Discrete Mathematics and Theoretical Computer Science DMTCS vol. **22**:4, 2020, #12

A new sufficient condition for a digraph to be Hamiltonian−A proof of Manoussakis conjecture

Samvel Kh. Darbinyan

[1] *Yerevan, Armenia*
[2] *Institute for Informatics and Automation Problems of NAS RA*

received 2020-02-11, *revised 2020-07-21,2020-11-05*, *accepted 2020-11-05*.

Y. Manoussakis (J. Graph Theory 16, 1992, 51-59) proposed the following conjecture.

Conjecture. *Let D be a 2-strongly connected digraph of order n such that for all distinct pairs of non-adjacent vertices x, y and w, z, we have $d(x) + d(y) + d(w) + d(z) \geq 4n - 3$. Then D is Hamiltonian.*

In this paper, we confirm this conjecture. Moreover, we prove that if a digraph D satisfies the conditions of this conjecture and has a pair of non-adjacent vertices $\{x, y\}$ such that $d(x) + d(y) \leq 2n - 4$, then D contains cycles of all lengths $3, 4, \ldots, n$.

Keywords: digraph, hamiltonian cycle, strong digraph, pancyclic digraph

1 Introduction

In this paper, we consider finite digraphs (directed graphs) without loops and multiple arcs. Every cycle and path are assumed simple and directed; its *length* is the number of its arcs. A digraph D is *Hamiltonian* if it contains a cycle passing through all the vertices of D. There are many conditions that guarantee that a digraph is Hamiltonian (see, e.g., Bang-Jensen and Gutin (Springer-Verlag, London, 2000), Bermond and Thomassen (1981), Kühn and Osthus (2012), Manoussakis (1992), Meyniel (1973)). Manoussakis (1992) the following theorem was proved.

Theorem 1.1 (Manoussakis (1992)). *Let D be a strong digraph of order $n \geq 4$. Suppose that D satisfies the following condition for every triple $x, y, z \in V(D)$ such that x and y are non-adjacent: If there is no arc from x to z, then $d(x) + d(y) + d^+(x) + d^-(z) \geq 3n - 2$. If there is no arc from z to x, then $d(x) + d(y) + d^-(x) + d^+(z) \geq 3n - 2$. Then D is Hamiltonian.*

Definition 1.2. *Let D be a digraph of order n. We say that D satisfies condition (M) when $d(x) + d(y) + d(w) + d(z) \geq 4n - 3$ for all distinct pairs of non-adjacent vertices x, y and w, z.*

ISSN subm. to DMTCS

Manoussakis (1992) proposed the following conjecture. This conjecture is an extension of Theorem 1.1.

Conjecture 1.3 (Manoussakis (1992)). *Let D be a 2-strong digraph of order n such that for all distinct pairs of non-adjacent vertices x, y and w, z we have $d(x) + d(y) + d(w) + d(z) \geq 4n - 3$. Then D is Hamiltonian.*

Manoussakis (1992) gave an example, which showed that if this conjecture is true, then the minimum degree condition is sharp. Notice that another examples can be found in a paper by Darbinyan (1983), where for any two integers $k \geq 2$ and $m \geq 1$, the author constructed a family of k-strong digraphs of order $4k + m$ with minimum degree $4k + m - 1$, which are not Hamiltonian. This result improves a conjecture of Thomassen (see Bermond and Thomassen (1981) Conjecture 1.4.1: Every 2-strong $(n-1)$-regular digraph of order n, except D_5 and D_7, is Hamiltonian). Moreover, when $m = 1$, then from these digraphs we can obtain k-strong non-Hamiltonian digraphs of order $n = 4k + 1$ with minimum degree equal to $n - 1$ and the minimal semi-degrees equal to $(n - 3)/2$. Thus, if in Conjecture 1.3 we replace $4n - 3$ with $4n - 4$, then for every n there are many digraphs of order n with high connectivity and high semi-degrees, for which Conjecture 1.3 is not true.

The *cycle factor* in a digraph D is a collection of pairwise vertex disjoint cycles $C_1, C_2, \ldots, C_l$ such that $\bigcup_{i=1}^{l} V(C_i) = V(D)$. It is clear that the existence of a cycle factor in a digraph D is a necessary condition for a digraph to be Hamiltonian. The following theorem gives a necessary and sufficient condition for the existence of a cycle factor in a digraph.

Theorem 1.4 (Yeo (1999)). *Let D be a digraph. Then D has a cycle factor if and only if $V(D)$ cannot be partitioned into subsets Y, Z, R_1, R_2 such that $A(Y \to R_1) = A(R_2 \to R_1 \cup Y) = \emptyset$, $|Y| > |Z|$ and Y is an independent set.*

Using theorem Theorem 1.4, it is not difficult to construct 2-strong digraphs satisfying the condition that $d(x) + d(y) + d(w) + d(z) \geq 4n - 4$ for every distinct pairs $\{x, y\}$, $\{w, z\}$ of non-adjacent vertices, but these digraphs do not even contain a cycle factor.

Thomassen suggested (see Bermond and Thomassen (1981)) the following two conjectures:

1. Conjecture 1.6.7. *Every 3-strong digraph of order n and with minimum degree at least $n+1$ is strongly Hamiltonian-connected.*

2. Conjecture 1.6.8. *Let D be a 4-strong digraph of order n such that the sum of the degrees of any pair of non-adjacent vertices is at least $2n + 1$. Then D is strongly Hamiltonian-connected.*

Investigating these conjectures, Darbinyan (1990) disproved the first conjecture (proving that for every integer $n \geq 9$ there exists a 3-strong non-strongly Hamiltonian-connected digraph of order n with the minimum degree at least $n + 1$) and for the second proved the following two theorems.

Theorem 1.5. *Any k-strong ($k \geq 1$) digraph D of order $n \geq 8$ satisfying the condition that the sum of degrees of any pair of non-adjacent vertices $x, y \in V(D) \setminus \{z\}$ at least $2n - 1$, where z is some vertex in $V(D)$, is Hamiltonian if and only if any $(k+1)$-strong digraph of order $n+1$ satisfying the condition that the sum of degrees of any pair of non-adjacent vertices at least $2n+3$ is strongly Hamiltonian-connected.*

Theorem 1.6. *Let D be a strong digraph of order $n \geq 3$. Suppose that $d(x) + d(y) \geq 2n - 1$ for every pair of non-adjacent vertices $x, y \in V(D) \setminus \{z\}$, where z is some vertex of $V(D)$. Then D contains a cycle of length at least $n - 1$.*

It is easy to see that if a digraph D satisfies the condition (M), then it contains at most one pair of non-adjacent vertices x, y such that $d(x) + d(y) \leq 2n - 2$. From this and Theorem 1.6, the following corollary immediately follows.

Corollary 1.7. *Let D be a strong digraph of order n satisfying condition (M). Then D contains a cycle of length at least $n - 1$ (in particular, D contains a Hamiltonian path).*

Corollary 1.7 was also later proved by Ning (2015).

It is worth to note that Darbinyan (2017), Darbinyan (2015) and Darbinyan and Karapetyan (2015) studied some properties in digraphs with the conditions of Theorem 1.1. They obtained the following results (in first two results D is a digraph of order n satisfying the degree condition of Theorem 1.1).

(i) (Darbinyan and Karapetyan (2015)). *If D is strong, then it contains a cycle of length $n - 1$ or D is isomorphic to the complete bipartite digraph $K^*_{n/2,n/2}$.*

(ii) (Darbinyan (2015)). *If D is strong, then it contains a Hamiltonian path in which the initial vertex dominates the terminal vertex or D is isomorphic to one tournament of order 5.*

(iii) (Darbinyan (2017)). *Let D be a digraph of order n and let Y be a non-empty subset of $V(D)$. Suppose that for every triple of the vertices $x, y, z \in Y$ such that x and y are non-adjacent: If there is no arc from x to z, then $d(x) + d(y) + d^+(x) + d^-(z) \geq 3n - 2$. If there is no arc from z to x, then $d(x) + d(y) + d^-(x) + d^+(z) \geq 3n - 2$. If there is a path from u to v and a path from v to u in D for every pair of distinct vertices $u, v \in Y$, then D has a cycle which contains at least $|Y| - 1$ vertices of Y.*

The last result is best possible in some situations and gives an answer to a question posted by Li et al. (2007).

Theorem 1.8 (Meyniel (1973)). *Let D be a strong digraph of order $n \geq 2$. If $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices x, y in D, then D is Hamiltonian.*

For a short proof of Theorem 1.8, see Bondy and Thomassen (1977). Darbinyan (1985) characterized those digraphs which satisfy Meyniel's condition, but are not pancyclic. Before stating the main result obtained by Darbinyan (1985), we need to define a family of digraphs.

