

Enumeration of Stack-Sorting Preimages via a Decomposition Lemma

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We give three applications of a recently-proven “Decomposition Lemma,” which allows one to count preimages of certain sets of permutations under West’s stack-sorting map s . We first enumerate the permutation class

$$s^{-1}(\text{Av}(231, 321)) = \text{Av}(2341, 3241, 45231),$$

finding a new example of an unbalanced Wilf equivalence. This result is equivalent to the enumeration of permutations sortable by $\mathbf{B} \circ s$, where \mathbf{B} is the bubble sort map. We then prove that the sets $s^{-1}(\text{Av}(231, 312))$, $s^{-1}(\text{Av}(132, 231)) = \text{Av}(2341, 1342, \underline{3241}, \underline{3142})$, and $s^{-1}(\text{Av}(132, 312)) = \text{Av}(1342, 3142, 3412, \underline{3421})$ are counted by the so-called “Boolean-Catalan numbers,” settling a conjecture of the current author and another conjecture of Hossain. This completes the enumerations of all sets of the form $s^{-1}(\text{Av}(\tau^{(1)}, \dots, \tau^{(r)}))$ for $\{\tau^{(1)}, \dots, \tau^{(r)}\} \subseteq S_3$ with the exception of the set $\{321\}$. We also find an explicit formula for $|s^{-1}(\text{Av}_{n,k}(231, 312, 321))|$, where $\text{Av}_{n,k}(231, 312, 321)$ is the set of permutations in $\text{Av}_n(231, 312, 321)$ with k descents. This allows us to prove a conjectured identity involving Catalan numbers and order ideals in Young’s lattice.

Keywords: Permutation pattern; stack-sorting; bubble sort; Pop-stack-sorting; unbalanced Wilf equivalence.

1 Introduction

1.1 Background

We use the word “permutation” to refer to an ordering of a set of positive integers written in one-line notation. Let S_n denote the set of permutations of the set $[n]$. If π is a permutation of length n , then the *standardization* of π is the permutation in S_n obtained by replacing the i^{th} -smallest entry in π with i for all i . We say a permutation is *standardized* if it is equal to its standardization.

Definition 1.1. Given $\tau \in S_m$, we say a permutation $\sigma = \sigma_1 \cdots \sigma_n$ *contains the pattern* τ if there exist indices $i_1 < \cdots < i_m$ in $[n]$ such that the standardization of $\sigma_{i_1} \cdots \sigma_{i_m}$ is τ . We say σ *avoids* τ if it does not contain τ . Let $\text{Av}(\tau^{(1)}, \tau^{(2)}, \dots)$ denote the set of standardized permutations that avoid the patterns $\tau^{(1)}, \tau^{(2)}, \dots$ (this list of patterns could be finite or infinite). A set of the form $\text{Av}(\tau^{(1)}, \tau^{(2)}, \dots)$ is called a *permutation class*. Let $\text{Av}_n(\tau^{(1)}, \tau^{(2)}, \dots) = \text{Av}(\tau^{(1)}, \tau^{(2)}, \dots) \cap S_n$.

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The investigation of permutation patterns began with Knuth’s analysis of a certain “stack-sorting algorithm” [38]. This analysis also marked the first use of what is now called the “kernel method,” a powerful enumerative tool that we employ throughout this article (see [4, 12, 13, 42] for more information about this technique). In his dissertation, West [47] defined a deterministic variant of Knuth’s algorithm. This variant is a function, which we call the “stack-sorting map” and denote by s , that sends permutations to permutations. The map s sends the empty permutation to itself. To define $s(\pi)$ when π is a nonempty permutation, we write $\pi = LnR$, where n is the largest entry in π . We then let $s(\pi) = s(L)s(R)n$. For example,

$$s(14253) = s(142) s(3) 5 = s(1) s(2) 4 3 5 = 12435.$$

There is now a large amount of literature concerning the stack-sorting map [7–11, 14–18, 21–35, 40, 43, 46–48].

We say a permutation π is t -stack-sortable if $s^t(\pi)$ is an increasing permutation, where s^t denotes the t -fold iterate of s . Let $\mathcal{W}_t(n)$ be the set of t -stack-sortable permutations in S_n , and let $W_t(n) = |\mathcal{W}_t(n)|$. Knuth initiated the study of pattern avoidance and the study of stack-sorting (and introduced the kernel method) with the following theorem.

