 finds the exit. The trajectory of R_2 (assuming it has not yet met R_1) is partitioned into four phases that we call the *deployment*, *pre-detour*, *detour* and *post-detour* phase. The description of the phases relies on the left-hand side of Figure 3.

Algorithm $\mathcal{B}(\chi, \phi)$ (with a linear detour). The phases of robot R_2 's trajectory (unless it is met by R_1 to go to the exit) are:

★ *Deployment phase:* Robot R_2 starts from the centre O of the disk and moves to an arbitrary position A on the boundary of the disk.

★ *Pre-detour phase:* R_2 moves clockwise along the boundary of the disk until having explored an arc of length χ . Let B be the point on the circle in which this phase ends.

★ *Detour phase:* Let D be the reflection of B with respect to AA' (where A' is the antipodal point of A). R_2 moves on a straight line towards the interior of the disk and towards the side where O lies, forming an angle of ϕ with line BD until it reaches the line AA' at point C . At C , R_2 turns around and follows the same straight line back to B . Note that by the restrictions on ϕ , C is indeed in the interior of the line segment AA' .

★ *Post-detour phase:* Robot R_2 continues moving clockwise on the arc BA' .

The trajectory of R_1 , until it finds the exit, is the reflection of R_2 's trajectory along the line AA' . When at time t_0 , R_1 finds the exit, it follows the Meeting Protocol defined above.

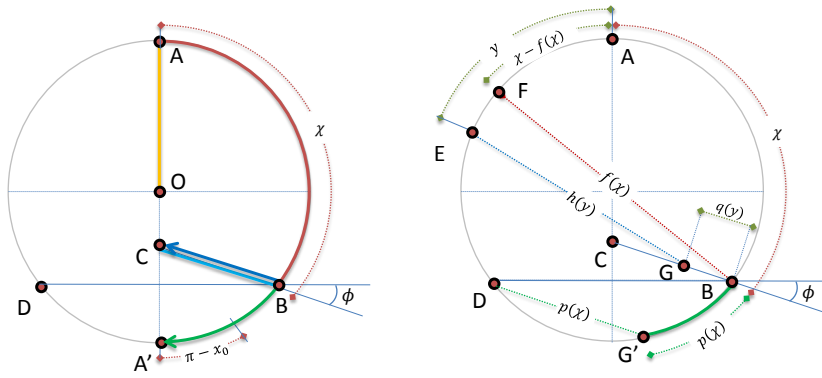


Fig. 3: Illustrations for Algorithm $\mathcal{B}(\chi, \phi)$.

In what follows we assume that R_1 finds the exit on the arc $A'A$. Subsequently, it catches R_2 on its trajectory and they return together to the exit. There are three cases to consider as to where R_2 can be caught by R_1 while moving on its trajectory. For all three cases, the reader can consult the right-hand side of Figure 3. As the time needed for the deployment phase is independent of where the exit is located, we ignore this extra cost of 1 during the case distinction.

Case 1: R_2 is caught during its pre-detour phase: The meeting point is anywhere on the arc AB . Recall that $\chi \leq x_0$, so by Lemma 2.1 the location F of the exit on the arc FA that maximizes the cost of $\mathcal{B}(\chi, \phi)$ is the one at distance $\chi - f(\chi)$ from A (see right-hand side of Figure 3). The cost then is $\overline{AB} + \overline{BF} = \overline{FA} + 2\overline{BF} = \chi + f(\chi)$.

Case 2: R_2 is caught during its detour phase: Let G be the point on BC where the robots meet. Further, let E be the position of the exit on the arc $A'A$, and let $y := \widehat{EA}$. In the following, $h(y) := \overline{EG}$ denotes the length of the trajectory of R_1 in its attempt to catch R_2 after it finds the exit. Also, $q(y) := \overline{BG}$ denotes the distance that R_2 travels on BC until it is caught by R_1 . Note that the functions h and q also depend on χ and ϕ ; however, while those are fixed, y varies with the position of the exit. Lemma 2.2 below states that $h(y)$ and $q(y)$ are well defined.

Lemma 2.2. $y \in [\chi - f(\chi), \chi]$ if and only if the meeting point G of the robots is on the line segment BC . Moreover, robot R_2 can be caught by R_1 only while moving from B to C .

Proof: If the exit is located at point F (i.e. $y = \chi - f(\chi)$), then the meeting point is B . We also observe that if the exit-position coincides with D (i.e. $y = \chi$), then the meeting point is C . Recall that $\phi \leq f(\chi)/2 = \widehat{DF}/2$, and hence point C is in the interior or on the boundary of the triangle FDB as it is depicted in right-hand side of Figure 3. Therefore, after time χ , robot R_2 would approach the exit if it was located anywhere on the arc DE . In particular, if $y \in [\chi - f(\chi), \chi]$ then the meeting point for Algorithm \mathcal{A} would be on the arc BA' . In Algorithm $\mathcal{B}(\chi, \phi)$, R_2 has a trajectory that brings it closer to the exit. This guarantees that the meeting point G always exists, and it lies in the line segment BC (we will soon derive a closed formula relating \widehat{EA} and \widehat{BG}). The previous argument guarantees that $h(y)$ is continuous and strictly decreasing in y . Notice that $\widehat{EA} + \widehat{EG} = \widehat{AB} + \widehat{BG}$ (since the two robots start from the same position A), which means that

$$y + h(y) = \chi + q(y) \quad (2)$$

Hence, $q(y) = y + h(y) - \chi$, and as already explained, $q(\chi - f(\chi)) = 0$ and $q(\chi) = \widehat{BC}$. By the mean value theorem, all values between 0 and \widehat{BC} are attainable for $q(y)$ and are attained while y ranges in $[\chi - f(\chi), \chi]$. ■

We conclude that if the exit is located at point E , then the cost of the algorithm is $y + 2h(y)$. Hence, in case 2, the cost of the algorithm is at most

$$\max_{y \in [\chi - f(\chi), \chi]} \{y + 2h(y)\}.$$

Function $h(y)$ is calculated later in Lemma 2.4. We emphasize again that $h(y)$ and $q(y)$ also depend on the fixed parameters χ and ϕ .

Case 3: R_2 is caught during its post-detour phase: Due to the already searched domain of the pre-detour phase, in this case the exit lies in the interior of the arc $A'D$ or coincides with A' . At time $t_d = \chi + 2q(\chi)$, robots R_1 and R_2 are located at points D and B , respectively. Then they move towards each other on the arc BD until R_1 finds the exit. Note that, since $\widehat{DB}/2 = \sin(\chi)$, we have $q(\chi) = \sin(\chi) / \cos(\phi)$. By the monotonicity of the cost of Algorithm \mathcal{A} , see also Figure 2, we have that the closer the exit is to D , the higher is the cost of the evacuation algorithm. In the limit (as the position of the exit approaches D), the cost of case 3 approaches t_d plus the time it takes R_1 to catch R_2 if the exit was located at D , and if they started moving from points D and B respectively. Let G' be the meeting point on the arc BD in this case, i.e. $\widehat{DG'} = \widehat{BG'}$. We define $p(x)$ to be the distance that R_1 needs to travel in the interior of the disk to catch R_2 , if the exit is located at distance x from A . Clearly⁽ⁱⁱⁱ⁾

$$p(x) := \text{unique } z \text{ satisfying } z = 2 \sin\left(x + \frac{z}{2}\right). \quad (3)$$

Note also that $\widehat{DG'} = p(\chi)$ so that the total cost in this case is at most

$$t_d + 2p(\chi) = \chi + 2 \sin(\chi) / \cos(\phi) + 2p(\chi).$$

⁽ⁱⁱⁱ⁾ Uniqueness of the root of the equation defining $p(x)$ is an easy exercise.