Definition 1.9. *For integers n and m, $(n+1)/2 < m \leq n-1$, let Φ_n^m denote the set of digraphs D, which satisfy the following conditions: (i) $V(D) = \{x_1, x_2, \ldots, x_n\}$; (ii) $x_n x_{n-1} \ldots x_2 x_1 x_n$ is a Hamiltonian cycle in D; (iii) for each k, $1 \leq k \leq n - m + 1$, the vertices x_k and x_{k+m-1} are not adjacent; (iv) $x_j x_i \notin A(D)$ whenever $2 \leq i + 1 < j \leq n$ and (v) the sum of degrees for any two distinct non-adjacent vertices is at least $2n - 1$.*

Theorem 1.10 (Darbinyan (1979), Darbinyan (1985)). *Let D be a strong digraph of order $n \geq 3$. Suppose that $d(x) + d(y) \geq 2n - 1$ for all pairs of distinct non-adjacent vertices x, y in D. Then either (a) D is pancyclic or (b) n is even and D is isomorphic to one of $K^*_{n/2,n/2}$, $K^*_{n/2,n/2} \setminus \{e\}$, where e is an arbitrary arc of $K^*_{n/2,n/2}$, or (c) $D \in \Phi^m_n$ (in this case D does not contain a cycle of length m).*

Later on, Theorem 1.10 was also proved by Benhocine (1986). Darbinyan (2019) investigated the pancyclicity of digraphs with the condition (M). Using Theorem 1.10 and the Moser theorem for a strong tournament to be pancyclic (see Harary and Moser (1966)), we proved the following theorem.

Theorem 1.11 (Darbinyan (2019)). *Let D be a 2-strong digraph of order $n \geq 6$ satisfying condition (M). Suppose that there exists a pair of non-adjacent vertices x, y in D such that $d(x) + d(y) \leq 2n - 4$. Then D contains cycles of all lengths $3, 4, \ldots, n - 1$.*

In this paper we confirm Conjecture 1.3.

Theorem 1.12. *Let D be a 2-strong digraph of order $n \geq 3$ satisfying condition (M). Then D is Hamiltonian.*

Theorem 1.12 also has the following immediate corollaries.

Corollary 1.13 (Woodall (1972)). *A digraph of order n is Hamiltonian if, for any two vertices x and y, either $x \to y$ or $d^+(x) + d^-(y) \geq n$.*

Corollary 1.14 (Nash-Williams (1969)). *Let D be a digraph of order $n \geq 2$. If for every vertex x, $d^+(x) \geq n/2$ and $d^-(x) \geq n/2$, then D is Hamiltonian.*

Note that Corollary 1.14 immediately follows from well-known theorem of Ghouila-Houri Ghouila-Houri (1960).

Corollary 1.15 (Ore (1960)). *Let G be a simple graph of order $n \geq 3$, in which the degree sum of any two non-adjacent vertices is at least n. Then G is Hamiltonian.*

As an immediate corollary of Theorems 1.12 and 1.11, we obtain the following theorem.

Theorem 1.16. *Let D be a 2-strong digraph of order $n \geq 6$ satisfying condition (M). Suppose that D contains a pair of non-adjacent vertices x, y such that $d(x) + d(y) \leq 2n - 4$. Then D is pancyclic.*

In view of Theorem 1.16, it is natural to set the following problem.

Problem 1.17. *Let D be a 2-strong connected digraph of order n satisfying condition (M). Suppose that $\{x, y\}$ is a pair of non-adjacent vertices in D such that $2n - 3 \leq d(x) + d(y) \leq 2n - 2$. Whether D is pancyclic?*

2 Terminology and notation

In this paper we consider finite digraphs without loops and multiple arcs. We shall assume that the reader is familiar with the standard terminology on digraphs and refer to the book Bang-Jensen and Gutin (Springer-Verlag, London, 2000) for terminology and notations not defined here. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. The *order* of D is the number of its vertices. For any $x, y \in V(D)$, we also write $x \to y$ if $xy \in A(D)$. We use the notations $\overrightarrow{a}[x, y] = 1$ if $xy \in A(D)$ and $\overrightarrow{a}[x, y] = 0$ if $xy \notin A(D)$. If $xy \in A(D)$, y is an *out-neighbour* of x and x is an *in-neighbour* of y. If $x \to y$ and $y \to z$, we write $x \to y \to z$. Two distinct vertices x and y are *adjacent* if $xy \in A(D)$ or $yx \in A(D)$ (or both). If there is no arc from x to y, we shall use the notation $xy \notin A(D)$.

We let $N^+(x)$, $N^-(x)$ denote the set of *out-neighbours*, respectively the set of *in-neighbours* of a vertex x in a digraph D. If $A \subseteq V(D)$, then $N^+(x, A) = A \cap N^+(x)$ and $N^-(x, A) = A \cap N^-(x)$. The *out-degree* of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the *in-degree* of x. Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$. The *degree* of the vertex x in D is defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, A) = d^+(x, A) + d^-(x, A)$). The subdigraph of D induced by a subset A of $V(D)$ is denoted by $D\langle A\rangle$. If z is a vertex of a digraph D, then the subdigraph $D\langle V(D) \setminus \{z\}\rangle$ is denoted by $D - z$.

For integers a and b, $a \leq b$, let $[a, b]$ denote the set of all integers, which are not less than a and are not greater than b.

The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and the arcs x_ix_{i+1}, $i \in [1, m-1]$ (respectively, x_ix_{i+1}, $i \in [1, m-1]$, and x_mx_1), is denoted by $x_1x_2 \cdots x_m$ (respectively, $x_1x_2 \cdots x_mx_1$). We say that $x_1x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)*-path*. Let x and y be two distinct vertices of a digraph D. Cycle that passing through x and y in D, we denote by $C(x, y)$.

A cycle (respectively, a path) that contains all the vertices of D, is a *Hamiltonian cycle* (respectively, is a *Hamiltonian path*). A digraph is *Hamiltonian* if it contains a Hamiltonian cycle. A digraph D is *strongly Hamiltonian-connected* if, for every ordered pair $\{x, y\}$ of distinct vertices of D there is a Hamiltonian path from x to y. A digraph D of order $n \geq 3$ is *pancyclic* if it contains cycles of all lengths m, $3 \leq m \leq n$. For a cycle $C = x_1x_2 \cdots x_kx_1$ of length k, the subscripts considered modulo k, i.e., $x_i = x_s$ for every s and i such that $i \equiv s \pmod{k}$. If P is a path containing a subpath from x to y, we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x, y]$ denotes the subpath of C from x to y. If $j < i$, then $\{x_i, \ldots, x_j\} = \emptyset$.

A digraph D is *strongly connected* (or just *strong*), if there exists a path from x to y and a path from y to x for every pair of distinct vertices x, y. A digraph D is k*-strongly connected* (or k*-strong*), where $k \geq 1$, if $|V(D)| \geq k + 1$ and $D\langle V(D) \setminus A\rangle$ is strongly connected for any subset $A \subset V(D)$ of at most $k - 1$ vertices.

For a pair of disjoint subsets A and B of $V(D)$, we define $A(A \to B) = \{xy \in A(D) \,|\, x \in A, y \in B\}$ and $A(A, B) = A(A \to B) \cup A(B \to A)$.

3 Auxiliary known results

Lemma 3.1 (Häggkvist and Thomassen (1976)). *Let D be a digraph of order $n \geq 3$ containing a cycle C of length m, $m \in [2, n-1]$. Let x be a vertex not contained in this cycle. If $d(x, V(C)) \geq m + 1$, then*

D contains a cycle of length k for all $k \in [2, m+1]$.

It is not difficult to prove the following lemma.

Lemma 3.2. *Let D be a digraph of order n. Assume that $xy \notin A(D)$ and the vertices x, y in D satisfy the degree condition $d^+(x) + d^-(y) \geq n - 2 + k$, where $k \geq 1$. Then D contains at least k internally disjoint (x, y)-paths of length two.*

The following results were proved by Darbinyan (2019) and its preliminary version presented at Emil Artin International Conference (Darbinyan (2018)).

Theorem 3.3. *Let D be a 2-strong digraph of order $n \geq 3$ satisfying condition (M). Suppose that $\{x, y\}$ is a pair of non-adjacent vertices in $V(D)$ such that $d(x) + d(y) \leq 2n - 2$. Then D is Hamiltonian if and only if D contains a cycle through the vertices x and y.*

Theorem 3.4. *Let D be a 2-strong digraph of order $n \geq 3$. Suppose that D contains at most one pair of non-adjacent vertices. Then D is Hamiltonian.*

Remark. *There is a strong non-Hamiltonian digraph of order $n \geq 5$, which is not 2-strong and has exactly one pair of non-adjacent vertices.*

Using Lemma 3.2, it is not difficult to prove the following lemma.

Lemma 3.5. *Let D be a 2-strong digraph of order $n \geq 3$ and let u, v be two distinct vertices in $V(D)$. If D contains no cycle through u and v, then u, v are not adjacent and there is no path of length two between them. In particular, $d(u) + d(v) \leq 2n - 4$.*

Theorem 3.6. *Let D be a 2-strong digraph of order $n \geq 3$ satisfying condition (M). Suppose that $\{u, v\}$ is a pair of non-adjacent vertices in $V(D)$ such that $d(u) + d(v) \leq 2n - 2$. Then D is Hamiltonian or D contains a cycle of length $n - 1$ passing through u and avoiding v (passing through v and avoiding u).*

As an immediate corollary of Theorems 3.6, 3.3 and Lemma 3.1, we obtain

Corollary 3.7. *Let D be a 2-strong non-Hamiltonian digraph of order $n \geq 3$ satisfying condition (M). Suppose that $\{u, v\}$ is a pair of non-adjacent vertices in $V(D)$ such that $d(u) + d(v) \leq 2n - 2$. Then $d(u) \leq n - 1$, $d(v) \leq n - 1$ and D contains at most one cycle of length two passing through u (v) .*

4 Preliminaries

Lemma 4.1. *Let D be a 2-strong digraph of order $n \geq 3$ satisfying condition (M). Suppose that $\{y, z\}$ is a pair of non-adjacent vertices in $V(D)$ such that $d(y) + d(z) \leq 2n - 2$ and $C = x_1 x_2 \dots x_{n-k} x_1$ is a cycle in D passing through y and avoiding z, where $2 \leq n - k \leq n - 2$. If the subdigraph $D\langle V(D) \setminus V(C)\rangle$ contains a cycle passing through z and $d(y, V(D) \setminus V(C)) = 0$, then D is Hamiltonian.*

Proof: Suppose, on the contrary, that $D\langle V(D) \setminus V(C)\rangle$ contains a cycle passing through z, but D is not Hamiltonian. Since D contains at most one cycle of length two passing through y (Corollary 3.7), from $d(y, V(D) \setminus V(C)) = 0$ it follows that $d(y) \leq n - k$. Let $y_1 y_2 \ldots y_s y_1$ be a cycle through z in $D\langle V(D) \setminus V(C)\rangle$, where $s \in [2, k]$.