Theorem 1.1 ([38]). *A permutation is 1-stack-sortable if and only if it avoids the pattern 231. Furthermore, $W_1(n) = |\text{Av}_n(231)| = C_n$, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n^{th} Catalan number.*

In his dissertation, West conjectured that $W_2(n) = \frac{2}{(n+1)(2n+1)} \binom{3n}{n}$. Zeilberger [48] later proved this formula, and other proofs have emerged over the past few decades [22, 24, 32–35]. Some authors have investigated the enumeration of 2-stack-sortable permutations according to various statistics [8, 14, 16, 24, 32]. There is little known about t -stack-sortable permutations when $t \geq 3$. The best known upper bound in general is given by the estimate $W_t(n) \leq (t+1)^{2n}$. The current author [28] proved that $\lim_{n \rightarrow \infty} W_3(n)^{1/n} < 12.53296$ and $\lim_{n \rightarrow \infty} W_4(n)^{1/n} < 21.97225$. Recently, Bóna [9] reproved the first of these two estimates.

Even more recently, the current author [24] proved a certain “Decomposition Lemma” and used it to give a new proof of Zeilberger’s formula for $W_2(n)$ that generalizes in order to count 2-stack-sortable permutations according to their number of descents and number of peaks. The proof also generalizes to the setting of 3-stack-sortable permutations, yielding a recurrence that generates the numbers $W_3(n)$ (more generally, it gives a recurrence for $W_3(n, k, p)$, the number of 3-stack-sortable permutations of length n with k descents and p peaks). This allowed the author to shed light on several conjectures of Bóna concerning 3-stack-sortable permutations, disproving one of them.

West [47] defined the *fertility* of a permutation π to be $|s^{-1}(\pi)|$, the number of preimages of π under s . He then went through a great deal of effort to compute the fertilities of the permutations of the forms

$$23 \cdots k1(k+1) \cdots n, \quad 12 \cdots (k-2)k(k-1)(k+1) \cdots n, \quad \text{and} \quad k12 \cdots (k-1)(k+1) \cdots n.$$

The fact that these permutations are of such specific forms indicates the difficulty of computing fertilities. Bousquet-Mélou [15] found a method for determining whether or not a given permutation is *sorted*, meaning that its fertility is positive. She asked for a general method for computing the fertility of any given permutation. The current author achieved this in even greater generality in [27–29] using new combinatorial objects called “valid hook configurations.” He and others have applied this method in order to prove several results about fertilities of permutations, many of which link the stack-sorting map to other fascinating combinatorial objects and sequences [23–31, 40, 43].

Define the fertility of a set A of permutations to be $|s^{-1}(A)|$. In [29], the current author computed the fertilities of many sets of the form $\text{Av}_n(\tau^{(1)}, \dots, \tau^{(r)})$ for $\tau^{(1)}, \dots, \tau^{(r)} \in S_3$. He also refined these enumerative results according to the statistics that count descents and peaks. Investigations of preimages of permutation classes appeared earlier in [16] and [21] and appeared more recently in [26]; this line of work is motivated by the observations that $\mathcal{W}_1(n) = s^{-1}(\text{Av}_n(21))$ and $\mathcal{W}_2(n) = s^{-1}(\text{Av}_n(231))$. Further motivation comes from the fact that preimages of permutation classes under s are often themselves permutation classes. For example, $s^{-1}(\text{Av}(321)) = \text{Av}(34251, 35241, 45231)$. Our goal in the present article is to compute fertilities of some sets of permutations that were not completed in [29]. We do this via the Decomposition Lemma, which, when combined with generating function tools such as the kernel method, gives a unified technique for computing fertilities.

