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# *A Note on Graphs of Dichromatic Number 2*

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Neumann-Lara and Škrekovski conjectured that every planar digraph is 2-colourable. We show that this conjecture is equivalent to the more general statement that all oriented $K_5$-minor-free graphs are 2-colourable.

**Keywords:** directed graphs, acyclic colouring, dichromatic number, planar graphs, $K_5$-minor-free graphs

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## 1 Introduction

Digraphs and graphs considered here are loopless, and without parallel or anti-parallel arcs. A directed edge starting in $u$ and ending in $v$ is denoted by $(u, v)$, $u$ is called its *tail* while $v$ is its *head*. In a digraph $D$, a vertex set $X \subseteq V(D)$ is called *acyclic* if the induced subdigraph $D[X]$ is acyclic. An *acyclic colouring* of $D$ with $k$ colours is a mapping $c : V(D) \to [k]$ such hat $c^{-1}(\{i\})$ is acyclic for all $i \in [k]$. The *dichromatic number* $\vec{\chi}(D)$ is defined as the minimal $k \geq 1$ for which such a colouring exists. For an undirected graph $G$, the dichromatic number $\vec{\chi}(G)$ is defined as the maximum dichromatic number an orientation of $G$ can have.

This notion has been introduced in 1982 by Neumann-Lara (1982), was rediscovered by Mohar (2003), and since then has received further attention, see Andres and Hochstättler (2015); Mohar and Wu (2016); Aboulker et al. (2019); Harutyunyan and Mohar (2017); Li and Mohar (2017); Bensmail et al. (2018); Harutyunyan et al. (2019) for some recent results.

In analogy to the famous Four-Colour-Theorem, the following intriguing conjecture was proposed by Neumann-Lara (1985) and independently by Škrekovski (see Bokal et al. (2004)).

**Conjecture 1.** *If $G$ is a planar graph, then $\vec{\chi}(G) \leq 2$.*

The strongest partial result obtained so far is due to Li and Mohar who showed the following:

**Theorem 1** (Li and Mohar (2017))**.** *Every planar digraph without directed triangles admits an acyclic 2-colouring.*

The purpose of this note is to show the following.

**Theorem 2.** *The following statements are equivalent:*

- *Every planar graph $G$ has $\vec{\chi}(G) \leq 2$.*

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- *Every $K_5$-minor-free graph $G$ fulfils $\vec{\chi}(G) \leq 2$. Moreover, any orientation of $G$ admits an acyclic $2$-colouring without monochromatic triangles.*

This strengthening is similar to the situation for undirected graph colourings, where it is known that all $K_5$-minor-free graphs are $4$-colourable (see Wagner (1937)).

# 2 $2$-Colourings of Planar Digraphs

In the following, we will use the term *planar triangulation* when we mean a maximal planar graph on at least three vertices. It is well-known that the latter (up to the choice of the outer face and reflections) admit combinatorially unique crossing-free embeddings in the plane or on the sphere, in which every face is bounded by a triangle (from now on called facial triangles). A frequent tool in our proof will be the following Lemma, which has already been used by Li and Mohar.

**Lemma 1** (Li and Mohar (2017))**.** *Let $D_1$ and $D_2$ be digraphs which intersect in a tournament. Suppose that $c_1 : V(D_1) \to [k]$, $c_2 : V(D_2) \to [k]$ are acyclic colourings such that $c_1|_{V(D_1)\cap V(D_2)} = c_2|_{V(D_1)\cap V(D_2)}$. Then the common extension of $c_1$ and $c_2$ to $V(D_1) \cup V(D_2)$ defines an acyclic $k$-colouring of $D_1 \cup D_2$.*

In this section we prepare the proof of Theorem 2 with some strengthend but equivalent formulations of Neumann-Lara's Conjecture. In the following, an oriented triangle is called *directed* if it forms a directed cycle, and otherwise *transitive*.