The following two lemmas summarize the above analysis and express $h(y)$ in explicit form (in dependence of χ and ϕ), respectively.

Lemma 2.3. For fixed constants χ and ϕ the evacuation time of Algorithm $\mathcal{B}(\chi, \phi)$ is

$$1 + \max \left\{ \chi + f(\chi), \max_{y \in [\chi - f(\chi), \chi]} \{y + 2h(y)\}, \chi + 2 \sin(\chi) / \cos(\phi) + 2p(\chi) \right\}, \quad (4)$$

where $h(y)$ (that also depends on the choice of χ and ϕ) denotes the time that a robot needs from the moment it finds the exit until it meets the other robot when following the meeting protocol.

Lemma 2.4. For every $\chi > 0$ and for every $\chi - f(\chi) \leq y \leq \chi$, the distance $h(y)$ that R_1 travels from A until finding R_2 when following the meeting protocol in Algorithm $\mathcal{B}(\chi, \phi)$ is

$$h(y) = \frac{2 + (\chi - y)^2 - 2 \cos(\chi + y) + 2(\chi - y) (\sin(\phi + y) - \sin(\phi - \chi))}{2(\chi - y - \sin(\phi - \chi) + \sin(\phi + y))}.$$

In particular, $h(y)$ is strictly decreasing for $0 \leq \phi \leq f(\chi)/2$.

Proof: We start by making some handy observations. For this we rely on Figure 4 (that is a continuation

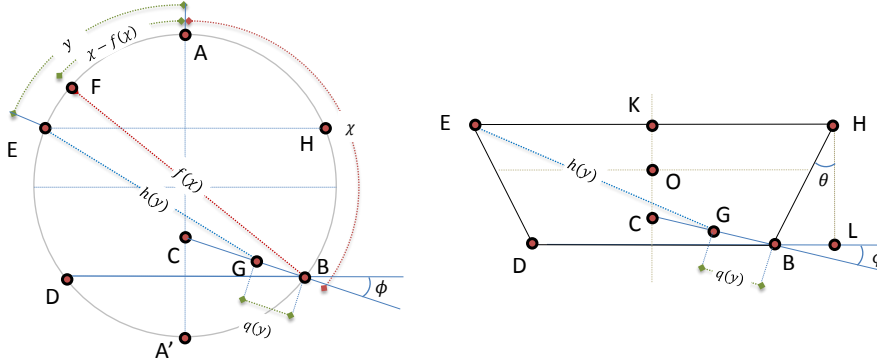


Fig. 4: The analysis of Algorithm $\mathcal{B}(\chi, \phi)$.

of Figure 3). Let H be the point that is symmetric to E with respect to AA' . Denote with L the projection of H onto the supporting line of DB . Set $\theta := \angle BHL$, and observe the following equation for θ .

$$\begin{aligned} \theta &= \frac{\pi}{2} - \angle EHB = \frac{\pi}{2} - \frac{\widehat{BE}}{2} = \frac{\pi}{2} - \frac{\widehat{BD} + \widehat{DE}}{2} \\ &= \frac{\pi}{2} - \frac{2(\pi - \chi) + \chi - y}{2} = \frac{\chi + y}{2} - \frac{\pi}{2} \end{aligned} \quad (5)$$

Our goal is to compute $h(y) = \overline{EG}$. For this we see that $\overrightarrow{EG} = \overrightarrow{EH} + \overrightarrow{HB} + \overrightarrow{BG}$, and therefore

$$\overline{EG}^2 = \overline{EH}^2 + \overline{HB}^2 + \overline{BG}^2 + 2(\overrightarrow{EH} \cdot \overrightarrow{HB} + \overrightarrow{EH} \cdot \overrightarrow{BG} + \overrightarrow{HB} \cdot \overrightarrow{BG}). \quad (6)$$

We have $\overline{EH} = 2 \sin(y)$, $\overline{HB} = 2 \sin\left(\frac{\chi-y}{2}\right)$, $\overline{BG} = q(y)$, and

$$\begin{aligned}\overrightarrow{EH} \cdot \overrightarrow{HB} &= \overline{EH} \overline{HB} \cos(\pi - \angle EHB) \\ \overrightarrow{EH} \cdot \overrightarrow{BG} &= \overline{EH} \overline{BG} \cos(\angle GBL) \\ \overrightarrow{HB} \cdot \overrightarrow{BG} &= \overline{HB} \overline{BG} \cos(\pi - \angle GBH).\end{aligned}$$

We also have

$$\begin{aligned}\cos(\pi - \angle EHB) &= \cos(\pi/2 + \theta) = \cos\left(\frac{\chi+y}{2}\right) \\ \cos(\angle GBL) &= \cos(\pi - \phi) = \cos(\phi) \\ \cos(\pi - \angle GBH) &= \cos\left(\pi + \frac{\pi}{2} + \theta - \phi\right) = -\cos\left(\frac{\chi+y}{2} - \phi\right).\end{aligned}$$

Substituting the above in (6), we obtain an equation between $h(y)$, $q(y)$, y , χ , and ϕ . In the latter equation we can substitute $q(y)$ using the meeting condition (2), obtaining this way the required closed formula for $h(y)$. The fact that $h(y)$ is strictly decreasing for $0 \leq \phi \leq f(\chi)/2$ follows from the proof of Lemma 2.2. \blacksquare

The first natural attempt in order to beat Algorithm \mathcal{A} would be to consider $\mathcal{B}(\chi, 0)$, i.e. make BC perpendicular to AA' in Figure 3. In light of Lemma 2.4 and using Lemma 2.3, we state the following claim to build some intuition for our next, improved, algorithm.

Claim 2.5. *The performance of algorithm $\mathcal{B}(\chi, 0)$ is optimized when $\chi = \chi_0 \approx 2.62359$, and its cost is $1 + 4.644 = 5.644$. The location of the exit inducing the worst case for $\mathcal{B}(\chi, 0)$ is when $y = \widehat{EA} \approx 0.837\pi$. The meeting point of the two robots takes place at point G (see Figure 3, and set $\phi = 0$), where $q(y) = \overline{BG} \approx 0.117 \approx 0.236\overline{BC}$. In particular, the cost of the algorithm, if the meeting point of the robots is during R_2 's pre-detour, detour and post-detour phase, is (approximately) 5.621, 5.644 and 5.644 respectively.*

Note that χ_0 of Claim 2.5 is strictly smaller than x_0 of Lemma 2.1. In other words, the previous claim is in coordination with our intuition that if the robots moved towards the interior of the disk a little before the critical position of the meeting point x_0 of Algorithm \mathcal{A} , then the cost of the algorithm could be improved.