1.2 Summary of Main Results

Section 2 is brief; its purpose is simply to state the Decomposition Lemma. In Section 3, we use the Decomposition Lemma to prove that

$$\sum_{n \geq 0} |s^{-1}(\text{Av}_n(231, 321))| x^n = \frac{1}{1 - xC(xC(x))},$$

where $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$ is the generating function of the Catalan numbers. This proves Conjecture 12.3 from [29]. This is notable because

$$s^{-1}(\text{Av}(231, 321)) = \text{Av}(2341, 3241, 45231).$$

Callan [20] proved that $\frac{1}{1 - xC(xC(x))}$ is the generating function of the permutation class $\text{Av}(4321, 4213)$, and Bloom and Vatter [6] later reproved this result with a clearer bijection. The authors of [39] showed that this expression is also the generating function for $\text{Av}(2413, 2431, 23154)$, which means that $\text{Av}(4321, 4213)$ and $\text{Av}(2413, 2431, 23154)$ form a pair of “unbalanced” Wilf equivalent permutation classes. The first naturally-occurring unbalanced Wilf equivalent pairs of finitely-based⁽ⁱ⁾ permutation classes were found only recently [5, 19]. Our theorem proves that $\text{Av}(2341, 3241, 45231)$ is also Wilf equivalent to $\text{Av}(4321, 4213)$ and $\text{Av}(2413, 2431, 23154)$, providing yet another example of a pair of unbalanced Wilf equivalent finitely-based classes.

Another motivation for the results of Section 3 comes from the study of the *bubble sort map* \mathbf{B} . This map sends the empty permutation to itself. If $\pi = LnR$, where n is the largest entry of π , then $\mathbf{B}(\pi) = \mathbf{B}(L)Rn$. The article [1] investigates permutations sortable via $s \circ \mathbf{B}$, showing that

$$(s \circ \mathbf{B})^{-1}(123 \cdots n) = \text{Av}_n(3241, 2341, 4231, 2431).$$

Therefore, it is natural to ask about permutations that are sortable by $\mathbf{B} \circ s$. It is known (see [1]) that $\mathbf{B}^{-1}(123 \cdots n) = \text{Av}_n(231, 321)$, so it follows from the main result of Section 3 that

$$(\mathbf{B} \circ s)^{-1}(123 \cdots n) = \text{Av}_n(2341, 3241, 45231)$$

⁽ⁱ⁾ We say a permutation class is *finitely-based* if it is of the form $\text{Av}(\tau^{(1)}, \dots, \tau^{(r)})$, where the list $\tau^{(1)}, \dots, \tau^{(r)}$ is finite.

and that

$$\sum_{n \geq 0} |(\mathbf{B} \circ s)^{-1}(123 \cdots n)|x^n = \frac{1}{1 - xC(xC(x))}.$$

In Section 4, we consider the chain of equalities

$$\begin{aligned} \sum_{n \geq 1} |s^{-1}(\text{Av}_n(132, 312))|x^n &= \sum_{n \geq 1} |s^{-1}(\text{Av}_n(231, 312))|x^n = \sum_{n \geq 1} |s^{-1}(\text{Av}_n(132, 231))|x^n \\ &= \frac{1 - 2x - \sqrt{1 - 4x - 4x^2}}{4x}. \end{aligned}$$

The first of these equalities was proven in [29], while the second was proven in [26]. We use the Decomposition Lemma to prove the third equality, completing the chain and proving a conjecture from [29]. Let us also remark that $s^{-1}(\text{Av}(231, 312))$ is the set of permutations that are sortable via one iteration of the map $\text{Pop} \circ s$, where Pop is the pop-stack-sorting map (see [2] and the references therein). Therefore, this theorem answers a deterministic variant of one of the problems considered in [44].

The expression $\frac{1 - 2x - \sqrt{1 - 4x - 4x^2}}{4x}$ is the generating function of OEIS sequence A071356 [41]. Hossain [36] has named these numbers ‘‘Boolean-Catalan numbers’’ and has discussed several interesting combinatorial properties that they enjoy. The above chain of inequalities is important because one can show that

$$s^{-1}(\text{Av}(132, 231)) = \text{Av}(2341, 1342, \underline{3241}, \underline{3142})$$

and

$$s^{-1}(\text{Av}(132, 312)) = \text{Av}(1342, 3142, 3412, \underline{3421}),$$

where the underlines are used to represent so-called ‘‘vincular patterns’’ (defined in Section 4). Therefore, we are able to use the stack-sorting map as a tool in order to enumerate sets of permutations avoiding certain vincular patterns. This is particularly noteworthy in the case of the set $s^{-1}(\text{Av}(132, 231)) = \text{Av}(2341, 1342, \underline{3241}, \underline{3142})$. Indeed, Hossain [37] *conjectured* that this set of permutations is enumerated by the Boolean-Catalan numbers. As far as we are aware, there is no known proof of this fact that does not make use of the stack-sorting map.