**Proposition 1.** *The following statements are equivalent:*

*(i) Neumann-Lara's Conjecture, i.e., every planar digraph has an acyclic $2$-colouring.*

*(ii) Every oriented planar triangulation admits an acyclic $2$-colouring without monochromatic facial triangles.*

*(iii) For any planar triangulation $T$, any facial triangle $a_1a_2a_3$ in $T$, and any non-monochromatic pre-colouring $p : \{a_1, a_2, a_3\} \to \{1, 2\}$, every orientation $\vec{T}$ of $T$ admits an acyclic $2$-colouring $c$ without monochromatic facial triangles such that $c(a_i) = p(a_i)$, $i \in \{1, 2, 3\}$.*

*(iv) For any planar triangulation $T$, any triangle $a_1a_2a_3$ in $T$, and any non-monochromatic pre-colouring $p : \{a_1, a_2, a_3\} \to \{1, 2\}$, every orientation $\vec{T}$ of $T$ admits an acyclic $2$-colouring $c$ without monochromatic triangles such that $c(a_i) = p(a_i)$, $i \in \{1, 2, 3\}$.*

***Proof:*** *$(i) \implies (ii)$: Suppose that every planar digraph is $2$-colourable and let $\vec{T}$ be an arbitrary orientation of a planar triangulation $T$. Looking at the orientation $\mathcal{O}_6$ of the octahedron graph depicted in Figure 1, it is easily observed that in any acyclic $2$-colouring the triangle bounding the outer face cannot be monochromatic. Now consider a crossing-free spherical embedding of $\vec{T}$. For every facial triangle in this embedding whose orientation is transitive, we take a copy of $\mathcal{O}_6$ and glue this copy into the face in such a way that the outer three edges of $\mathcal{O}_6$ are identified with the three edges of the facial triangle (to make the orientations of the identified edges compatible, it might be necessary to reflect and rotate the embedding of $\mathcal{O}_6$ shown in Figure 1). This creates a crossing-free embedding of a planar oriented triangulation $\vec{T}^{\Delta}$. By assumption, $\vec{T}^{\Delta}$ admits an acyclic $2$-colouring. This colouring restricted to the vertices of the subdigraph $\vec{T}$ clearly is still valid. Furthermore, no triangle in $\vec{T}$ can be monochromatic:*

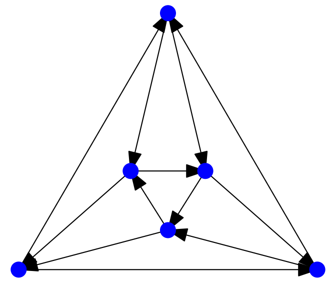

**Fig. 1:** The octahedron-orientation $\mathcal{O}_6$.

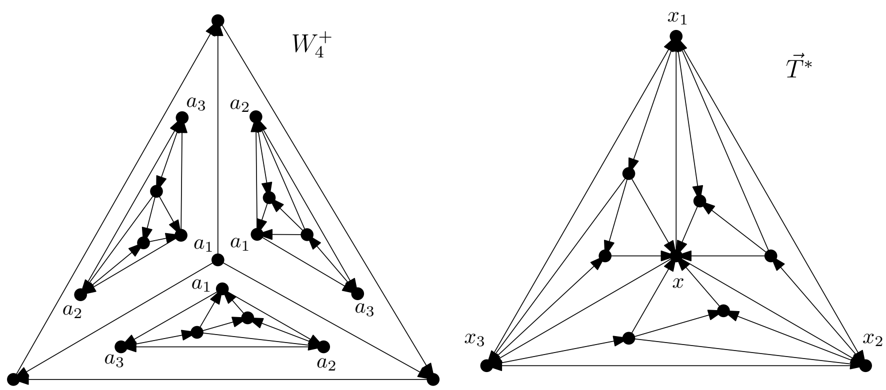


**Fig. 2:** Left: Glueing three copies of $\vec{T}$ into the faces of $W_4^+$. Right: The arising directed planar triangulation $\vec{T}^*$.