2.3 New evacuation algorithm $\mathcal{C}(\chi, \phi, \lambda)$

Claim 2.5 is instructive for the following reason. Note that the worst meeting point G for Algorithm $\mathcal{B}(\chi_0, 0)$ satisfies $\overline{BG} \approx 0.236\overline{BC}$. This suggests that if we consider algorithm $\mathcal{B}(\chi_0, \phi)$ instead, where $\phi > 0$, then we would be able to improve the cost if the meeting point happened during the detour phase of R_2 . On one hand, this further suggests that we can decrease the detour position χ_0 (note that the increasing in χ cost $\chi + f(\chi)$ is always a lower bound to the performance of our algorithms when $\chi < x_0$). On the other hand, that would have a greater impact on the cost when the meeting point is in the post-detour phase of R_2 , as in this case the cost of moving from B to C and back to B would be $2 \sin(\chi) / \cos(\phi)$ instead of just $2 \sin(\chi)$. A compromise to this would be to follow the linear detour trajectory of R_2 in $\mathcal{B}(\chi_0, \phi)$ only up to a certain threshold-distance λ , after which the robot should reach the diameter segment AA' along a linear segment perpendicular to segment AA' then return to

the detour point B along a linear segment. Thus the detour forms a triangle. This in fact completes the high level description of Algorithm $\mathcal{C}(\chi, \phi, \lambda)$ that we formally describe below.

For an instance of Algorithm $\mathcal{C}(\chi, \phi, \lambda)$, we fix χ , ϕ , and λ , with $\chi \in [\pi/2, x_0]$, $\phi \in [0, f(\chi)/2]$, and $\lambda \in [0, \sin(\chi)/\cos(\phi)]$. As before, we assume without loss of generality that R_1 finds the exit. The trajectory of robot R_2 (that has neither found the exit nor met R_1 after R_1 has found the exit) can be partitioned into roughly the same four phases as for Algorithm $\mathcal{B}(\chi, \phi)$; so we again call them *deployment*, *pre-detour*, *detour* and *post-detour* phases. The description of the phases refers to the left-hand side of Figure 5, which is a partial modification of Figure 3.

Algorithm $\mathcal{C}(\chi, \phi, \lambda)$ (with a triangular detour). The phases of robot R_2 's trajectory (unless it is met by R_1 to go to the exit) are:

- ★ *Deployment phase*: Same as in Algorithm $\mathcal{B}(\chi, \phi)$. At time 1, R_2 is at point A .
 - ★ *Pre-detour phase*: Same as in Algorithm $\mathcal{B}(\chi, \phi)$. In additional time χ , R_2 is in point B .
 - ★ *Detour phase*: This phase is further split into three subphases.
 - ◇ *Subphase-1*: Up to additional time λ , R_2 moves along a line segment exactly as in the detour phase of Algorithm $\mathcal{B}(\chi, \phi)$. Let G be the position of the robot at the end of this phase.
 - ◇ *Subphase-2*: Let C be the projection of G onto AA' . R_2 follows line segment GC until it reaches point C .
 - ◇ *Subphase-3 (Recovering phase)*: Robot follows line segment CB back to point B .
 - ★ *Post-detour phase*: Same as in Algorithm $\mathcal{B}(\chi, \phi)$. After additional time $\widehat{EA'}$, R_2 reaches point A' .
- At the same time R_1 follows a trajectory that is the reflection of R_2 's trajectory along the line AA' until it finds the exit. The meeting protocol that R_1 follows once it finds the exit is the same as for Algorithms \mathcal{A} and $\mathcal{B}(\chi, \phi)$.

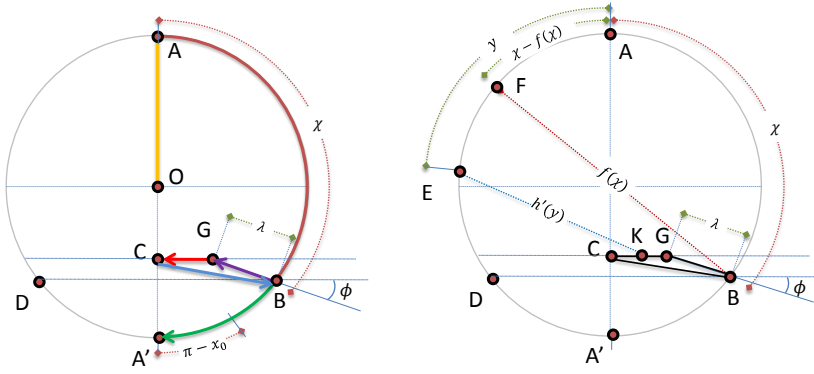


Fig. 5: Illustrations for Algorithm $\mathcal{C}(\chi, \phi, \lambda)$.

Obviously, Algorithm $\mathcal{C}\left(\chi, \phi, \frac{\sin(\chi)}{\cos(\phi)}\right)$ is identical to Algorithm $\mathcal{B}(\chi, \phi)$. Notice that an immediate consequence of the definition is that if robot R_1 finds the exit and meets R_2 during its detour subphase-2 in some point K (as in the right-hand side of Figure 5), then

$$\widehat{EA} + \widehat{EK} = \widehat{AB} + \widehat{BG} + \widehat{GK}. \quad (7)$$

When R_1 finds the exit somewhere on the arc $A'A$, it catches R_2 on its trajectory so that they return together to the exit. Note that since robots meet at point C , if the exit is not in the arc DB , it is impossible for a robot to be caught by the other robot in subphase-3 of its detour phase. Hence, there are four cases as to where R_2 can be caught by R_1 that found the exit. As before, we omit the extra cost 1 which is the time needed for the deployment phase during the case distinction.

Case 1: R_2 is caught in its pre-detour phase: The cost of Algorithm $\mathcal{C}(\chi, \phi, \lambda)$ is at most $\chi + f(\chi)$, exactly as in the analogous case of Algorithm $\mathcal{B}(\chi, \phi)$.

Case 2: R_2 is caught in its detour subphase-1: As in case 2 of the analysis of Algorithm $\mathcal{B}(\chi, \phi)$, if E is the position of the exit, then $y = \widehat{EA}$ satisfies $y \geq \chi - f(\chi)$ (the relevant figure in this case remains the left-hand side of Figure 4). As long as $q(y)$ remains less than λ , the cost of the algorithm remains $y + 2h(y)$. In order to find the maximum y for which this formula remains valid, we need to solve the equation $\lambda = q(y)$. This is possible by recalling that $h(y) = \chi + q(y) - y$, and by invoking the formula of $h(y)$ as it appears in Lemma 2.4. By the monotonicity of $h(y)$, we have that there exists a unique ψ satisfying $h(\psi) = \chi + \lambda - \psi$. It follows that the cost of the algorithm in this case is at most $\max_{\chi - f(\chi) \leq y \leq \psi} \{y + 2h(y)\}$.