Applying the Decomposition Lemma yet again in Section 5, we prove that

$$|s^{-1}(\text{Av}_{n,k}(231, 312, 321))| = \frac{1}{n+1} \binom{n-k-1}{k} \binom{2n-2k}{n},$$

where $\text{Av}_{n,k}(231, 312, 321)$ is the set of permutations in $\text{Av}_n(231, 312, 321)$ with k descents. This yields a new proof of Theorem 6.2 from [29]. It also allows us to prove Conjecture 6.1 from that same article, which is an identity involving Catalan numbers and orderings on integer compositions and partitions.

There are 64 subsets $\{\tau^{(1)}, \dots, \tau^{(r)}\}$ of S_3 ; for each one, we have the problem of enumerating the permutations in $s^{-1}(\text{Av}(\tau^{(1)}, \dots, \tau^{(r)}))$. Many of these enumerations were accomplished in [16, 29, 38, 48]. The main results of Sections 3 and 4 below finish the last remaining cases with the single exception of the set $\{321\}$.

Remark 1.1. The permutation classes $\text{Av}(231, 321)$, $\text{Av}(132, 231)$, and $\text{Av}(231, 312, 321)$ have very simple descriptions, so one might think that the enumerations of their preimages under s could be computed directly and easily without additional tools. This is not the case. As mentioned above, even computing the fertilities of extremely simple permutations like $23 \cdots k1(k+1) \cdots n$ takes some work without the Decomposition Lemma or valid hook configurations. Even with the machinery of valid hook configurations, the author could not figure out how to compute these fertilities in the paper [29].

1.3 General Proof Strategy

The *tail length* of a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$, denoted $\text{tl}(\pi)$, is the largest integer $\ell \in \{0, \dots, n\}$ such that $\pi_i = i$ for all $i \in \{n - \ell + 1, \dots, n\}$. Our strategy for proving the results in Sections 3–5 is to use the Decomposition Lemma in order to count various sets of permutations according to the statistic tls defined by $\text{tls}(\pi) = \text{tl}(s(\pi))$. One interesting property of this statistic is that it is closely related to the statistic zeil that Zeilberger used when he counted 2-stack-sortable permutations. For $\sigma \in S_n$, $\text{zeil}(\sigma)$ is the largest positive integer m such that $n, n - 1, \dots, n - m + 1$ appear in decreasing order in σ . It is proven in [26] that

$$\text{zeil}(\sigma) = \min\{\text{rmax}(\sigma), \text{tls}(\sigma)\},$$

where $\text{rmax}(\sigma)$ is the number of right-to-left maxima of σ . Our enumerations according to the statistic tls will yield generating function equations involving catalytic variables that we can remove using the kernel method.

2 The Decomposition Lemma

A *descent* of a permutation $\pi = \pi_1 \cdots \pi_n$ is an index $i \in [n - 1]$ such that $\pi_i > \pi_{i+1}$. The *plot* of π is the diagram showing the points (i, π_i) for all $i \in [n]$. For example, the image on the left in Figure 1 is the plot of 31542678. A *hook* of a permutation is obtained by starting at a point (i, π_i) in the plot of π , drawing a vertical line segment moving upward, and then drawing a horizontal line segment to the right that connects with a point (j, π_j) . This only makes sense if $i < j$ and $\pi_i < \pi_j$. The point (i, π_i) is called the *southwest endpoint* of the hook, while (j, π_j) is called the *northeast endpoint*. Let $\text{SW}_i(\pi)$ be the set of hooks of π with southwest endpoint (i, π_i) . The right image in Figure 1 shows a hook of 31542678. This hook is in $\text{SW}_3(31542678)$ because its southwest endpoint is $(3, 5)$.