*This follows by definition if the triangle forms a directed cycle. If the orientation is transitive, by definition of $\vec{T}^\Delta$, a monochromatic colouring would contradict the fact that $\mathcal{O}_6$ has no acyclic 2-colouring with the outer three vertices being coloured the same.*

$(ii) \implies (iii)$***:*** *Suppose that $(ii)$ holds. Let $\vec{T}$ be an orientation of a planar triangulation $T$ and let $a_1a_2a_3$ be the vertices of a facial triangle $t$ of $T$, equipped with a non-monochromatic pre-colouring $p : \{a_1, a_2, a_3\} \to \{1, 2\}$.*

***Case 1:** $t$ **is not directed in** $\vec{T}$.* *Let $\overleftarrow{T}$ denote the directed triangulation obtained from $\vec{T}$ by reversing the orientations of all arcs. It is apparent that the acyclic 2-colorings of $\vec{T}$ and $\overleftarrow{T}$ coincide, hence (possibly by relabelling, moving from $\vec{T}$ to $\overleftarrow{T}$ or exchanging colors 1 and 2), we may assume w.l.o.g. that we have the orientation $(a_1, a_2), (a_1, a_3), (a_2, a_3) \in E(\vec{T})$, and that $(p(a_1), p(a_2), p(a_3)) \in \{(1, 1, 2), (1, 2, 1)\}$.*

*Now consider a plane embedding of $\vec{T}$ in which $a_1a_2a_3$ forms the bounding triangle of the outer face and where $a_1, a_2, a_3$ appear in clockwise order. We now define a new oriented planar triangulation $\vec{T}^*$ as follows: We consider the embedded tournament $W_4^+$ of order 4 as shown in Figure 2, consisting of a directed triangle $x_1, (x_1, x_2), x_2, (x_2, x_3), x_3, (x_3, x_1)$ bounding the outer face and a central vertex $x$, which is a source. Into each of the three inner faces of this embedding, we can now glue a copy of $\vec{T}$ with the described embedding in such a way that all orientations of identified edges agree. The vertex*

*$a_1$ from each copy now is identified with the center $x$ of $W_4^+$. The arising oriented planar triangulation $\vec{T}^*$ according to assumption admits an acyclic 2-colouring $c^* : V(\vec{T}^*) \to \{1,2\}$ without monochromatic facial triangles. By relabelling the colours, we may assume that $c^*(x) = 1$. Because the directed triangle $x_1x_2x_3$ bounding the outer face of $\vec{T}^*$ is not monochromatic, there have to be $i \neq j \in \{1,2,3\}$ such that*[(i)] *$c^*(x_i) = 1, c^*(x_{i+1}) = 2$, while $c^*(x_j) = 2, c^*(x_{j+1}) = 1$. For every $k \in \{i,j\}$ let $\vec{T}^k$ be the copy of $\vec{T}$ which has been glued into the face $x, x_k, x_{k+1}$ of $W_4^+$, and let $c_k$ be the 2-coloring induced by $c^*$ on the copy $\vec{T}^k$. Clearly, $c_i$ and $c_j$ both define acyclic colorings of (copies of) $\vec{T}$. Furthermore, there are no monochromatic facial triangles in $\vec{T}$ with respect to $c_i, c_j$: Every bounded facial triangle of $\vec{T}$ in the corresponding copy also forms a bounded facial triangle of $\vec{T}^*$, while the outer triangles of the copies contain the arcs $(x_i, x_{i+1})$ respectively $(x_j, x_{j+1})$ and are thus not monochromatic under $c_i$ respectively $c_j$. From the way we glued the three copies of $\vec{T}$ we conclude that $c_i(a_1) = c^*(x) = 1, c_i(a_2) = c^*(x_i) = 1, c_i(a_3) = c^*(x_{i+1}) = 2$, and $c_j(a_1) = c^*(x) = 1, c_j(a_2) = c^*(x_j) = 2, c_j(a_3) = c^*(x_{j+1}) = 1$.*