By Theorem 3.3 we have that D contains no cycle through y and z. Therefore, for each pair of integers i and j, where $i \in [1, n-k]$ and $j \in [1, s]$, $\overrightarrow{a}[x_i, y_j] + \overrightarrow{a}[y_{j-1}, x_{i+1}] \leq 1$ (here, $y_0 = y_s$ and $x_{n-k+1} = x_1$). This implies that for every $j \in [1, s]$ we have

$$d^-(y_j, V(C)) + d^+(y_{j-1}, V(C)) = \sum_{i=1}^{n-k} (\overrightarrow{a}[x_i, y_j] + \overrightarrow{a}[y_{j-1}, x_{i+1}]) \leq n - k.$$

Hence,

$$d(y_1, V(C)) + \cdots + d(y_s, V(C)) = \sum_{j=1}^{s} (d^-(y_j, V(C)) + d^+(y_{j-1}, V(C))) \leq s(n-k). \qquad (1)$$

Since there is at most one cycle of length two through z (y) (Corollary 3.7), it follows that for $A := V(D) \setminus V(C)$ and for every $y_j \in \{y_1, \ldots, y_s\} \setminus \{z, y_1\}$ (we may assume that $y_1 \neq z$) the following holds:

$$d(z, A) \leq k, \quad d(y_1, A) \leq 2k - 2 \quad \text{and} \quad d(y_j, A) \leq 2(k-2) + 1 = 2k - 3.$$

Therefore,

$$d(y_1, A) + \cdots + d(y_s, A) \leq (s-2)(2k-3) + k + 2k - 2 = 2ks - 3s - k + 4.$$

Combining this with (1), we obtain

$$d(y_1) + \cdots + d(y_s) \leq ns + ks - 3s - k + 4.$$

The last inequality together with $d(y) \leq n - k$ implies that

$$d(y_1) + \cdots + d(y_s) + sd(y) \leq 2ns - 3s - k + 4. \qquad (2)$$

Notice that $\{y, y_1\}, \ldots, \{y, y_s\}$ are s distinct pairs of non-adjacent vertices. We will consider the cases when s is even and s is odd separately.

Assume first that s is even. Using condition (M) and (2), we obtain

$$s(4n-3)/2 \leq d(y_1) + \cdots + d(y_s) + sd(y) \leq 2ns - 3s - k + 4.$$

Therefore, $2ns - 1.5s \leq 2ns - 3s - k + 4$, i.e., $1.5s + k \leq 4$. The last inequality is impossible, since $k \geq s \geq 2$.

Assume next that s is odd. Then $s \geq 3$. Since $d(y) \leq n - k$, and $d(z) \leq n - 1$ by Corollary 3.7 (we may assume that $z \neq y_s$), from condition (M) it follows that $d(y) + d(y_s) \geq 2n + k - 2$. Now, by condition (M) and (2) we have,

$$(s-1)(4n-3)/2 + 2n + k - 2 \leq d(y_1) + \cdots + d(y_{s-1}) + d(y_s) + sd(y)$$

$$\leq 2ns - 3s - k + 4.$$

Hence,

$$2n(s-1) - 1.5(s-1) + 2n + k - 2 \leq 2ns - 3s - k + 4.$$

This means that $1.5s + 2k \leq 4.5$, which is a contradiction. This contradiction completes the proof of Lemma 4.1. □

Lemma 4.2. *Let D be a 2-strong digraph of order $n \geq 3$ satisfying condition (M). Suppose that $\{y, z\}$ is a pair of non-adjacent vertices in $V(D)$ such that $d(y) + d(z) \leq 2n - 2$ and $C = x_1 x_2 \ldots x_{n-2} z x_1$ is a cycle of length $n-1$ passing through z and avoiding y in D. Then either D is Hamiltonian or for every $k \in [2, n-3]$, the following holds:*

$$A(\{x_1, \ldots, x_{k-1}\} \to \{x_{k+1}, \ldots, x_{n-2}\}) \neq \emptyset.$$

Proof: Suppose that D is not Hamiltonian. Since D is 2-strong, $n \geq 5$. Then by Theorem 3.3, there is no cycle through y and z. Therefore, we have that if $x_i \to y$ with $i \in [1, n-3]$, then $d^+(y, \{x_{i+1}, \ldots, x_{n-2}\}) = 0$ (for otherwise, $x_1 \ldots x_i y x_j \ldots x_{n-2} z x_1$, where $j \in [i+1, n-2]$, is a cycle through y and z, a contradiction). Let $x_r \to y \to x_p$, $1 \leq p < r \leq n-2$, and p, r be chosen so that p is minimal and r is maximal with these properties. Then

$$d(y, \{x_1, \ldots, x_{p-1}\}) = d(y, \{x_{r+1}, \ldots, x_{n-2}\}) = 0. \tag{3}$$

If $p = 1$ and $r = n-2$, then by a similar argument as above, we conclude that if $x_i \to z$ with $i \in [1, n-3]$, then $d^+(z, \{x_{i+1}, \ldots, x_{n-2}\}) = 0$. Assume that $p \geq 2$ or $r \leq n-3$. Observe that $Q := y x_p \ldots x_r y$ is a cycle through y which does not contain z, and $d(y, V(D) \setminus V(Q)) = 0$ because of (3). Therefore by Lemma 4.1, the subdigraph $D\langle V(D) \setminus V(Q)\rangle$ contains no cycle through z since D is not Hamiltonian. This implies that

$$d^-(z, \{x_1, \ldots, x_{p-1}\}) = d^+(z, \{x_{r+1}, \ldots, x_{n-2}\}) = 0$$

since $x_{n-2} \to z \to x_1$. From the last equalities it follows that if there are i, j such that $x_i \to z$ and $z \to x_j$ with $i < j$, then $i \geq p$, $j \leq r$ and $y x_p \ldots x_i z x_j \ldots x_r y$ is a cycle passing through y and z, a contradiction. Thus, we may assume that for every pair of integers i and j, $1 \leq i < j \leq n-2$,

$$\text{if} \quad x_i \to y, \quad \text{then} \quad y x_j \notin A(D) \quad \text{and if} \quad x_i \to z, \quad \text{then} \quad z x_j \notin A(D). \tag{4}$$

Now suppose that the theorem is not true. Then D is not Hamiltonian and there is an integer $k \in [2, n-3]$ such that

$$A(\{x_1, \ldots, x_{k-1}\} \to \{x_{k+1}, \ldots, x_{n-2}\}) = \emptyset. \tag{5}$$

It is easy to see that there are vertices x_m and x_l such that $y \to x_m$, $z \to x_l$ and

$$d^+(y, \{x_{m+1}, \ldots, x_{n-2}\}) = d^+(z, \{x_{l+1}, \ldots, x_{n-2}\}) = 0. \tag{6}$$

Then by (4),

$$d^-(y, \{x_1, \ldots, x_{m-1}\}) = d^-(z, \{x_1, \ldots, x_{l-1}\}) = 0. \tag{7}$$

Assume first that $m \leq l$. Since D is 2-strong, (4) and (7) imply that $2 \leq m \leq l \leq n-3$. Now from (5), (6) and (7) it follows that:

(i) if $k \le m$ or $k \ge l$, then (respectively)

$$A(\{x_1, x_2, \dots, x_{k-1}\} \to \{y, z, x_{k+1}, x_{k+2}, \dots, x_{n-2}\}) = \emptyset$$

or

$$A(\{y, z, x_1, x_2, \dots, x_{k-1}\} \to \{x_{k+1}, x_{k+2}, \dots, x_{n-2}\}) = \emptyset,$$

(ii) if $m < k < l$, then $A(\{y, x_1, x_2, \dots, x_{k-1}\} \to \{z, x_{k+1}, x_{k+2}, \dots, x_{n-2}\}) = \emptyset$. Thus, in each case we have that $D - x_k$ is not strong, which contradicts the condition that D is 2-strongly connected.

Assume next that $m > l$. This case is similar to the first case and we omit the details. Lemma 4.2 is proved. □

The following lemma is proved by Darbinyan (2019). We present its proof for completeness.