Case 3: R_2 is caught in its detour subphase-2: In this case, the relevant figure is the right-hand side of Figure 5. Let the exit be at point E , and let K denote the meeting point of the robots on the line segment GC . We set $h'(y) := \widehat{EK}$, whose value is determined by Lemma 2.6 (a). We conclude that in this case the cost of the algorithm is at most $\max_{\psi \leq y \leq \chi} \{y + 2h'(y)\}$.

Case 4: R_2 is caught in its post-detour phase: Let t_d again be the total time a robot needs until it enters its post-detour phase. As in case 3 of Algorithm $\mathcal{B}(\chi, \phi)$, the cost of Algorithm $\mathcal{C}(\chi, \phi, \lambda)$ for this case is at most $t_d + 2p(\chi)$. It thus remains to show how to calculate $t_d = \widehat{AB} + \widehat{BG} + \widehat{GC} + \widehat{CB}$, which is done in Lemma 2.6 (b).

Lemma 2.6. *The following statements hold for Algorithm $\mathcal{C}(\chi, \phi, \lambda)$:*

(a) *Suppose that R_1 finds the exit and meets R_2 in its detour subphase-2. Then the time $h'(y)$ that R_1 needs from finding the exit until meeting R_2 is $h'(y) = N(\chi, y, \lambda, \phi) / D(\chi, y, \lambda, \phi)$, where*

$$\begin{aligned} N(\chi, y, \lambda, \phi) := & 2 + \lambda^2 + (\lambda + \chi - y)^2 + 2\lambda(\sin(\phi - \chi) - \sin(\phi + y)) \\ & + 2(\lambda + \chi - y)(\sin(\chi) + \sin(y) - \lambda \cos(\phi)) - 2\cos(\chi + y), \quad \text{and} \\ D(\chi, y, \lambda, \phi) := & 2(\lambda + \chi - y + \sin(\chi) + \sin(y) - \lambda \cos(\phi)). \end{aligned}$$

(b) *Suppose that R_1 finds the exit and meets R_2 in its post-detour phase. Then the total time that R_2 spends in its detour phase is*

$$\lambda + \sin(\chi) - \lambda \cos(\phi) + \sqrt{\sin^2(\chi) + \lambda^2 \sin^2(\phi)}.$$

Proof: As an illustration of the proof we refer to Figure 6, which is a continuation of Figure 5. Let H be the symmetric point of E with respect to diameter AA' . As in Figure 4, L is the projection of H onto the supporting line of DB , and θ denotes the angle $\angle BHL$, whose value is given by (5). Further, G' and C'

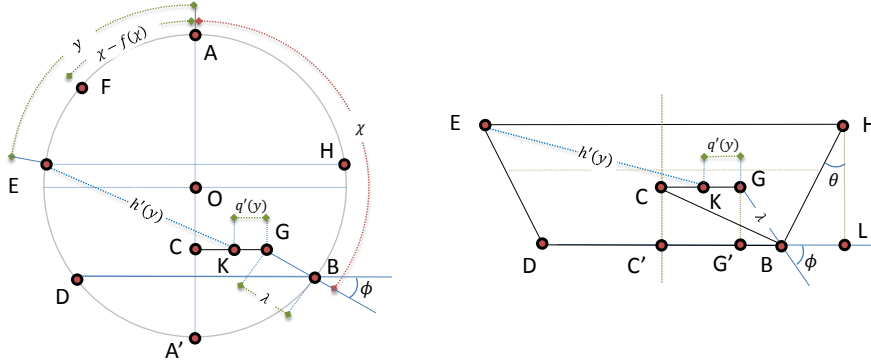


Fig. 6: The analysis of Algorithm $\mathcal{C}(\chi, \phi, \lambda)$.

are the projections of G and C , respectively, onto DL . The calculations below follow the spirit of the arguments in Lemma 2.4.

(a) As before, y denotes the distance of the exit from point A . We have that $\overrightarrow{EK} = \overrightarrow{EH} + \overrightarrow{HB} + \overrightarrow{BG} + \overrightarrow{GC}$, and therefore

$$\begin{aligned} \overline{EK}^2 &= \overline{EH}^2 + \overline{HB}^2 + \overline{BG}^2 + \overline{GC}^2 + \\ &2\left(\overline{EH} \cdot \overline{HB} + \overline{EH} \cdot \overline{BG} + \overline{EH} \cdot \overline{GC} + \overline{HB} \cdot \overline{BG} + \overline{HB} \cdot \overline{GC} + \overline{BG} \cdot \overline{GC}\right), \end{aligned} \quad (8)$$

where $\overline{EK} = h'(y)$, $\overline{EH} = 2 \sin(y)$, $\overline{HB} = 2 \sin\left(\frac{\chi - y}{2}\right)$, $\overline{BG} = \lambda$, and $\overline{GC} = q'(y)$. The inner products $\overline{EH} \cdot \overline{HB}$, $\overline{EH} \cdot \overline{BG}$, $\overline{HB} \cdot \overline{BG}$ are obtained exactly as in Lemma 2.4. For the remaining inner products we see that

$$\begin{aligned} \overline{EH} \cdot \overline{GC} &= \overline{EH} \overline{GC} \cos(\pi) = -\overline{EH} \overline{GC}, \\ \overline{HB} \cdot \overline{GC} &= \overline{HB} \overline{GC} \cos\left(\frac{\pi}{2} - \theta\right) = \overline{HB} \overline{GC} \sin(\theta), \quad \text{and} \\ \overline{BG} \cdot \overline{GC} &= \overline{BG} \overline{GC} \cos(\phi). \end{aligned}$$

Substituting the above in (8), we obtain an equation for $h'(y)$ as a function of $q'(y)$, y , χ , and ϕ . In that equation we can substitute $q'(y)$ using the meeting condition (7), according to which $q'(y) = y + h'(y) - \chi - \lambda$. Resolving the resulting equation for $h'(y)$ then gives the desired formula.

(b) We need to determine $\overline{BG} + \overline{GC} + \overline{CB}$, where $\overline{BG} = \lambda$. First, we observe that $\overline{GC} = \overline{G'C'} = \overline{BC'} - \overline{BG'} = \sin(\chi) - \lambda \cos(\phi)$. In order to calculate \overline{CB} , we see that $\overrightarrow{BC} = \overrightarrow{BG} + \overrightarrow{GC}$. Hence we obtain

$$\begin{aligned} \overline{BC}^2 &= \overline{BG}^2 + \overline{GC}^2 + 2\overline{BG} \cdot \overline{GC} \\ &= \lambda^2 + (\sin(\chi) - \lambda \cos(\phi))^2 + 2\lambda(\sin(\chi) - \lambda \cos(\phi)) \cos(\phi) \\ &= \sin^2(\chi) + \lambda^2 \sin^2(\phi), \end{aligned}$$

which concludes our claim. \blacksquare

Before stating our main theorem, we summarize the total time required by Algorithm $\mathcal{C}(\chi, \phi, \lambda)$ in the following lemma.