*Since $(p(a_1), p(a_2), p(a_3)) \in \{(1,1,2),(1,2,1)\}$, we find that either $c_i$ or $c_j$ is an acyclic coloring of $\vec{T}$ without monochromatic facial triangles extending the pre-coloring $p$ of $a_1a_2a_3$, as desired.*

***Case 2:** $t$ **is directed in** $\vec{T}$. After relabelling (and possibly exchanging the colours 1 and 2), we may suppose that $(a_1,a_2),(a_2,a_3),(a_3,a_1) \in E(\vec{T})$ and $p(a_1) = 1, p(a_2) = 1, p(a_3) = 2$. Consider the orientation $\vec{T}_e$ obtained from $\vec{T}$ by reversing the edge $e = (a_3, a_1)$. By the first case, we know that $\vec{T}_e$ admits an acyclic 2-colouring $c_e$ without monochromatic facial triangles which extends $p$. Because the endpoints of $e$ receive different colours, it follows directly that $c_e$ also defines an acyclic 2-colouring of $\vec{T}$ with the required properties, and the claim follows also in this case.*

*$(iii) \implies (iv)$**:** We prove the statement by induction on the number of vertices. In the base case, where $\vec{T}$ is an oriented triangle, the statement clearly holds true. Now let $n \geq 4$, and assume that the statement holds for all triangulations with less than $n$ vertices. Let $\vec{T}$ be an arbitrary orientation of some planar triangulation $T$ with $n$ vertices. If $T$ is 4-connected, then the only triangles in $T$ are the facial triangles and therefore the claim follows from $(iii)$. Therefore we may suppose that $T$ is not 4-connected, i.e., there exists a* separating triangle *$x_1x_2x_3$ in $T$. Consider some plane crossing-free embedding of $T$. Here, $x_1x_2x_3$ separates the vertices in its interior ($V_{in} \subseteq V(T)$) from those in its exterior ($V_{out} \subseteq V(T)$). Let $\vec{T}_{out} := \vec{T}[V_{out} \cup \{x_1,x_2,x_3\}]$ and $\vec{T}_{in} := \vec{T}[V_{in} \cup \{x_1,x_2,x_3\}]$. Both form oriented planar triangulations on less than $n$ vertices. To prove that $\vec{T}$ satisfies the inductive claim, let $t = a_1a_2a_3$ be a given triangle in $T$ equipped with a non-monochromatic pre-colouring $p : \{a_1,a_2,a_3\} \to \{1,2\}$. We must either have $\{a_1,a_2,a_3\} \subseteq V_{out} \cup \{x_1,x_2,x_3\}$ or $\{a_1,a_2,a_3\} \subseteq V_{in} \cup \{x_1,x_2,x_3\}$. Assume that we are in the first case, the second case is completely analogous. Then, by the induction hypothesis, there exists an acyclic colouring $c_{out} : V(\vec{T}_{out}) \to \{1,2\}$ without monochromatic triangles such that $c(a_i) = p(a_i), i \in \{1,2,3\}$. The restriction of $c_{out}$ to $x_1x_2x_3$ now defines a non-monochromatic pre-colouring for $\vec{T}_{in}$, and it follows from the induction hypothesis that there exists an acyclic 2-colouring $c_2$ of $\vec{T}_{in}$ without monochromatic triangles which agrees with $c_{out}$ on $V(\vec{T}_{out}) \cap V(\vec{T}_{in}) = \{x_1,x_2,x_3\}$. By Lemma 1, the common extension of $c_{out}, c_{in}$ to $V(\vec{T})$ now defines an acyclic 2-colouring of $\vec{T}$, extending the given pre-colouring $p$ of $t$ and without monochromatic triangles. This verifies the inductive claim.*

*$(iv) \Rightarrow (i)$**:** This follows since every planar graph is a subgraph of a planar triangulation.* □

[(i)] Here summation is to be understood modulo 3, i.e., $3 + 1 := 1$

Because any edge in a planar triangulation lies on a triangle, we directly obtain the following.