Lemma 4.3. *Let D be a 2-strong digraph of order $n \ge 3$ satisfying condition (M). Suppose that $\{y, z\}$ is a pair of non-adjacent vertices in $V(D)$ such that $d(y) + d(z) \le 2n - 2$ and $C = x_1x_2 \dots x_{n-2}zx_1$ is a cycle of length $n - 1$ passing through z and avoiding y in D. If $x_a \to x_b$ and there are integers l, s, f, t such that $1 \le l \le a < s \le f < b \le t \le n - 2$ and $\{x_f, x_t\} \to y \to \{x_l, x_s\}$, then D is Hamiltonian.*

Proof: Suppose, on the contrary, that D is not Hamiltonian. By Theorem 3.3, D contains no cycle through y and z. Therefore, there are no integers i and j, $1 \le i < j \le n - 2$, such that $x_i \to y \to x_j$ (for otherwise, $x_1 \dots x_iyx_j \dots x_{n-2}zx_1$ is a cycle through y and z). Since the arcs yx_l, yx_s, x_fy, x_ty are in D and $l \le a < s \le f < b \le t$, it is easy to check that:

(i) if $z \to x_i$ with $i \in [a + 1, f]$, then $C(y, z) = yx_l \dots x_ax_b \dots x_{n-2}zx_i \dots x_fy$;

(ii) if $x_j \to z$ with $j \in [s, b - 1]$, then $C(y, z) = x_1 \dots x_ax_b \dots x_tyx_s \dots x_jzx_1$. Thus, in both cases we have a contradiction. Therefore,

$$d^+(z, \{x_{a+1}, \dots, x_f\}) = d^-(z, \{x_s, \dots, x_{b-1}\}) = 0,$$

in particular, $d(z, \{x_s, \dots, x_f\}) = 0$ and the vertices z and x_s (z and x_f) are not adjacent. The last equality together with the fact that D contains at most one cycle of length two passing through z (Corollary 3.7) implies that

$$d(z) = d(z, \{x_1, \dots, x_{s-1}\}) + d(z, \{x_{f+1}, \dots, x_{n-2}\}) \le n + s - f - 2. \tag{8}$$

Now we consider the vertex x_s. It is not difficult to check that:

(iii) if $x_i \to x_s$ with $i \in [1, l - 1]$, then $C(y, z) = x_1 \dots x_ix_s \dots x_fyx_l \dots x_ax_b \dots x_{n-2}zx_1$;

(iv) if $x_s \to x_j$ with $j \in [t + 1, n - 2]$, then $C(y, z) = x_1 \dots x_ax_b \dots x_tyx_s\, x_j \dots x_{n-2}zx_1$. In both cases we have a contradiction. Therefore, we may assume that

$$d^-(x_s, \{x_1, \dots, x_{l-1}\}) = d^+(x_s, \{x_{t+1}, \dots, x_{n-2}\}) = 0.$$

This implies that

$$\begin{aligned} d(x_s) &= d^+(x_s, \{x_1, \dots, x_{l-1}\}) + d^-(x_s, \{x_{t+1}, \dots, x_{n-2}\}) + d(x_s, \{x_l, \dots, x_t\}) + d(x_s, \{y\}) \\ &\le l - 1 + n - 2 - t + 2(t - l + 1) = n + t - l - 1. \end{aligned} \tag{9}$$

Without loss of generality, we may assume that l, f are chosen as maximal as possible and s, t are chosen as minimal as possible, i.e.,

$$d(y,\{x_{l+1},\dots,x_{s-1}\}) = d(y,\{x_{f+1},\dots,x_{t-1}\}) = 0.$$

This, since D contains at most one cycle of length two passing through y, implies that

$$d(y) = d(y,\{x_1,\dots,x_l\}) + d(y,\{x_s,\dots,x_f\}) + d(y,\{x_t,\dots,x_{n-2}\})$$
$$\leq l+f-s+1+n-2-t+2 = n+l+f-s-t+1.$$

Since $\{y,z\}$ and $\{x_s,z\}$ are two distinct pairs of non-adjacent vertices, from (8), (9), the last inequality and condition (M) it follows that

$$4n-3 \leq d(y)+2d(z)+d(x_s) \leq n+l+f-s-t+1+2n+2s-2f-4+n+t-l-1$$
$$= 4n-4-(f-s) \leq 4n-4,$$

which is a contradiction. Lemma 4.3 is proved. □

5 Proof of Theorem 1.12

Recall the statement of Theorem 1.12.

Theorem 1.12. *Let D be a 2-strong digraph of order $n \geq 3$ satisfying condition (M). Then D is Hamiltonian.*

Proof: By Theorem 3.4, the theorem is true if D contains at most one pair of non-adjacent vertices. We may therefore assume that D contains at least two distinct pairs of non-adjacent vertices. If the degrees sum of any two non-adjacent vertices at least $2n-1$, then by Meyniel's theorem, the theorem is true. We may therefore assume that D contains a pair of non-adjacent vertices, say y,z, such that $d(y)+d(z) \leq 2n-2$. By Theorem 3.3, to prove the theorem, it suffices to prove that D contains a cycle through y and z. If $d(y)+d(z) \geq 2n-3$, then by Lemma 3.5 we have that D contains a cycle trough y and z, which, in turn, implies that D is Hamiltonian (by Theorem 3.3). Thus, we may assume that $d(y)+d(z) \leq 2n-4$. By Theorem 3.6 we have that either D is Hamiltonian or D contains a cycle of length $n-1$ passing through z and avoiding y (passing through y and avoiding z).

Suppose that D is not Hamiltonian, i.e., D contains no cycle through y and z. Let $C := x_1x_2\dots x_{n-2}zx_1$ be a cycle of length $n-1$ in D, which does not contain y. Let q be the maximum integer such that $y \to x_q$ and k be the minimum integer such that $x_k \to y$. Since D is 2-strong and contains no cycle passing through y and z, it follows that $k \geq q$ and there are some integers p, r, $1 \leq p < q \leq k < r \leq n-2$, such that $x_r \to y \to x_p$ and

$$d(y,\{x_1,\dots,x_{p-1}\}) = d(y,\{x_{q+1},\dots,x_{k-1}\}) = d(y,\{x_{r+1},\dots,x_{n-2}\})$$
$$= d^-(y,\{x_p,\dots,x_{q-1}\}) = d^+(y,\{x_{k+1},\dots,x_r\}) = 0. \quad (10)$$

Note that if D contains a cycle of length two passing trough y, then $k = q$, otherwise $k > q$, $yx_k \notin A(D)$ and $x_q y \notin A(D)$. Therefore, it is not difficult to see that

$$d(y) = d^+(y, \{x_p, \dots, x_q\}) + d^-(y, \{x_k, \dots, x_r\}) \leq q - p + r - k + 2. \tag{11}$$

In order to prove the theorem, it is convenient for the digraph D and the cycle C to prove the following claims.

Claim 5.1. *If $p \geq 2$, then $d^-(x_{n-2}, \{z, x_1, \dots, x_{p-1}\}) = 0$.*

Proof: Notice that $Q := yx_p \dots x_r y$ is a cycle passing through y and avoiding z. By (10) we have that $d(y, V(D) \setminus V(Q)) = 0$. Now by Lemma 4.1 , the induced subdigraph $D\langle V(D) \setminus V(Q)\rangle$ contains no cycle through z. Then, since $x_{n-2} \to z \to x_1$, we have

$$d^-(z, \{x_1, \dots, x_{p-1}\}) = 0 \quad \text{and} \quad A(\{z, x_1, \dots, x_{p-1}\} \to \{x_{r+1}, \dots, x_{n-2}\}) = \emptyset.$$

The first equality together with 2-connectedness of D implies that there is an integer $t \in [p, n-3]$ such that $x_t \to z$. The last equality means that if $r \leq n-3$, then $d^-(x_{n-2}, \{z, x_1, \dots, x_{p-1}\}) = 0$. Assume that $r = n-2$, i.e., $x_{n-2} \to y$. In this case, we have that if $x_i \to x_{n-2}$ with $i \in [1, p-1]$ (respectively, $z \to x_{n-2}$), then $C(y, z) = x_1 \dots x_i x_{n-2} y x_p \dots x_t z x_1$ (respectively, $C(y, z) = yx_p \dots x_t z x_{n-2} y$), which is a contradiction. This proves that $d^-(x_{n-2}, \{z, x_1, \dots, x_{p-1}\}) = 0$. □

Claim 5.2. *Suppose that $k \geq q+1$ and $x_h \to x_l$, where $h \in [q, k-1]$ and $l \in [k+1, n-2]$. Then $d^-(x_k, \{x_1, \dots, x_{q-1}\}) = 0$.*

Proof: Assume that Claim 5.2 is not true. Then for some $i \in [1, q-1]$, $x_i \to x_k$. Then, since the arcs yx_q, $x_k y$, $x_h x_l$ are in D and $i < q \leq h < k < l$, we have a cycle $C(y, z) = x_1 \dots x_i x_k y x_q \dots x_h x_l \dots x_{n-2} z x_1$, which contradicts our initial supposition. □

Claim 5.3. *Suppose that $k \geq q+1$, $x_h \to x_l$ with $h \in [q, k-1]$ and $l \in [k+1, r]$ (possibly, $r = n-2$). Then there is an integer $f \geq 0$ such that $l + f \leq r$, $x_{l+f} \to y$, $d(y, \{x_l, \dots, x_{l+f-1}\}) = 0$ (possibly, $\{x_l, \dots, x_{l+f-1}\} = \emptyset$). Moreover, either there is a vertex x_g with $g \in [l+f+1, n-2]$ such that $x_k \to x_g$ or there is a vertex x_c with $c \in [k, l-1]$ such that $x_c \to z$.*