Lemma 2.7. *The cost of Algorithm $\mathcal{C}(\chi, \phi, \lambda)$ can be expressed as*

$$1 + \max \left\{ \begin{array}{ll} \chi + f(\chi) & (\text{pre-detour phase}) \\ \max_{\chi - f(\chi) \leq y \leq \psi} \{y + 2h(y)\} & (\text{detour subphase-1}) \\ \max_{\psi \leq y \leq \chi} \{y + 2h'(y)\} & (\text{detour subphase-2}) \\ \chi + \lambda + \sin(\chi) - \lambda \cos(\phi) + \sqrt{\sin^2(\chi) + \lambda^2 \sin^2(\phi) + 2p(\chi)} & (\text{post-detour phase}) \end{array} \right\},$$

where the functions f and p are as in (1) and (3), respectively; functions $h(y)$ and $h'(y)$ are expressed explicitly in Lemmas 2.4 and 2.6 (a), respectively; and ψ is the unique solution to the equation $h(\psi) = \chi + \lambda - \psi$.

In Lemma 2.7, the phases in parentheses in the maximization expression are the ones in which R_1 meets R_2 after having found the exit. Using the statement of Lemma 2.7 and numerical optimization, we obtain the following improved upper bound.

Theorem 2.8. *For $\chi_0 = 2.631865$, $\phi_0 = 0.44916$ and $\lambda_0 = 0.05762$, the evacuation algorithm $\mathcal{C}(\chi_0, \phi_0, \lambda_0)$ has cost no more than 5.628.*

Proof: We examine the cost of our algorithm depending on where the meeting point of the two robots occurs. The guidelines of the analysis are suggested by Lemma 2.7. Also the deployment cost of 1 will be added at the end. Any calculations below are numerical, and were performed using MATHEMATICA.

For the given parameters, we see that $f(\chi_0) = 1.99603$, $p(\chi_0) = 0.506932$, $\chi_0 - f(\chi_0) = 0.63584$, and $\psi = 0.755204$. If the meeting point is during the pre-detour phase, then the cost is $\chi_0 + f(\chi_0) < 4.62791$ (note that $\chi_0 < x_0$). If the meeting point is in the post-detour phase, then the cost is $\chi_0 + \lambda_0 + \sin(\chi_0) - \lambda_0 \cos(\phi_0) + \sqrt{\sin^2(\chi_0) + \lambda_0^2 \sin^2(\phi_0) + 2p(\chi_0)} < 4.627965$.

For the more interesting intermediate cases, we have

$$h(y) = \frac{-0.5y^2 + 3.45042y + a(y) + b(y) - 6.61768}{y - 0.900812 \sin(y) - 0.434209 \cos(y) - 3.45042}$$

$$h'(y) = \frac{-0.5y^2 + 3.12552y + (y - 3.12552) \sin(y) - 0.847858 \cos(y) - 5.74387}{y - \sin(y) - 3.12552},$$

where $a(y) := (0.900812y - 2.85875) \sin(y)$ and $b(y) := (0.434209y - 2.01566) \cos(y)$. Hence the cost in the detour subphase-2 is bounded from above by

$$\max_{0.63584 \leq y \leq 0.755204} \{y + 2h(y)\} < 4.627972$$

and the cost in the detour subphase-3 is bounded from above by

$$\max_{0.755204 \leq y \leq 2.631865} \{y + 2h'(y)\} < 4.627961.$$

This completes the proof of the theorem. \blacksquare

3 Lower Bound

In this section we show that any evacuation algorithm for two robots in the face-to-face model takes time at least $3 + \frac{\pi}{6} + \sqrt{3} \approx 5.255$. We first prove a result of independent interest about an evacuation problem on a hexagon.

Theorem 3.1. *Consider a hexagon of radius 1 with an exit placed at an unknown vertex. Then the worst case evacuation time for two robots starting at any two arbitrary vertices of the hexagon is at least $2 + \sqrt{3}$.*

Proof: Assume an arbitrary deterministic algorithm \mathcal{D} for the problem. \mathcal{D} solves the problem for any input, i.e., any placement of the exit. We construct two inputs for \mathcal{D} and show that for at least one of them, the required evacuation time is at least $2 + \sqrt{3}$. First, we let \mathcal{D} run without placing an exit at any vertex, so as to find out in which order the robots are exploring all the vertices of the hexagon. We label the vertices of the hexagon according to this order (if two vertices are explored simultaneously then we just order them arbitrarily). Let t be the time when the fifth vertex, v_5 , of the hexagon is visited by some robot, say R_1 , i.e., robot R_1 is at vertex v_5 at time t , and four more vertices of the hexagon have already been visited. In other words, v_5 and v_6 are the only vertices^(iv) that are guaranteed to not have been explored at time $t - \varepsilon$, for any sufficiently small $\varepsilon > 0$. Note that we must have $t \geq 2$, since at least one of the two robots must have visited at least three vertices by time t (and hence must have walked at least the two segments between the first and the second, and between the second and the third vertex visited in its trajectory).

The first input I_1 we construct has the exit placed at vertex v_6 . Until robot R_1 reaches v_5 , the algorithm \mathcal{D} processes I_1 identically to the case in which there is no exit; further, at time t , robot R_1 needs additional time at least 1 just to reach the exit. If $t \geq 1 + \sqrt{3}$, then this input gives an evacuation time of at least $2 + \sqrt{3}$.

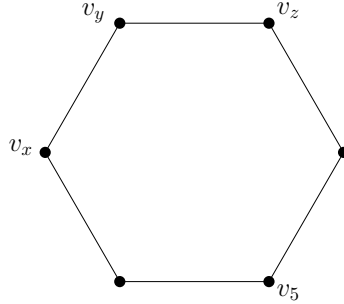


Fig. 7: Vertices of the hexagon as visited by algorithm \mathcal{D} ; t is the time when the fifth vertex v_5 is visited by some robot, say R_1 . One of the vertices adjacent to v_5 has not been visited yet by a robot (here $v_6 \notin \{v_x, v_y, v_z\}$).

Hence assume that $2 \leq t < 1 + \sqrt{3}$. Let v_x , v_y , and v_z be the three vertices that are non-adjacent to v_5 in the hexagon (see Figure 7). Note that the minimum distance between v_5 and any of v_x , v_y , and v_z is at least $\sqrt{3}$. If $v_6 \in \{v_x, v_y, v_z\}$ then on input I_1 , \mathcal{D} needs evacuation time at least $t + \sqrt{3} \geq 2 + \sqrt{3}$, as R_1 still has to reach the exit.

^(iv) It might be that v_4 and v_5 are explored simultaneously, or that v_5 and v_6 are explored simultaneously. In the former case v_6 is explored strictly after v_5 while in the latter v_4 is explored strictly before v_5 .