Fig. 1: The left image is the plot of 31542678. The right image shows this plot along with a single hook.

As mentioned above, the tail length $\text{tl}(\pi)$ of a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$ is the largest integer $\ell \in \{0, \dots, n\}$ such that $\pi_i = i$ for all $i \in \{n - \ell + 1, \dots, n\}$. The *tail* of π is the sequence of points

$(n - \text{tl}(\pi) + 1, n - \text{tl}(\pi) + 1), \dots, (n, n)$ in the plot of π . For example, the tail length of the permutation 31542678 shown in Figure 1 is 3, and the tail of this permutation is $(6, 6), (7, 7), (8, 8)$. We say a descent d of π is *tail-bound* if every hook in $\text{SW}_d(\pi)$ has its northeast endpoint in the tail of π . The tail-bound descents of 31542678 are 3 and 4.

Suppose H is a hook of a permutation $\pi = \pi_1 \cdots \pi_n$ with southwest endpoint (i, π_i) and northeast endpoint (j, π_j) . Let $\pi_U^H = \pi_1 \cdots \pi_i \pi_{j+1} \cdots \pi_n$ and $\pi_S^H = \pi_{i+1} \cdots \pi_{j-1}$. The permutations π_U^H and π_S^H are called the *H-unsheltered subpermutation* of π and the *H-sheltered subpermutation* of π , respectively. For example, if $\pi = 31542678$ and H is the hook shown on the right in Figure 1, then $\pi_U^H = 3158$ and $\pi_S^H = 426$. Note that the northeast endpoint of the hook does not contribute to either the unsheltered subpermutation nor the sheltered subpermutation. In all of the cases we consider in this paper, the plot of π_S^H lies completely below the hook H in the plot of π (it is “sheltered” by the hook H).

Lemma 2.1 (Decomposition Lemma, [24]). *If d is a tail-bound descent of a permutation $\pi \in S_n$, then*

$$|s^{-1}(\pi)| = \sum_{H \in \text{SW}_d(\pi)} |s^{-1}(\pi_U^H)| \cdot |s^{-1}(\pi_S^H)|.$$

3 Enumerating $s^{-1}(\text{Av}(231, 321))$

In his dissertation, West [47] proved that a permutation is in $s^{-1}(\text{Av}(231))$ (that is, it is 2-stack-sortable) if and only if it avoids the pattern 2341 and avoids any 3241 pattern that is not part of a 35241 pattern. As mentioned in the introduction, $s^{-1}(\text{Av}(321)) = \text{Av}(34251, 35241, 45231)$. Combining these two facts, it is straightforward to check that

$$s^{-1}(\text{Av}(231, 321)) = \text{Av}(2341, 3241, 45231). \quad (1)$$

In this section, we use the Decomposition Lemma to prove the following theorem.

Theorem 3.1. *We have*

$$\sum_{n \geq 0} |s^{-1}(\text{Av}_n(231, 321))| x^n = \frac{1}{1 - xC(xC(x))},$$

where $C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$ is the generating function of the sequence of Catalan numbers.

This is notable because it tells us that we can use the stack-sorting map and the Decomposition Lemma as tools in order to enumerate the permutation class $\text{Av}(2341, 3241, 45231)$, which, as far as we are aware, has not been enumerated yet. As mentioned in the introduction, $\frac{1}{1 - xC(xC(x))}$ is also the generating function of the sequence that enumerates the permutation classes $\text{Av}(4321, 4213)$ [6, 20] and $\text{Av}(2413, 2431, 23145)$ [39]. In particular, Theorem 3.1 tells us that the permutation classes

$$\text{Av}(4321, 4213) \quad \text{and} \quad \text{Av}(2341, 3241, 45231)$$

form a new example of an unbalanced Wilf equivalence, as defined in [5, 19].

Proof of Theorem 3.1: Let

$$\mathcal{D}_\ell(n) = \{\pi \in \text{Av}_{n+\ell}(231, 321) : \text{tl}(\pi) = \ell\} \quad \text{and} \quad \mathcal{D}_{\geq \ell}(n) = \{\pi \in \text{Av}_{n+\ell}(231, 321) : \text{tl}(\pi) \geq \ell\}.$$

Let $B_\ell(n) = |s^{-1}(\mathcal{D}_\ell(n))|$ and $B_{\geq \ell}(n) = |s^{-1}(\mathcal{D}_{\geq \ell}(n))|$.