Proof: By Claim 5.2,

$$d^-(x_k, \{x_1, \dots, x_{q-1}\}) = 0. \tag{12}$$

Since $l \leq r$ and $x_r \to y$, obviously there is an integer $f \geq 0$ such that $l + f \leq r$, $x_{l+f} \to y$, $d^-(y, \{x_l, \dots, x_{l+f-1}\}) = 0$ (possibly $\{x_l, \dots, x_{l+f-1}\} = \emptyset$). This together with $d^+(y, \{x_l, \dots, x_{l+f-1}\}) = 0$ implies that

$$d(y, \{x_l, \dots, x_{l+f-1}\}) = 0. \tag{13}$$

Now suppose that the claim is not true. Then

$$d^+(x_k, \{x_{l+f+1}, \dots, x_{n-2}\}) = 0 \text{ and } d^-(z, \{x_k, \dots, x_{l-1}\}) = 0. \tag{14}$$

The second equality of (14) together with $d^+(y, \{x_k, \dots, x_{l-1}\}) = 0$ and the fact that there is no path of length two between y and z (Lemma 3.5) implies that the vertices x_k, z are not adjacent and

$$d(z, \{x_k, \dots, x_{l-1}\}) + d(y, \{x_k, \dots, x_{l-1}\}) \leq l - k.$$

This together with (13), (10) and the fact that there is at most one cycle of length two through z (Corollary 3.7) implies that

$$d(y) + d(z) = d^+(y, \{x_p, \dots, x_q\}) + d(y, \{x_k, \dots, x_{l-1}\}) + d(z, \{x_k, \dots, x_{l-1}\})$$
$$+d^-(y, \{x_{l+f}, \dots, x_r\}) + d(z, \{x_1, \dots, x_{k-1}\}) + d(z, \{x_l, \dots, x_{n-2}\})$$
$$\leq q - p + 1 + l - k + r - l - f + 1 + k - 1 + n - 2 - l + 2$$
$$= n + q + r + 1 - p - l - f.$$

Now consider the vertex x_k. Note that $d(x_k, \{y\}) = 1$ since $k \geq q + 1$. Using (12) and the first equality of (14), we obtain

$$d(x_k) = d^+(x_k, \{x_1, \dots, x_{q-1}\}) + d(x_k, \{x_q, \dots, x_{l+f}\}) + d^-(x_k, \{x_{l+f+1}, \dots, x_{n-2}\})$$
$$+d^+(x_k, \{y\}) \leq q - 1 + 2l + 2f - 2q + n - 2 - l - f + 1 = n + l + f - q - 2.$$

Combining the last two inequalities, $d(z) \leq n - 1$ (Corollary 3.7) and $r \leq n - 2$, we obtain

$$d(y) + d(z) + d(x_k) + d(z) \leq 3n + r - p - 2 \leq 4n - 4 - p,$$

which contradicts condition (M), since $\{y, z\}$, $\{z, x_k\}$ are two distinct pairs of non-adjacent vertices. This contradiction completes the proof of Claim 5.3. □

Claim 5.4. *If* $p \geq 2$*, then* $A(\{x_1, \dots, x_{p-1}\} \to \{x_{k+1}, \dots, x_{n-2}\}) = \emptyset$.

Proof: Suppose, on the contrary, that $p \geq 2$ and $x_a \to x_b$ with $a \in [1, p-1]$ and $b \in [k+1, n-2]$. Let b be the maximum with these properties, i.e.,

$$A(\{x_1, \dots, x_{p-1}\} \to \{x_{b+1}, \dots, x_{n-2}\}) = \emptyset. \tag{15}$$

Notice that $Q := yx_p \dots x_r y$ is a cycle in D and $d(y, V(D) \setminus V(Q)) = 0$ by (10). Therefore by Lemma 4.1, the subdigraph $D\langle V(D) \setminus V(Q)\rangle$ does not contain a cycle through z. In particular,

$$d^-(z, \{x_1, \dots, x_{p-1}\}) = 0, \tag{16}$$

and if $r \leq n - 3$, then

$$d^+(z, \{x_{r+1}, \dots, x_{n-2}\}) = 0 \quad \text{and} \quad A(\{x_1, \dots, x_{p-1}\} \to \{x_{r+1}, \dots, x_{n-2}\}) = \emptyset. \tag{17}$$

By Claim 5.1, we have

$$d^-(x_{n-2}, \{z, x_1, \dots, x_{p-1}\}) = 0. \tag{18}$$

From (17) and (18) it follows that $b \leq r$ and, if $r = n - 2$, then $b \leq n - 3$. In both cases we have that $b \leq n - 3$.

If $x_i \to z$ with $i \in [p, b-1]$, then $C(y, z) = x_1 \dots x_a x_b \dots x_r y x_p \dots x_i z x_1$, a contradiction. We may therefore assume that $d^-(z, \{x_p, \dots, x_{b-1}\}) = 0$. This together with (16) implies that

$$d^-(z, \{x_1, \dots, x_{b-1}\}) = 0. \tag{19}$$

Applying Lemma 4.2 to the vertex x_b, we obtain that

$$A(\{x_1, \dots, x_{b-1}\} \to \{x_{b+1}, \dots, x_{n-2}\}) \neq \emptyset.$$

Let $x_s \to x_t$, where $s \in [1, b-1]$ and $t \in [b+1, n-2]$. Choose t maximal with these properties, i.e.,

$$A(\{x_1, \dots, x_{b-1}\} \to \{x_{t+1}, \dots, x_{n-2}\}) = \emptyset. \tag{20}$$

From (15) it follows that $s \geq p$, i.e., $s \in [p, b-1]$. If $x_i \to y$ with $i \in [b, t-1]$, then $C(y, z) = x_1 \dots x_a x_b \dots x_i y x_p \dots x_s x_t \dots x_{n-2} z x_1$, a contradiction. We may therefore assume that $d^-(y, \{x_b, \dots, x_{t-1}\}) = 0$. This together with $d^+(y, \{x_b, \dots, x_{t-1}\}) = 0$ implies that

$$d(y, \{x_b, \dots, x_{t-1}\}) = 0. \tag{21}$$

In particular, the vertices x_b and y are not adjacent, $t \leq r$ and $b \leq r-1$ since $b+1 \leq t \leq r$ (i.e., $A(\{x_p, \dots, x_{b-1}\} \to \{x_{r+1}, \dots, x_{n-2}\}) = \emptyset$). Using Lemma 4.3, we obtain

$$A(\{x_p, \dots, x_{q-1}\} \to \{x_{k+1}, \dots, x_r\}) = \emptyset \quad \text{and} \quad d^-(x_{k+1}, \{x_p, \dots x_{q-1}\}) = 0. \tag{22}$$

Then, since $t \leq r$ and (20), we have that $A(\{x_p, \dots, x_{q-1}\} \to \{x_{b+1}, \dots, x_{n-2}\}) = \emptyset$. This together with (15) implies that

$$A(\{x_1, \dots, x_{q-1}\} \to \{x_{b+1}, \dots, x_{n-2}\}) = \emptyset.$$

Therefore, $s \geq q$, i.e., $s \in [q, b-1]$. Since $b \leq r-1$, and x_b, y are not adjacent, there is an integer $f \geq 0$ such that $d^-(y, \{x_b, \dots, x_{b+f}\}) = 0$ and $x_{b+f+1} \to y$. Then, since (21) and $d^+(y, \{x_b, \dots, x_{b+f}\}) = 0$ we have that $t \leq b+f+1$ and

$$d(y, \{x_b, \dots, x_{b+f}\}) = 0. \tag{23}$$

This together with (10) implies that

$$\begin{aligned} d(y) &= d^+(y, \{x_p, \dots, x_q\}) + d^-(y, \{x_k, \dots, x_{b-1}\}) + d^-(y, \{x_{b+f+1}, \dots, x_r\}) \\ &\leq q - p + 1 + b - k + r - b - f = q + r + 1 - p - k - f. \end{aligned} \tag{24}$$

From (19), $d^+(y, \{x_k, \dots, x_{b-1}\}) \leq 1$ and the fact that there is no path of length two between y and z (Lemma 3.5) it follows that

$$d(y, \{x_k, \dots, x_{b-1}\}) + d(z, \{x_k, \dots, x_{b-1}\}) \leq b - k + 1.$$

This together with (10), (23) and the fact that there is at most one cycle of length two through z (Corollary 3.7) implies that

$$\begin{aligned} d(y) + d(z) &= d^+(y, \{x_p, \dots, x_q\}) + d(y, \{x_k, \dots, x_{b-1}\}) + d(z, \{x_k, \dots, x_{b-1}\}) \\ &+ d^-(y, \{x_{b+f+1}, \dots, x_r\}) + d(z, \{x_1, \dots, x_{k-1}\}) + d(z, \{x_b, \dots, x_{n-2}\}) \\ &\leq q - p + 1 + b - k + 1 + r - b - f + k - 1 + n - 2 - b + 2 \\ &= n + 1 + q - p + r - b - f. \end{aligned} \tag{25}$$

Since $t \leq b+f+1$ and (20), it follows that

$$A(\{x_p,\dots,x_{b-1}\} \to \{x_{b+f+2},\dots,x_{n-2}\}) = \emptyset. \tag{26}$$

In particular, from $b \geq k+1$ and (26) it follows that

$$d^+(x_k,\{x_{b+f+2},\dots,x_{n-2}\}) = 0. \tag{27}$$

We will consider the cases $b \geq k+2$, $b = k+1$ separately.

Case 1. $b \geq k+2$.