Therefore, assume that $v_6 \notin \{v_x, v_y, v_z\}$ and hence $\{v_x, v_y, v_z\} \subset \{v_1, v_2, v_3, v_4\}$. Note that, since $t < 1 + \sqrt{3}$, on input I_1 , robot R_1 has visited at most one of v_x, v_y , and v_z at time t . Hence, the other robot, R_2 has visited at least two of them. For the second input I_2 that we construct, we place the exit on the vertex v^* that is the last vertex among v_x, v_y , and v_z in the visiting order of the vertices by R_2 . Let t^* be the time when R_2 reaches v^* , and note that at least until t^* , the algorithm \mathcal{D} behaves identical on the two inputs I_1 and I_2 . As R_2 has visited at least one vertex before visiting v^* , we have $t^* \geq 1$.

Next we claim that R_1 and R_2 cannot meet between time t^* and t . The claim below states that this is impossible. Namely, we can prove:

Claim 3.2. *Let t, t^* , and I_2 be as defined in the preceding paragraphs. If $t < 1 + \sqrt{3}$, then on input I_2 , R_1 and R_2 do not meet between time t^* and time t .*

Proof: (Claim 3.2) Assume on the contrary, that on input I_2 , R_1 and R_2 do meet at some time t' at point P , with $t^* \leq t' < t$. Observe that on input I_2 , robot R_1 continues until time t' as on input I_1 but having met R_2 at time t' might continue differently after time t' . Let $t_B = t' - t^*$ be the time that R_2 uses on input I_2 to get from the exit v^* to P , and let $t_A = t - t'$ be the time that R_1 uses on input I_1 to get to vertex v_5 from P . As v^* and v_5 are at distance at least $\sqrt{3}$, and since $t^* \geq 1$, we have $\sqrt{3} \leq t_A + t_B = t - t' + t' - t^* = t - t^* \leq t - 1$. So we obtain $\sqrt{3} + 1 \leq t$, which contradicts the assumption that $t < 1 + \sqrt{3}$. This proves the claim. ■

Having proved the claim, we conclude that on input I_2 , R_1 continues until time t as on input I_1 . Hence R_1 needs at least $t + \sqrt{3} \geq 2 + \sqrt{3}$ time to reach the exit on input I_2 , which completes the proof of the theorem. ■

It is worth noting that the lower bound from Theorem 3.1 matches the upper bound of evacuating a regular hexagon, when the initial starting vertices may be chosen by the algorithm. Consider a hexagon $ABCDEF$ and suppose that the trajectory of one robot, as long as no exit was found, is $ABDC$. Similarly, the other robot follows the symmetric trajectory $FECD$; cf. left-hand side of Fig. 8. By symmetry it is sufficient to consider exits at vertices A, B , or C . An exit at C is reached by each robot independently, while both robots proceed to an exit at A or B after meeting at point M , the intersection of the segments BD and EC . Altogether, they need a total time of at most $\max\{1 + 4/\sqrt{3}, 1 + (2 + \sqrt{7})/\sqrt{3}, 1 + \sqrt{3} + 1\}$ to evacuate from the hexagon. An interested reader may verify that, in each case, the evacuation time of this algorithm is always upper bounded by $2 + \sqrt{3}$.

In the above algorithm, the robots meet at M , regardless of whether the exit has been already found or not. The idea of our algorithm for disk evacuation presented in the previous section was influenced by this non-intuitive presence of a forced meeting.

Combining Theorem 3.1 with the fact that hexagon edges correspond to arcs of length $\pi/6$, we obtain the following lower bound for our evacuation problem.

Theorem 3.3. *Assume you have a unit disk with an exit placed somewhere on the boundary. The worst case evacuation time for two robots starting at the centre in the face-to-face model is at least $3 + \frac{\pi}{6} + \sqrt{3} \approx 5.255$.*

Proof: It takes 1 time unit for the robots to reach the boundary of the disk. By time $t = 1 + \frac{\pi}{6}$, any algorithm could have explored at most $\frac{2\pi}{6}$ of the boundary of the disk. Hence for any ε with $0 < \varepsilon < t$, there exists a regular hexagon with all vertices on the boundary of the disk and all of whose vertices

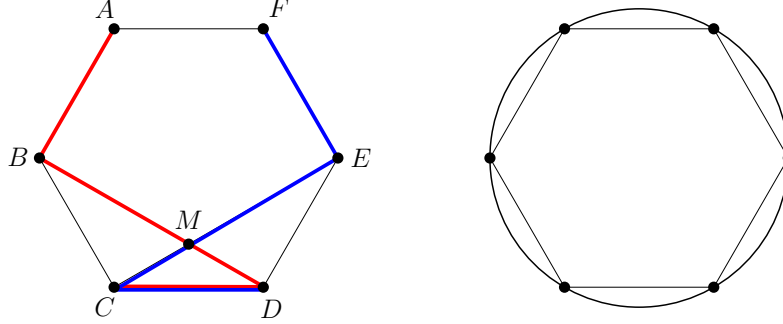


Fig. 8: The trajectories for R_1 (red) and R_2 (blue) for the hexagon evacuation algorithm having evacuation time $2 + \sqrt{3}$, while the exit has not been found, are depicted on the left. Right-hand side: At time $1 + \frac{\pi}{6} - \varepsilon$, there is regular hexagon all of whose vertices are unexplored and lie on the boundary of the disk.

are unexplored at time $t - \varepsilon$; see the right-hand side of Figure 8. As the exit might as well be on any vertex of this hexagon, invoking Theorem 3.1 implies a lower bound of $1 + \frac{\pi}{6} + 2 + \sqrt{3}$ for the worst case of evacuating both robots. ■

Our lower bound technique could be extended to $n \geq 7$, and it might improve the lower bound value. However, the case analysis becomes extensive, very complicated, and it would not provide any more insight into the lower bound question.

4 Conclusion

In this paper we studied evacuating two robots from a disk, where the robots can collaborate using face-to-face communication. Unlike evacuation for two robots in the wireless communication model, for which the tight bound $1 + \frac{2\pi}{3} + \sqrt{3}$ is proved in Czyzowicz et al. (2014), the evacuation problem for two robots in the face-to-face model is much harder to solve. We gave a new non-trivial algorithm for the face-to-face communication model, by this improving the upper bound of 5.740 in Czyzowicz et al. (2014) to 5.628 (the upper bound has since been improved to 5.625 in Brandt et al. (2017) and to 5.6234 in Disser and Schmitt (2019)). We used a novel, non-intuitive idea of a forced meeting between the robots, regardless of whether the exit was found before the meeting (although Brandt et al. (2017); Disser and Schmitt (2019) have since demonstrated that by avoiding a forced meeting one can improve the upper bound). We also provided a different analysis that improves the lower bound in Czyzowicz et al. (2014). Further tightening of the upper and lower bounds remains a challenging open question. Along the same lines, results pertaining to the generalization of our auxiliary problem of evacuating from an n -gon is another interesting open problem.