**Corollary 1.** *Under the assumption of Neumann-Lara's Conjecture, every orientation of a planar graph admits an acyclic* $2$*-colouring without monochromatic triangles which can be chosen to extend any given pre-colouring of an edge or any non-monochromatic pre-colouring of a triangle.*

# 3 $K_5$-Minor-Free Graphs

Given a pair $G_1, G_2$ of undirected graphs such that $V(G_1) \cap V(G_2)$ forms a clique of size $i$ in both $G_1$ and $G_2$, and such that $|V(G_1)|, |V(G_2)| > i$, the graph $G$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$ is called the *proper* $i$*-sum* of $G_1$ and $G_2$. A graph obtained from $G$ by deleting some (possibly all or none) of the edges in $E(G_1) \cap E(G_2)$ is said to be an $i$*-sum* of $G_1$ and $G_2$. The central tool in proving Theorem 2 is the following classical result due to Wagner. By $V_8$ we denote the so-called *Wagner graph*, that is the graph obtained from $C_8$ by joining any two diagonally opposite vertices by an edge.

**Theorem 3** (Wagner (1937))**.** *A simple graph is* $K_5$*-minor-free if and only if it can be obtained from planar graphs and copies of* $V_8$ *by means of repeated* $i$*-sums with* $i \in \{0, 1, 2, 3\}$.

$V_8$ is triangle-free and admits an acyclic 2-colouring for any orientation. Moreover, it can be easily checked that such a colouring can be chosen to extend any given pre-colouring of two adjacent vertices. We are now in the position to prove Theorem 2.

**Proof of Theorem 2:** Assume that Neumann-Lara's Conjecture holds true. We have to prove that every oriented $K_5$-minor-free graph admits an acyclic 2-colouring without monochromatic triangles. In fact, we prove the following slightly stronger statement:
*Every orientation of a* $K_5$*-minor-free graph admits an acyclic* $2$*-colouring without monochromatic triangles which can be chosen to extend any given pre-colouring of an edge or any non-monochromatic pre-colouring of a triangle.*
Assume towards a contradiction that there exists a $K_5$-minor-free graph $G$ which does not satisfy this claim and choose $G$ minimal with respect to the number of vertices, and among all such graphs maximal with respect to the number of edges.

By Corollary 1, we know that the claim is fulfilled by all planar graphs and by $V_8$, and so it follows from Theorem 3 that $G$ is the $i$-sum of two $K_5$-minor-free graphs $G_1, G_2$ with fewer vertices, where $0 \leq i \leq 3$. By the minimality assumption, we therefore know that $G_1, G_2$ satisfy the claim. Clearly, every super-graph of $G$ does not satisfy the assertion as well. Therefore, by the assumed edge-maximality, $G$ must in fact be the proper $i$-sum of $G_1$ and $G_2$. Let $C = V(G_1) \cap V(G_2)$ denote the clique in the intersection of $G_1$ and $G_2$.

Consider some orientation $\vec{G}$ of $G$ for which the above claim fails, and denote by $\vec{G}_1, \vec{G}_2$ the induced orientations on the subgraphs $G_1, G_2$. Let $X \subseteq V(G)$ be the vertex-set of an edge or of a triangle in $G$ with a given pre-colouring $p : X \to \{1, 2\}$ which is not monochromatic in case that $|X| = 3$, and which cannot be extended to an acyclic 2-coloring of $\vec{G}$ without monochromatic triangles. Clearly, $X$ is completely contained in either $G_1$ or $G_2$, let us assume w.l.o.g. $X \subseteq V(G_1)$.

Let now $c_1 : V(G_1) \to \{1, 2\}$ be an acyclic coloring of $\vec{G}_1$ without monochromatic triangles such that $c_1|_X = p$. We distinguish three cases depending on the size of $C$ in order to find an acyclic coloring $c_2 : V(G_2) \to \{1, 2\}$ of $\vec{G}_2$ without monochromatic triangles such that $c_1|_C = c_2|_C$:

**Case 1.** $|C| \in \{0, 1\}$. Since $G_2$ satisfies the claim, there is an acyclic 2-coloring $c_2 : V(G_2) \to \{1, 2\}$ of $\vec{G}_2$ without monochromatic triangles. Now possibly after exchanging colors 1 and 2 in the coloring $c_2$, we have that $c_1|_C = c_2|_C$.