Then by the first equality of (22) we have

$$d^-(x_{b-1},\{x_p,\dots,x_{q-1}\}) = 0. \tag{28}$$

Using the fact that there is no path of length two between y and z (Lemma 3.5) and (19), we obtain that $d(x_{b-1},\{y,z\}) \leq 1$. This together with $d^+(x_{b-1},\{x_{b+f+2},\dots,x_{n-2}\}) = 0$ (by (26)) and (28) implies that

$$\begin{aligned} d(x_{b-1}) = d(x_{b-1},\{x_1,\dots,x_{p-1}\}) + d^+(x_{b-1},\{x_p,\dots,x_{q-1}\}) + d(x_{b-1},\{x_q,\dots,x_{b+f+1}\}) \\ +d^-(x_{b-1},\{x_{b+f+2},\dots,x_{n-2}\}) + d(x_{b-1},\{y,z\}) \leq 2p-2+q-p \\ +2b+2f+2-2q+n-2-b-f-1+1 = n+p-q+b+f-2. \end{aligned} \tag{29}$$

Now we divide this case into the following subcases.

Subcase 1.1. *The vertices x_{b-1} and y are not adjacent.*

Then $\{y,x_{b-1}\}$ and $\{y,z\}$ are two distinct pairs of non-adjacent vertices. Since $p \geq 2$, $r \leq n-2$, $f \geq 0$ and $k \geq q$, combining (25), (24) and (29), we obtain

$$\begin{aligned} d(y)+d(z)+d(y)+d(x_{b-1}) \leq n+1+q-p+r-b-f+q+r+1-p-k-f \\ +n+p-q+b+f-2 = 2n+2r+q-p-k-f \leq 4n-4-(k-q)-f-p, \end{aligned}$$

which contradicts condition (M).

Subcase 1.2. *The vertices x_{b-1} and y are adjacent.*

Then $x_{b-1} \to y$. Therefore by Lemma 3.5 and (19), the vertices z and x_{b-1} are not adjacent. Since $d(z) \leq n-2$ (because of $d(z,\{y,x_{b-1}\}) = 0$ and Corollary 3.7) and $r \leq n-2$, from (25) and (29) it follows that

$$\begin{aligned} d(y)+d(z)+d(x_{b-1})+d(z) \leq n+1+q-p+r-b-f+n+p-q+b+f-2+n-2 \\ = 3n-3+r \leq 4n-5, \end{aligned}$$

which contradicts condition (M). The discussion of Case 1 is completed.

Case 2. $b = k+1$.

We divide this case into the following subcases.

Subcase 2.1. $s \leq k-1$.

Then $k \geq q+1$ since $s \geq q$. Then $yx_k \notin A(D)$ by the definition of q and k. Recall that the vertices z, x_k are not adjacent by (19) and Lemma 3.5. Now it is easy to see that $d(z) \leq n-2$. Since $x_s \to x_t$ with $s \in [q, k-1]$ and $t \in [b+1, n-2]$, by Claim 5.2 we have that $d^-(x_k, \{x_1, \ldots, x_{q-1}\}) = 0$. This together with (27) and $b = k+1$ implies that

$$d(x_k) = d^+(x_k, \{x_1, \ldots, x_{q-1}\}) + d(x_k, \{x_q, \ldots, x_{b+f+1}\}) + d^-(x_k, \{x_{b+f+2}, \ldots, x_{n-2}\})$$
$$+d^+(x_k, \{y\}) \leq q-1+2b+2f+2-2q+n-2-b-f-1+1 = n+k-q+f.$$

This together with (24) and $d(z) \leq n-2$, we obtain

$$d(y) + d(x_k) + 2d(z) \leq q+r+1-p-k-f+n+k-q+f+2n-4$$
$$= 3n+r-p-3 \leq 4n-5-p,$$

which is a contradiction since $\{y, z\}$ and $\{x_k, z\}$ are two distinct pairs of non-adjacent vertices.

Subcase 2.2. $s = k$.

From $b = k+1$, $t \in [b+1 = k+2, b+f+1]$ and (23) it follows that

$$d(y, \{x_{k+1}, \ldots, x_{t-1}\}) = 0, \tag{30}$$

in particular, the vertices y and x_{k+1} are not adjacent. Observe that $R := yx_p \ldots x_k x_t \ldots$ $x_r y$ is a cycle in D passing through y, avoiding z and $d(y, V(D) \setminus V(R)) = 0$. By Lemma 4.1, the induced subdigraph $D\langle V(D) \setminus V(R) \rangle$ contains no cycle through z. In particular, this means that

$$A(\{x_{k+1}, \ldots, x_{t-1}\} \to \{x_{r+1}, \ldots, x_{n-2}\}) = \emptyset, \text{ hence } d^+(x_{k+1}, \{x_{r+1}, \ldots, x_{n-2}\}) = 0, \tag{31}$$

for otherwise, if $x_i \to x_j$ with $i \in [k+1, t-1]$ and $j \in [r+1, n-2]$, then $H := x_1 \ldots x_a x_{k+1} \ldots x_i x_j$ $\ldots x_{n-2} z x_1$ is a cycle in $D\langle V(D) \setminus V(R) \rangle$ through z, a contradiction.

Subcase 2.2.1. There is an integer $l \in [b+f+2, n-2]$ such that $x_{k+1} \to x_l$ and

$$d^+(x_{k+1}, \{x_{l+1}, \ldots, x_{n-2}\}) = 0. \tag{32}$$

Then $b+f+2 \leq n-2$, and $l \leq r$ because of the first equality of (31). Recall that $t \leq b+f+1 \leq l-1$. Hence, $l \geq t+1$. If $x_i \to z$ with $i \in [t, l-1]$, then $C(y, z) = x_1 \ldots x_a x_{k+1} x_l \ldots x_r y x_q \ldots x_k x_t \ldots x_i z x_1$, a contradiction. We may therefore assume that $d^-(z, \{x_t, \ldots, x_{l-1}\}) = 0$. This together with $d^+(y, \{x_t, \ldots, x_{l-1}\}) = 0$ and the fact that there is no path of length two between y and z implies that

$$d(y, \{x_t, \ldots, x_{l-1}\}) + d(z, \{x_t, \ldots, x_{l-1}\}) \leq l-t.$$

Combining this, (10) and (30), we obtain

$$d(y) + d(z) = d^+(y, \{x_p, \ldots, x_q\}) + d^-(y, \{x_k\}) + d(y, \{x_t, \ldots, x_{l-1}\}) + d(z, \{x_t, \ldots, x_{l-1}\})$$

$$+d^-(y,\{x_l,\dots,x_r\})+d(z,\{x_1,\dots,x_{t-1}\})+d(z,\{x_l,\dots,x_{n-2}\})$$
$$\leq q-p+1+1+l-t+r-l+1+t-1+n-2-l+2$$
$$\leq n+2+q+r-p-l. \tag{33}$$

For the vertex x_{k+1}, using (32) and the second equality of (22), we obtain

$$d(x_{k+1})=d(x_{k+1},\{x_1,\dots,x_{p-1}\})+d^+(x_{k+1},\{x_p,\dots,x_{q-1}\})+d(x_{k+1},\{x_q,\dots,x_l\})$$
$$+d^-(x_{k+1},\{x_{l+1},\dots,x_{n-2}\})+d(x_{k+1},\{z\})$$
$$\leq 2p-2+q-p+2l-2q+n-2-l+2=n-2+p-q+l.$$

This together with (33), (24), $r\leq n-2$, $k\geq q$ and $p\geq 2$ implies that

$$d(y)+d(z)+d(y)+d(x_{k+1})\leq n+2+q+r-p-l+q+r+1-p-k-f+n-2+p-q+l$$
$$=2n+1+q+2r-p-k-f\leq 4n-3-(k-q)-p-f\leq 4n-5,$$

which contradicts condition (M) since $\{y,z\}$ and $\{y,x_{k+1}\}$ are two distinct pairs of non-adjacent vertices.

Subcase 2.2.2. There is no $l\in[b+f+2,n-2]$ such that $x_{k+1}\to x_l$.

Then $d^+(x_{k+1},\{x_{b+f+2},\dots,x_{n-2}\})=0$. This together with the second equality of (22) implies that

$$d(x_{k+1})=d(x_{k+1},\{x_1,\dots,x_{p-1}\})+d^+(x_{k+1},\{x_p,\dots,x_{q-1}\})$$
$$+d(x_{k+1},\{x_q,\dots,x_{b+f+1}\})+d^-(x_{k+1},\{x_{b+f+2},\dots,x_{n-2}\})+d(x_{k+1},\{z\})$$
$$\leq 2p-2+q-p+2b+2f+2-2q+n-2-b-f-1+2$$
$$=n-1+p-q+b+f.$$

Combining this, $b=k+1$, (24) and $d(z)\leq n-2$, we obtain

$$2d(y)+d(x_{k+1})+d(z)\leq 2q+2r+2-2p-2k-2f+n-1+p-q+b+f$$
$$+n-2=2n+q+2r-p-k-f\leq 4n-4-(k-q)-p-f,$$

which contradicts condition (M). In each case we obtain a contradiction and hence the discussion of Case 2 is completed. This completes the proof of Claim 5.4. □

Now we are ready to complete the proof of the main result.