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References

- R. Ahlswede and I. Wegener. *Search problems*. Wiley-Interscience, 1987.
- S. Alpern and A. Beck. Asymmetric rendezvous on the line is a double linear search problem. *Mathematics of Operations Research*, 24(3):604–618, 1999.
- S. Alpern and S. Gal. *The theory of search games and rendezvous*, volume 55. Springer, 2003.
- S. Angelopoulos, C. Dürr, and T. Lidbetter. The expanding search ratio of a graph. *Discrete Applied Mathematics*, 260:51–65, 2019.
- R. Baeza-Yates, J. Culberson, and G. Rawlins. Searching in the plane. *Inf. Comput.*, 106(2):234–252, Oct. 1993.
- R. Baeza Yates, J. Culberson, and G. Rawlins. Searching in the plane. *Information and Computation*, 106(2):234–252, 1993.
- A. Beck. On the linear search problem. *Israel Journal of Mathematics*, 2(4):221–228, 1964.
- A. Beck. More on the linear search problem. *Israel Journal of Mathematics*, 3(2):61–70, 1965.
- A. Beck and M. Beck. Son of the linear search problem. *Israel Journal of Mathematics*, 48(2-3):109–122, 1984.
- A. Beck and D. Newman. Yet more on the linear search problem. *Israel Journal of Mathematics*, 8(4): 419–429, 1970.
- A. Beck and P. Warren. The return of the linear search problem. *Israel Journal of Mathematics*, 14(2): 169–183, 1973.
- R. Bellman. An optimal search. *Siam Review*, 5(3):274–274, 1963.
- S. Benkoski, M. Monticino, and J. Weisinger. A survey of the search theory literature. *Naval Research Logistics (NRL)*, 38(4):469–494, 1991.
- A. Bonato and R. Nowakowski. *The game of cops and robbers on graphs*. American Mathematical Soc., 2011.
- A. Bonato, K. Georgiou, C. MacRury, and P. Pralat. Probabilistically faulty searching on a half-line. In *14th Latin American Theoretical Informatics Symposium, 25 – 29 May 2020, University of Sao Paulo, Sao Paulo, Brazil, (to appear)*.
- P. Borowiecki, S. D. 0001, D. Dereniowski, and L. Kuszner. Distributed evacuation in graphs with multiple exits. In J. Suomela, editor, *Structural Information and Communication Complexity - 23rd International Colloquium, SIROCCO 2016, Helsinki, Finland, July 19-21, 2016, Revised Selected Papers*, volume 9988 of *Lecture Notes in Computer Science*, pages 228–241, 2016. ISBN 978-3-319-48313-9.
- S. Brandt, F. Laufenberg, Y. Lv, D. Stolz, and R. Wattenhofer. Collaboration without communication: Evacuating two robots from a disk. In *Algorithms and Complexity - 10th International Conference, CIAC 2017, Athens, Greece, May 24-26, 2017. Proceedings*, pages 104–115, 2017.

- S. Brandt, K.-T. Foerster, B. Richner, and R. Wattenhofer. Wireless evacuation on m rays with k searchers. *Theoretical Computer Science*, 811:56–69, 2020.
- H. Chuangpishit, S. Mehrabi, L. Narayanan, and J. Opatrny. Evacuating an equilateral triangle in the face-to-face model. In J. Aspnes, A. Bessani, P. Felber, and J. Leitão, editors, *The 21st International Conference on Principles of Distributed Systems (OPODIS), 18-20 December 2017, Lisboa, Portugal*, volume 95 of *LIPICs*, pages 11:1–11:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017.
- H. Chuangpishit, K. Georgiou, and P. Sharma. Average case-worst case tradeoffs for evacuating 2 robots from the disk in the face-to-face model. In *International Symposium on Algorithms and Experiments for Sensor Systems, Wireless Networks and Distributed Robotics*, pages 62–82. Springer, 2018.
- J. Czyzowicz, L. Gasieniec, T. Gorry, E. Kranakis, R. Martin, and D. Pajak. Evacuating robots from an unknown exit located on the perimeter of a disc. In *Distributed Computing - 28th International Symposium, DISC 2014*, pages 122–136. Springer, Austin, Texas, 2014.
- J. Czyzowicz, K. Georgiou, E. Kranakis, L. Narayanan, J. Opatrny, and B. Vogtenhuber. Evacuating robots from a disk using face-to-face communication (extended abstract). In V. T. Paschos and P. Widmayer, editors, *Algorithms and Complexity - 9th International Conference, CIAC 2015, Paris, France, May 20-22, 2015. Proceedings*, volume 9079 of *Lecture Notes in Computer Science*, pages 140–152. Springer, 2015a. ISBN 978-3-319-18172-1.
- J. Czyzowicz, E. Kranakis, D. Krizanc, L. Narayanan, J. Opatrny, and S. M. Shende. Wireless autonomous robot evacuation from equilateral triangles and squares. In S. Papavassiliou and S. Ruehrup, editors, *Ad-hoc, Mobile, and Wireless Networks - 14th International Conference, ADHOC-NOW 2015, Athens, Greece, June 29 - July 1, 2015, Proceedings*, volume 9143 of *Lecture Notes in Computer Science*, pages 181–194. Springer, 2015b.
- J. Czyzowicz, K. Georgiou, M. Godon, E. Kranakis, D. Krizanc, W. Rytter, and M. Włodarczyk. Evacuation from a disc in the presence of a faulty robot. In *International Colloquium on Structural Information and Communication Complexity*, pages 158–173. Springer, 2017a.
- J. Czyzowicz, E. Kranakis, D. Krizanc, L. Narayanan, J. Opatrny, and S. M. Shende. Linear search with terrain-dependent speeds. In D. Fotakis, A. Pagourtzis, and V. T. Paschos, editors, *Algorithms and Complexity - 10th International Conference, CIAC 2017, Athens, Greece, May 24-26, 2017, Proceedings*, volume 10236 of *Lecture Notes in Computer Science*, pages 430–441, 2017b. ISBN 978-3-319-57585-8; 978-3-319-57586-5.
- J. Czyzowicz, S. Dobrev, K. Georgiou, E. Kranakis, and F. MacQuarrie. Evacuating two robots from multiple unknown exits in a circle. *Theoretical Computer Science*, 709:20–30, 2018a.
- J. Czyzowicz, K. Georgiou, R. Killick, E. Kranakis, D. Krizanc, L. Narayanan, J. Opatrny, and S. Shende. Priority evacuation from a disk using mobile robots. In Z. Lotker and B. Patt-Shamir, editors, *Structural Information and Communication Complexity - 25th International Colloquium, SIROCCO 2018*, volume 11085 of *Lecture Notes in Computer Science*, pages 392–407. Springer, 2018b.