Suppose $\pi \in \mathcal{D}_\ell(n+1)$ is such that $\pi_{n+1-i} = n+1$ (see Figure 2). Note that $n+1-i$ is a tail-bound descent of π because every point in the plot of π that is higher than $(n+1-i, n+1)$ is in the tail of π . The Decomposition Lemma tells us that $|s^{-1}(\pi)|$ is equal to the number of triples (H, μ, λ) , where $H \in \text{SW}_{n+1-i}(\pi)$, $\mu \in s^{-1}(\pi_U^H)$, and $\lambda \in s^{-1}(\pi_S^H)$. Choosing H amounts to choosing the number $j \in \{1, \dots, \ell\}$ such that the northeast endpoint of H is $(n+1+j, n+1+j)$. The permutation π and the choice of H determine the permutations π_U^H and π_S^H . On the other hand, the choices of H and the permutations π_U^H and π_S^H uniquely determine π . It follows that $B_\ell(n+1)$, which is the number of ways to choose an element of $s^{-1}(\mathcal{D}_\ell(n+1))$, is also the number of ways to choose j , the permutations π_U^H and π_S^H , and the permutations μ and λ . Let us fix a choice of j .

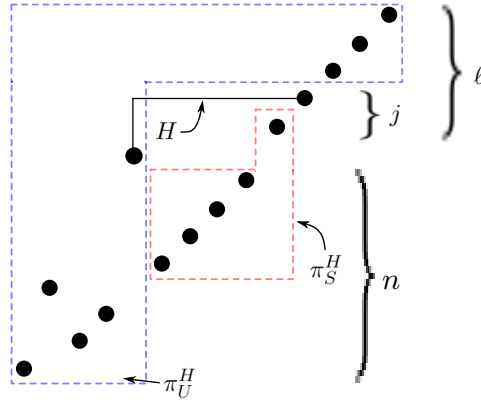


Fig. 2: An example of a permutation $\pi \in \mathcal{D}_5(9)$ and a hook H with $i = 4$ and $j = 2$ (in the notation of the proof of Theorem 3.1).

Because π avoids 231, π_U^H must be a permutation of the set

$$\{1, \dots, n-i\} \cup \{n+1\} \cup \{n+2+j, \dots, n+\ell+1\},$$

while π_S^H must be a permutation of $\{n-i+1, \dots, n+j\} \setminus \{n+1\}$. In fact, π_S^H is the increasing permutation of the $(i+j-1)$ -element set $\{n-i+1, \dots, n+j\} \setminus \{n+1\}$ because π avoids 321. According to Theorem 1.1, there are C_{i+j-1} choices for λ . Choosing π_U^H is equivalent to choosing its standardization, which is an element of $\mathcal{D}_{\geq \ell-j+1}(n-i)$. Any element of $\mathcal{D}_{\geq \ell-j+1}(n-i)$ can be chosen as the standardization of π_U^H . Also, π_U^H has the same fertility as its standardization. Combining these facts, we find that the number of choices for π_U^H and μ is $|s^{-1}(\mathcal{D}_{\geq \ell-j+1}(n-i))| = B_{\geq \ell-j+1}(n-i)$. We obtain the recurrence relation

$$B_\ell(n+1) = \sum_{i=1}^n \sum_{j=1}^{\ell} C_{i+j-1} B_{\geq \ell-j+1}(n-i). \quad (2)$$

Let

$$G_\ell(x) = \sum_{n \geq 0} B_{\geq \ell}(n)x^n \quad \text{and} \quad I(x, y) = \sum_{\ell \geq 0} G_\ell(x)y^\ell.$$

Note that

$$G_\ell(0) = B_{\geq \ell}(0) = |s^{-1}(123 \cdots \ell)| = C_\ell$$

by Theorem 1.1. Our goal is to understand the generating function

$$I(x, 0) = G_0(x) = \sum_{n \geq 0} B_{\geq 0}(n)x^n = \sum_{n \geq 0} |s^{-1}(\text{Av}_n(231, 321))|x^n.$$