By Claim 5.4, if $p\geq 2$, then $A(\{x_1,\dots,x_{p-1}\}\to\{x_{k+1},\dots,x_{n-2}\})=\emptyset$. Similarly, if $r\leq n-3$, then $A(\{x_1,\dots,x_{q-1}\}\to\{x_{r+1},\dots,x_{n-2}\})=\emptyset$. Using Lemma 4.3, we obtain $A(\{x_p,\dots,x_{q-1}\}\to\{x_{k+1},\dots,x_r\})=\emptyset$. From the last three equalities it follows that

$$A(\{x_1,\dots,x_{q-1}\}\to\{x_{k+1},\dots,x_{n-2}\})=\emptyset. \tag{34}$$

From (34) and Lemma 4.2 it follows that $k \geq q+1$. Applying Lemma 4.2 to the vertices x_q and x_k, we obtain

$$A(\{x_1,\dots,x_{q-1}\} \to \{x_{q+1},\dots,x_{n-2}\}) \neq \emptyset,\ A(\{x_1,\dots,x_{k-1}\} \to \{x_{k+1},\dots,x_{n-2}\}) \neq \emptyset.$$

Let $x_a \to x_b$ and $x_h \to x_l$ with $a \in [1, q-1]$, $b \in [q+1, n-2]$, $h \in [1, k-1]$ and $l \in [k+1, n-2]$. Choose b maximal and h minimal with these properties, i.e.,

$$A(\{x_1,\dots,x_{q-1}\} \to \{x_{b+1},\dots,x_{n-2}\}) = A(\{x_1,\dots,x_{h-1}\} \to \{x_{k+1},\dots,x_{n-2}\}) = \emptyset. \quad (35)$$

From (34) it follows that $b \leq k$ and $h \geq q$, i.e., $b \in [q+1, k]$ and $h \in [q, k-1]$. If $h \leq b-1$, then $C(y,z) = x_1 \dots x_a x_b \dots x_k y x_q \dots x_h x_l \dots x_{n-2} z x_1$, a contradiction. We may therefore assume that $h \geq b$, which in turn implies that $k \geq q+2$. By Lemma 4.2, $A(\{x_1,\dots,x_{b-1}\} \to \{x_{b+1},\dots,x_{n-2}\}) \neq \emptyset$. Let $x_s \to x_t$, where $s \in [1, b-1]$ and $t \in [b+1, n-2]$. Choose t maximal with this property, i.e.,

$$A(\{x_1,\dots,x_{b-1}\} \to \{x_{t+1},\dots,x_{n-2}\}) = \emptyset. \quad (36)$$

From (35) it follows that $s \geq q$ and $t \leq k$, i.e., $s \in [q, b-1]$ and $t \in [b+1, k]$. We may assume that l (recall that $x_h \to x_l$, $l \geq k+1$) is chosen so that

$$d^+(x_h, \{x_{k+1},\dots,x_{l-1}\}) = 0. \quad (37)$$

We consider the cases $l \leq r$ and $l \geq r+1$ separately.

Case 1. $l \leq r$.

For this case, it is not difficult to check that the conditions of Claim 5.3 hold. Therefore, there is an integer $f \geq 0$ such that $l+f \leq r$, $x_{l+f} \to y$, $d(y, \{x_l,\dots,x_{l+f-1}\}) = 0$ (possibly, $\{x_l,\dots,x_{l+f-1}\} = \emptyset$), and either there is a vertex x_g with $g \in [l+f+1, n-2]$ such that $x_k \to x_g$ or there is a vertex x_c with $c \in [k, l-1]$ such that $x_c \to z$.

Assume first that $t \geq h+1$. Then, since the arcs yx_q, $x_a x_b$, $x_s x_t$, $x_h x_l$, $x_k y$, $x_{l+f} y$ are in D and $1 \leq a \leq q-1 < s < b \leq h < t \leq k < l \leq l+f \leq r \leq n-2$, we have that $C(y,z) = x_1 \dots x_a x_b \dots x_h x_l \dots x_{l+f} y x_q \dots x_s x_t \dots x_c z x_1$, or $C(y,z) = x_1 \dots x_a x_b \dots x_h x_l \dots x_{l+f} y x_q \dots x_s x_t \dots x_k x_g \dots x_{n-2} z x_1$ when $x_c \to z$ or when $x_k \to x_g$ respectively. In each case we have a contradiction.

Assume next that $t \leq h$. By Lemma 4.2, $A(\{x_1,\dots,x_{t-1}\} \to \{x_{t+1},\dots,x_{n-2}\}) \neq \emptyset$. Let $x_{s_1} \to x_{t_1}$, where $s_1 \in [1, t-1]$ and $t_1 \in [t+1, n-2]$. Choose t_1 maximal with this property, i.e.,

$$A(\{x_1,\dots,x_{t-1}\} \to \{x_{t_1+1},\dots,x_{n-2}\}) = \emptyset. \quad (38)$$

From (36) (respectively, from (35)) it follows that $s_1 \geq b$, i.e., $s_1 \in [b, t-1]$ (respectively, $t_1 \leq k$, i.e., $t_1 \in [t+1, k]$). If $t_1 \geq h+1$, then $C(y,z) = x_1 \dots x_a x_b \dots x_{s_1} x_{t_1} \dots x_k y x_q \dots x_s x_t \dots x_h x_l \dots x_{n-2} z x_1$, a contradiction. We may therefore assume that $t_1 \leq h$. By Lemma 4.2,

$$A(\{x_1,\dots,x_{t_1-1}\} \to \{x_{t_1+1},\dots,x_{n-2}\}) \neq \emptyset.$$

Let $x_{s_2} \to x_{t_2}$, where $s_2 \in [1, t_1-1]$ and $t_2 \in [t_1+1, n-2]$. Choose t_2 maximal with this property, i.e.,

$$A(\{x_1,\dots,x_{t_1-1}\} \to \{x_{t_2+1},\dots,x_{n-2}\}) = \emptyset.$$

From (38) (respectively, from (35)) it follows that $s_2 \geq t$, i.e., $s_2 \in [t, t_1 - 1]$ (respectively, $t_2 \leq k$, i.e., $t_2 \in [t_1 + 1, k]$).

Assume first that $t_2 \geq h + 1$. Then it is not difficult to see that $C(y, z) = x_1 \dots x_a x_b \dots x_{s_1} x_{t_1} \dots x_h x_l \dots x_{l+f} y x_q \dots x_s x_t \dots x_{s_2} x_{t_2} \dots x_c z x_1$ or $C(y, z) = x_1 \dots x_a x_b \dots x_{s_1} x_{t_1} \dots x_h x_l \dots x_{l+f} y x_q \dots x_s x_t \dots x_{s_2} x_{t_2} \dots x_k x_g \dots x_{n-2} z x_1$ when $x_c \to z$ or when $x_k \to x_g$, respectively. In each case we have a contradiction.

Continuing this process, we finally conclude that for some $m \geq 0$, $t_m \in [h + 1, k]$ (here, $t_0 = t$) since all the vertices $x_t, x_{t_1}, \dots, x_{t_m}$ are distinct and in $\{x_{q+1}, \dots, x_k\}$. We already have constructed a cycle $C(y, z)$ when $m \in \{0, 1, 2\}$. Assume that $m \geq 3$. By the above arguments we have that:

If $m \geq 3$ is odd, then $C(y, z) = x_1 \dots x_a x_b \dots x_{s_1} x_{t_1} \dots x_{s_m} x_{t_m} \dots x_k y x_q \dots x_s x_t \dots x_{s_2} x_{t_2} \dots x_{s_{m-1}} x_{t_{m-1}} \dots x_h x_l \dots x_{n-2} z x_1$.

If $m \geq 4$ is even, then $C(y, z) = x_1 \dots x_a x_b \dots x_{s_1} x_{t_1} \dots x_{s_{m-1}} x_{t_{m-1}} \dots x_h x_l \dots x_{l+f} y x_q \dots x_s x_t \dots x_{s_2} x_{t_2} \dots x_{s_m} x_{t_m} \dots x_c z x_1$ or $C(y, z) = x_1 \dots x_a x_b \dots x_{s_1} x_{t_1} \dots x_{s_{m-1}} x_{t_{m-1}} \dots x_h x_l \dots x_{l+f} y x_q \dots x_s x_t \dots x_{s_2} x_{t_2} \dots x_{s_m} x_{t_m} \dots x_k x_g \dots x_{n-2} z x_1$ when $x_c \to z$ or when $x_k \to x_g$, respectively. In all cases we have a cycle through y and z, which contradicts our supposition and hence the discussion of Case 1 is completed.

Case 2. $l \geq r + 1$.

Then $r \leq n - 3$. Recall that $h \in [b, k - 1]$, $x_h \to x_l$ and $x_s \to x_t$, where $l \leq n - 2$, $s \in [q, b - 1]$ and $t \in [b + 1, k]$. Note that $\{y, x_h\}$, $\{y, z\}$ are two distinct pairs of non-adjacent vertices.

Subcase 2.1. $t \geq h + 1$.

Since $s \in [q, b - 1]$ and $t \in [h + 1, k]$, we have that $Q := y x_p \dots x_s x_t \dots x_r y$ is a cycle in D and $d(y, V(D) \setminus V(Q)) = 0$. If $a \leq p - 1$, then $H := x_1 \dots x_a x_b \dots x_h x_l \dots x_{n-2} z x_1$ is a cycle in $D\langle V(D) \setminus V(Q)\rangle$ passing through z, which contradicts Lemma 4.1. We may therefore assume that $a \geq p$, i.e., $a \in [p, q - 1]$.