**Case 2.** $|C| = 2$. Let $e' = u'v' \in E(G_1) \cap E(G_2)$ be the unique edge contained in $C$. Let $p' : \{u', v'\} \to \{1, 2\}$ be the pre-coloring of the edge $e'$ in $G_2$ defined by $p'(u') := c_1(u'), p'(v') := c_1(v')$. Since $G_2$ satisfies the claim, there exists an acyclic coloring $c_2 : V(G_2) \to \{1, 2\}$ of $\vec{G}_2$ without monochromatic triangles which extends $p'$, that is $c_2|_C = p' = c_1|_C$.

**Case 3.** $|C| = 3$. Let $t' = x_1x_2x_3$ be the triangle in $G_2$ formed by the three vertices in $C$. Let $p' : \{x_1, x_2, x_3\} \to \{1, 2\}$ be the pre-coloring of $t'$ defined by $p'(x_i) := c_1(x_i), i = 1, 2, 3$. Since the coloring $c_1$ of $\vec{G}_1$ has no monochromatic triangles and since $t'$ is also a triangle in $G_1$, we find that the pre-coloring $p'$ is not monochromatic. Hence, since $G_2$ satisfies the claim, there has to be an acyclic coloring $c_2 : V(G_2) \to \{1, 2\}$ of $\vec{G}_2$ without monochromatic triangles which extends $p'$, that is $c_2|_C = p' = c_1|_C$.

In each of the above cases we have found a pair $c_1, c_2$ of acyclic 2-colorings of $\vec{G}_1$ respectively $\vec{G}_2$, each without monochromatic triangles, such that $c_1|_X = p$ and $c_2|_C = c_1|_C$. Let $c : V(G) \to \{1, 2\}$ be the common extension of $c_1$ and $c_2$ to $V(G) = V(G_1) \cup V(G_2)$. By Lemma 1, $c$ is an acyclic 2-coloring of $\vec{G}$. We furthermore have $c|_X = c_1|_X = p$, and since every triangle in $G$ is a triangle in $G_1$ or $G_2$, there are no monochromatic triangles in the coloring $c$ of $G$. This clearly contradicts our initial assumption that the pre-coloring $p : X \to \{1, 2\}$ cannot be extended to an acyclic 2-coloring of $\vec{G}$ without monochromatic triangles. This shows that the initial assumption was false and concludes the proof of the Theorem. □

## 4 Conclusion

A natural question that comes out from the discussion in this paper is the following.

**Question 1.** *What is the largest minor-closed class $\mathcal{G}_2$ of undirected graphs with dichromatic number at most 2?*

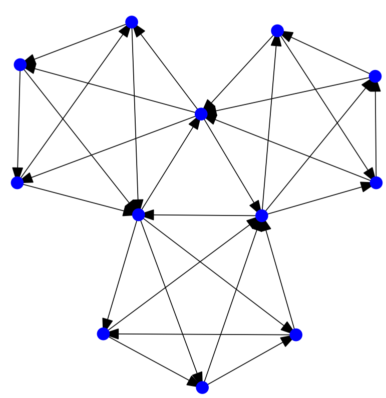

**Fig. 3:** An oriented $K_{3,3}$-minor-free graph without an acyclic 2-colouring.

While $\vec{\chi}(K_6) = 2$, it is known that $\vec{\chi}(K_7) = 3$. Therefore, $\mathcal{G}_2$ is a subclass of the $K_7$-minor-free graphs. However, $\mathcal{G}_2$ seems to be a lot smaller than this class. In fact, there are examples of $K_{3,3}$-minor-free graphs with dichromatic number greater than 2, see Figure 3. In every 2-coloring of the vertices of this oriented graph, two vertices of the central triangle have to be colored the same, whence the three vertices attached to this edge must form a monochromatic directed triangle.

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