- J. Czyzowicz, K. Georgiou, R. Killick, E. Kranakis, D. Krizanc, L. Narayanan, J. Opatrny, and S. M. Shende. God save the queen. In H. Ito, S. Leonardi, L. Pagli, and G. Prencipe, editors, *9th International Conference on Fun with Algorithms, FUN 2018, June 13-15, 2018, La Maddalena, Italy*, volume 100 of *LIPICs*, pages 16:1–16:20. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2018c. ISBN 978-3-95977-067-5.
- J. Czyzowicz, K. Georgiou, R. Killick, E. Kranakis, D. Krizanc, M. Lafond, L. Narayanan, J. Opatrny, and S. Shende. Energy Consumption of Group Search on a Line. In C. Baier, I. Chatzigiannakis, P. Flocchini, and S. Leonardi, editors, *46th International Colloquium on Automata, Languages, and Programming (ICALP 2019)*, volume 132 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 137:1–137:15, Dagstuhl, Germany, 2019a. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. ISBN 978-3-95977-109-2. doi: 10.4230/LIPIcs.ICALP2019.137. URL <http://drops.dagstuhl.de/opus/volltexte/2019/10713>.
- J. Czyzowicz, K. Georgiou, and E. Kranakis. Group search and evacuation. In P. Flocchini, G. Prencipe, and N. Santoro, editors, *Distributed Computing by Mobile Entities; Current Research in Moving and Computing*, chapter 14, pages 335–370. Springer, 2019b.
- E. Demaine, S. Fekete, and S. Gal. Online searching with turn cost. *Theoretical Computer Science*, 361(2):342–355, 2006.
- Y. Disser and S. Schmitt. Evacuating two robots from a disk: a second cut. In K. Censor-Hillel and M. Flammini, editors, *International Colloquium on Structural Information and Communication Complexity*, volume 11639 of *Lecture Notes in Computer Science*, pages 200–214. Springer, 2019.
- J. Dobbie. A survey of search theory. *Operations Research*, 16(3):525–537, 1968.
- Y. Emek, T. Langner, J. Uitto, and R. Wattenhofer. Ants: Mobile finite state machines. *CoRR*, abs/1311.3062, 2013.
- Y. Emek, T. Langner, J. Uitto, and R. Wattenhofer. Solving the ants problem with asynchronous finite state machines. In J. Esparza, P. Fraigniaud, T. Husfeldt, and E. Koutsoupias, editors, *Automata, Languages, and Programming - 41st International Colloquium, ICALP 2014, Copenhagen, Denmark, July 8-11, 2014, Proceedings, Part II*, volume 8573 of *Lecture Notes in Computer Science*, pages 471–482. Springer, 2014. ISBN 978-3-662-43950-0.
- O. Feinerman, A. Korman, A. Lotker, and J.-S. Sereni. Collaborative search on the plane without communication. In *Proceedings of PODC*, pages 77–86, 2012.
- P. Flocchini, G. Prencipe, N. Santoro, and P. Widmayer. Gathering of asynchronous robots with limited visibility. *Theoretical Computer Science*, 337(1):147 – 168, 2005. ISSN 0304-3975. doi: <http://dx.doi.org/10.1016/j.tcs.2005.01.001>. URL <http://www.sciencedirect.com/science/article/pii/S0304397505000149>.
- P. Flocchini, G. Prencipe, and N. Santoro. *Distributed Computing by Mobile Entities; Current Research in Moving and Computing*, volume 11340. Springer, 2019.

- K. Georgiou, G. Karakostas, and E. Kranakis. Search-and-fetch with one robot on a disk. In M. Chrobak, A. F. Anta, L. Gasieniec, and R. Klasing, editors, *Algorithms for Sensor Systems - 12th International Symposium on Algorithms and Experiments for Wireless Sensor Networks, ALGOSENSORS 2016*, volume 10050 of *Lecture Notes in Computer Science*, pages 80–94. Springer, 2016.
- K. Georgiou, G. Karakostas, and E. Kranakis. Search-and-fetch with 2 robots on a disk: Wireless and face-to-face communication models. *Discrete Mathematics & Theoretical Computer Science*, Vol. 21 no. 3, June 2019a. URL <https://dmtcs.episciences.org/5528>.
- K. Georgiou, E. Kranakis, N. Leonardos, A. Pagourtzis, and I. Papaioannou. Optimal cycle search despite the presence of faulty robots. In F. Dressler and C. Scheideler, editors, *Algorithms for Sensor Systems - 15th International Symposium on Algorithms and Experiments for Wireless Sensor Networks, ALGOSENSORS 2019, Munich, Germany, September 12-13, 2019, Revised Selected Papers*, volume 11931 of *Lecture Notes in Computer Science*, pages 192–205. Springer, 2019b.
- E. J. Kranakis, D. K. Krizanc, M. L. Georgiou, R. Killick, L. Narayanan, J. Opatrny, and S. Shende. Time-energy tradeoffs for evacuation by two robots in the wireless model. In K. Censor-Hillel and M. Flammini, editors, *Structural Information and Communication Complexity - 26th International Colloquium, SIROCCO 2019, L'Aquila, Italy, July 1-4, 2019, Proceedings*, volume 11639 of *Lecture Notes in Computer Science*, pages 185–199. Springer, 2019. ISBN 978-3-030-24921-2; 978-3-030-24922-9.
- I. Lamprou, R. Martin, and S. Schewe. Fast two-robot disk evacuation with wireless communication. In C. Gavoille and D. Ilcinkas, editors, *Distributed Computing - 30th International Symposium, DISC 2016, Paris, France, September 27-29, 2016. Proceedings*, volume 9888 of *Lecture Notes in Computer Science*, pages 1–15. Springer, 2016.
- C. Lenzen, N. Lynch, C. Newport, and T. Radeva. Trade-offs between selection complexity and performance when searching the plane without communication. In M. M. Halldórsson and S. Dolev, editors, *ACM Symposium on Principles of Distributed Computing, PODC '14, Paris, France, July 15-18, 2014*, pages 252–261. ACM, 2014.
- A. López-Ortiz and G. Sweet. Parallel searching on a lattice. In *Proceedings of the 13th Canadian Conference on Computational Geometry (CCCG'01)*, pages 125–128, 2001.
- P. Nahin. *Chases and Escapes: The Mathematics of Pursuit and Evasion*. Princeton University Press, 2012.
- D. Pattanayak, H. Ramesh, P. S. Mandal, and S. Schmid. Evacuating two robots from two unknown exits on the perimeter of a disk with wireless communication. In P. Bellavista and V. K. Garg, editors, *Proceedings of the 19th International Conference on Distributed Computing and Networking, ICDCN 2018, Varanasi, India, January 4-7, 2018*, pages 20:1–20:4. ACM, 2018.
- D. Pattanayak, H. Ramesh, and P. S. Mandal. Chauffeur-ing a crashed robot from a disk. In F. Dressler and C. Scheideler, editors, *Algorithms for Sensor Systems - 15th International Symposium on Algorithms and Experiments for Wireless Sensor Networks, ALGOSENSORS 2019, Munich, Germany, September 12-13, 2019, Revised Selected Papers*, volume 11931 of *Lecture Notes in Computer Science*, pages 177–191. Springer, 2019.

L. Stone. *Theory of optimal search*. Academic Press New York, 1975.