Assume first that $b \leq h - 1$. Then $q + 1 \leq b \leq h - 1 \leq k - 2$ and $k \geq q + 3$. From the first equality of (35) it follows that $d^-(x_h, \{x_1, \dots, x_{q-1}\}) = 0$. This equality together with (37) implies that

$$d(x_h) = d^+(x_h, \{x_1, \dots, x_{q-1}\}) + d(x_h, \{x_q, \dots, x_k\}) + d^-(x_h, \{x_{k+1}, \dots, x_{l-1}\})$$

$$+d(x_h, \{x_l, \dots, x_{n-2}\}) + d(x_h, \{z\}) \leq q - 1 + 2k - 2q + l - 1 - k + 2n - 2l - 2 + 2$$

$$= 2n - 2 - q + k - l.$$

This together with (11) and $d(z) \leq n - 1$ implies that

$$2d(y) + d(x_h) + d(z) \leq 2q - 2p + 2r - 2k + 4 + 2n - 2 - q + k - l + n - 1$$

$$\leq 4n - 2 + (r - l) + (q - k) - 2p,$$

which contradicts condition (M).

Assume that $b = h$, i.e., $x_a \to x_h$. We may assume that a is chosen so that $d^-(x_h, \{x_1, \dots, x_{a-1}\}) = 0$. This and (37) imply that

$$d(x_h) = d^+(x_h, \{x_1, \dots, x_{a-1}\}) + d(x_h, \{x_a, \dots, x_k\}) + d^-(x_h, \{x_{k+1}, \dots, x_{l-1}\})$$

$$+d(x_h, \{x_l, \ldots, x_{n-2}\}) + d(x_h, \{z\}) \le a - 1 + 2k - 2a + l - 1 - k + 2n - 2l - 2 + 2$$
$$= 2n - 2 - a + k - l. \quad (39)$$

Since $a \ge p$, it is not difficult to check that if $z \to x_i$ with $i \in [a+1, s]$, then $C(y,z) = yx_p \ldots x_a x_h x_l \ldots x_{n-2} z x_i \ldots x_s x_t \ldots x_k y$, which is a contradiction. Therefore, $d^+(z, \{x_{a+1}, \ldots, x_s\}) = 0$. This together with $d^-(y, \{x_{a+1}, \ldots, x_s\}) = 0$ and the fact that there is no path of length two between y and z implies that

$$d(y, \{x_{a+1}, \ldots, x_s\}) + d(z, \{x_{a+1}, \ldots, x_s\}) \le s - a.$$

Using this and (10), we obtain

$$d(y) + d(z) = d^+(y, \{x_p, \ldots, x_a\}) + d(y, \{x_{a+1}, \ldots, x_s\}) + d(z, \{x_{a+1}, \ldots, x_s\})$$
$$+d^-(y, \{x_k, \ldots, x_r\}) + d(z, \{x_1, \ldots, x_a\}) + d(z, \{x_{s+1}, \ldots, x_{n-2}\})$$
$$\le a - p + 1 + s - a + r - k + 1 + a + n - 2 - s + 1 = n + 1 + a - p + r - k.$$

Combining this, (11) and (39), we obtain

$$2d(y) + d(z) + d(x_h)$$
$$\le 3n + 1 + 2r - 2p + q - l - k \le 4n - 2 - (l - r) - (k - q) - 2p < 4n - 6,$$

which contradicts condition (M) and hence the discussion of Subcase 2.1 is completed.

Subcase 2.2. $t \le h$.

Then $b \le h - 1$ since $h \ge t \ge b + 1$.

Assume first that $t = h$. Then $x_s \to x_h \to x_l$. By Lemma 4.2,

$$A(\{x_1, \ldots, x_{h-1}\} \to \{x_{h+1}, \ldots, x_{n-2}\}) \ne \emptyset.$$

Let $x_i \to x_j$, where $i \in [1, h-1]$ and $j \in [h+1, n-2]$. From the second equality of (35) it follows that $j \le k$, i.e., $j \in [h+1, k]$. By (36) we have that $i \ge b$, i.e., $i \in [b, h-1]$. Therefore, $C(y,z) = x_1 \ldots x_a x_b \ldots x_i x_j \ldots x_k y x_q \ldots x_s x_h x_l \ldots x_{n-2} z x_1$, a contradiction.

Assume next that $t \le h-1$. From the maximality of b and t it follows that $d^-(x_h, \{x_1, \ldots, x_{b-1}\}) = 0$. This last equality together with (37) implies that

$$d(x_h) = d^+(x_h, \{x_1, \ldots, x_{b-1}\}) + d(x_h, \{x_b, \ldots, x_k\}) + d^-(x_h, \{x_{k+1}, \ldots, x_{l-1}\})$$
$$+d(x_h, \{x_l, \ldots, x_{n-2}\}) + d(x_h, \{z\}) \le b - 1 + 2k - 2b + l - 1 - k + 2n - 2l - 2 + 2$$
$$= 2n - l - 2 + k - b.$$

This together with (11), $d(z) \le n - 1$ and $r \le n - 3$ implies that

$$2d(y) + d(x_h) + d(z) \le 2q - 2p + 2r - 2k + 4 + 2n - l - 2 + k - b + n - 1$$
$$\le 4n - 2 - (l - r) - (k - q) - (b - q) - 2p,$$

which contradicts condition (M), since $k - q \ge 0$, $b - q \ge 1$. The discussion of Case 2 is completed. Theorem 1.12 is proved. □

Acknowledgements

The author would like to thank the anonymous referees for thoroughly review and many helpful comments and suggestions which improved substantially the rewriting of this paper. We also thank PhD P. Hakobyan for formatting the manuscript of this paper.

References

J. Bang-Jensen and G. Gutin. *Digraphs: Theory, Algorithms and Applications*, Springer-Verlag, London, 2000.

A. Benhocine. Pancyclism and Meyniel's conditions. *Discrete Math.*, 58:113–120, 1986.

J.-C. Bermond and C. Thomassen. Cycles in digraphs − a survey. *J. Graph Theory*, 5(1):1–43, 1981.

J. Bondy and C. Thomassen. A short proof of Meyniel's theorem. *Discrete Math.*, 19:195–197, 1977.

S. Darbinyan. On Hamiltonian bypasses in digraphs with the condition of Y. Manoussakis. In *Procceedings Conference Computer Science and Information Tecnologies, (CSIT 2015)*. Yerevan, 2015, pp. 53-63. doi: 10.1109/CSITechnol.2015.7358250.

S. Darbinyan. On pancyclic digraphs. *Preprint of the Computing Center of Akademy of Sciences of Armenia*, 1979.

S. Darbinyan. Disproof of a conjecture of Thomassen. *Akad. Nauk Armyan. SSR Dokl.*, 76(2):51–54, 1983.

S. Darbinyan. Pancyclicity of digraphs with the Meyniel condition. *Studia Sci. Math. Hungar., (Ph.D. Thesis, Institute Mathematici Akad. Nauk BSSR, Minsk, 1981)*, 20(1-4):95–117, 1985.

S. Darbinyan. Hamiltonian and strongly Hamilton-connected digraphs. *Akad. Nauk Armyan. SSR Dokl., (for a detailed proof see arXiv: 1801.05166v1, 16 Jan. 2018)*, 91(1):3–6, 1990.

S. Darbinyan. On cyclability of digraphs with a Manoussakis'-type condition. *Transactions of IIAP NAS RA, Mathematical Problems of Computer Science*, 47:15–29, 2017.

S. Darbinyan. Some remarks on Manoussakis conjecture for a digraph to be hamiltonian. *Emil Artin International Conference, Yerevan, Armenia, May 27-June 2*, pages 39–40, 2018.

S. Darbinyan. On the Manoussakis conjecture for a digraph to be hamiltonian. *Transactions of IIAP NAS RA, Mathematical Problems of Computer Science*, 51:21–38, 2019.

S. Darbinyan and I. Karapetyan. On pre-Hamiltonian cycles in Hamiltonian digraphs. *Transactions of IIAP NAS RA, Mathematical Problems of Computer Science*, 43:5–25, 2015.

A. Ghouila-Houri. Une condition suffisante d'existence d'un circuit hamiltonien. *C. R. Acad. Sci. Paris Ser. A-B*, 251:495–497, 1960.

R. Häggkvist and C. Thomassen. On pancyclic digraphs. *J. Combin. Theory Ser.B*, 20:20–40, 1976.

F. Harary and L. Moser. The theory of round robin tournaments. *Amer. Math. Monthly*, 73:231–246, 1966.

D. Kühn and D. Osthus. A survey on Hamilton cycles in directed graphs. *European J. Combin.*, 33: 750–766, 2012.

H. Li, E. Flandrin, and J. Shu. A sufficient condition for cyclability of directed graphs. *Discrete math.*, 307:1291–1297, 2007.

Y. Manoussakis. Directed hamiltonian graphs. *J. Graph Theory*, 16(1):51–59, 1992.

M. Meyniel. Une condition suffisante d'existence d'un circuit hamiltonien dans un graphe oriente. *J. Combin. Theory Ser.B*, 14:137–147, 1973.

C. S. J. A. Nash-Williams. Hamilton circuits in graphs and digraphs, the many facets of graph theory. *Springer Verlag Lecture Notes*, 110:237–243, 1969.

B. Ning. Notes on a conjecture of Manoussakis concerning Hamilton cycles in digraphs. *Information Processing Letters*, 115:221–224, 2015.

O. Ore. Note on Hamilton circuits. *Amer. Math. Monthly*, 67:55, 1960.

D. Woodall. Sufficient condition for circuits in graphs. *Proc. London Math. Soc.*, 24:739–755, 1972.

A. Yeo. How close to regular must a semicomplete multipartite digraph to be secure hamiltonicity? *Graphs Combin.*, 15:481–493, 1999.