The recurrence (2) tells us that

$$\sum_{n \geq 0} B_\ell(n+1)x^n = \sum_{j=1}^{\ell} \sum_{n \geq 0} \sum_{i=1}^n C_{i+j-1} B_{\geq \ell-j+1}(n-i)x^n = \sum_{j=1}^{\ell} \left(\sum_{i \geq 1} C_{i+j-1} x^i \right) G_{\ell-j+1}(x),$$

so

$$\begin{aligned} \sum_{\ell \geq 0} \sum_{n \geq 0} B_\ell(n+1)x^n y^\ell &= \sum_{\ell \geq 0} \sum_{j=1}^{\ell} \left(\sum_{i \geq 1} C_{i+j-1} x^i \right) G_{\ell-j+1}(x) y^\ell \\ &= \left(\sum_{j \geq 1} \sum_{i \geq 1} C_{i+j-1} x^i y^{j-1} \right) (I(x, y) - I(x, 0)) = x \frac{C(x) - C(y)}{x - y} (I(x, y) - I(x, 0)). \end{aligned} \quad (3)$$

On the other hand,

$$\begin{aligned} \sum_{\ell \geq 0} \sum_{n \geq 0} B_\ell(n+1)x^n y^\ell &= \sum_{\ell \geq 0} \sum_{n \geq 0} B_{\geq \ell}(n+1)x^n y^\ell - \sum_{\ell \geq 0} \sum_{n \geq 0} B_{\geq \ell+1}(n)x^n y^\ell \\ &= \frac{1}{x} \sum_{\ell \geq 0} (G_\ell(x) - C_\ell) y^\ell - \frac{1}{y} \sum_{\ell \geq 0} G_{\ell+1}(x) y^{\ell+1} = \frac{I(x, y) - C(y)}{x} - \frac{I(x, y) - I(x, 0)}{y}. \end{aligned} \quad (4)$$

Combining (3) and (4) and rearranging terms, we get the equation

$$x(I(x, y) - I(x, 0)) \left(x \frac{C(x) - C(y)}{x - y} - \frac{1}{x} + \frac{1}{y} \right) = I(x, 0) - C(y). \quad (5)$$

We now employ the kernel method (see [4, 12, 13, 42] for more on this method). There is a unique power series $Y = Y(x)$ such that $Y(x) = x + O(x^2)$ and $x \frac{C(x) - C(Y)}{x - Y} - \frac{1}{x} + \frac{1}{Y} = 0$. Substituting this into (5), we find that $I(x, 0) = C(Y)$. For ease of notation, let $u = I(x, 0)$ and $v = \frac{1}{1 - xC(xC(x))}$. We wish to show that $u = v$.

Combining the equation $x \frac{C(x) - C(Y)}{x - Y} - \frac{1}{x} + \frac{1}{Y} = 0$ with the standard Catalan functional equation $YC(Y)^2 + 1 - C(Y) = 0$ yields

$$u = C(Y) = C(x) + \frac{(x - Y)^2}{x^2 Y} = C(x) + \frac{(x - \frac{u-1}{u^2})^2}{x^2 \frac{u-1}{u^2}}.$$

If we solve for $C(x)$ and use the functional equation $xC(x)^2 + 1 - C(x) = 0$, we obtain

$$x \left(u - \frac{(x - \frac{u-1}{u^2})^2}{x^2 \frac{u-1}{u^2}} \right)^2 + 1 - \left(u - \frac{(x - \frac{u-1}{u^2})^2}{x^2 \frac{u-1}{u^2}} \right) = 0.$$

We can simplify this last equation to find that $Q(u, x) \frac{1 - u + xu^2}{x^3 u^4 (u - 1)^2} = 0$, where Q is the polynomial given by

$$Q(a, x) = 1 - 3a + (3 + 2x)a^2 + (-1 - 4x + 2x^2)a^3 + (2x - 2x^2 + x^3)a^4. \quad (6)$$

It is easy to see that $1 - u + xu^2 \neq 0$, so $Q(u, x) = 0$.

We have $1 - \frac{1}{v} = xC(xC(x))$, so

$$C(x) \left(1 - \frac{1}{v} \right)^2 = x^2 C(x) C(xC(x))^2 = x(C(xC(x)) - 1),$$

where the last equality follows from the Catalan functional equation. This means that

$$1 - x - C(x) \left(1 - \frac{1}{v} \right)^2 = 1 - xC(xC(x)) = \frac{1}{v}.$$

It follows that $C(x) = \frac{1 - \frac{1}{v} - x}{(1 - \frac{1}{v})^2}$, so we can use the Catalan functional equation to find that

$$x \left(\frac{1 - \frac{1}{v} - x}{(1 - \frac{1}{v})^2} \right)^2 + 1 - \frac{1 - \frac{1}{v} - x}{(1 - \frac{1}{v})^2} = 0.$$

After simplifying this equation, we find that $Q(v, x) \frac{1}{(v-1)^4} = 0$, where Q is the polynomial in (6). Hence, $Q(v, x) = 0$.

We now know that $Q(u, x) = Q(v, x) = 0$. There are four Laurent series $F(x)$ satisfying $Q(F(x), x) = 0$, but only one satisfies $F(x) = 1 + x + O(x^2)$. This proves that $u = v$, as desired. \square

Remark 3.1. Now that we have determined $I(x, 0)$, one could use (5) to find the bivariate generating function $I(x, y)$, which counts the permutations in the set

$$s^{-1}(\text{Av}(231, 321)) = \text{Av}(2341, 3241, 45231)$$

according to length and an additional statistic.

As mentioned in the introduction, Theorem 3.1 is equivalent to the enumeration of permutations that are sortable via the map $\mathbf{B} \circ s$, where \mathbf{B} is the bubble sort map.

4 Enumerating $s^{-1}(\text{Av}(132, 231))$

Many analogues and generalizations of classical permutation patterns have emerged over the past few decades. One of the most notable notions is that of a “vincular pattern,” which appeared first in [3] and has garnered a large amount of attention ever since. We refer the reader to the survey [45] and the references therein for further background. A *vincular pattern* is a permutation pattern in which some consecutive entries can be underlined. We say a permutation *contains* a vincular pattern if it contains an occurrence of the permutation pattern in which underlined entries are consecutive. For example, saying that a permutation $\sigma = \sigma_1 \cdots \sigma_n$ contains the vincular pattern $\underline{3}241$ means that there are indices $i_1 < i_2 < i_3 < i_4$ such that $\sigma_{i_4} < \sigma_{i_2} < \sigma_{i_1} < \sigma_{i_3}$ and $i_2 = i_1 + 1$. We say a permutation *avoids* a vincular pattern τ if it does not contain τ . Let $\text{Av}(\tau^{(1)}, \tau^{(2)}, \dots)$ be the set of standardized permutations avoiding the vincular patterns $\tau^{(1)}, \tau^{(2)}, \dots$, and let $\text{Av}_n(\tau^{(1)}, \tau^{(2)}, \dots) = \text{Av}(\tau^{(1)}, \tau^{(2)}, \dots) \cap S_n$.

There are many fascinating combinatorial properties of the OEIS sequence A071356 [41]; several are listed in [36], where the numbers in this sequence are named “Boolean-Catalan numbers.” One can define this sequence via its generating function

$$\frac{1 - 2x - \sqrt{1 - 4x - 4x^2}}{4x}.$$

Hossain [37] has conjectured that the Boolean-Catalan numbers enumerate the permutations in

$$\text{Av}(2341, 1342, \underline{3}241, \underline{3}142).$$

We will see that this set is precisely $s^{-1}(\text{Av}(132, 231))$. We will then enumerate these permutations via the Decomposition Lemma, proving Hossain’s conjecture. It is interesting that the only known proof of this conjecture makes heavy use of the stack-sorting map.

The main theorem of this section is the following.

Theorem 4.1. *We have*

$$\begin{aligned} \sum_{n \geq 1} |s^{-1}(\text{Av}_n(132, 312))|x^n &= \sum_{n \geq 1} |s^{-1}(\text{Av}_n(231, 312))|x^n = \sum_{n \geq 1} |s^{-1}(\text{Av}_n(132, 231))|x^n \\ &= \frac{1 - 2x - \sqrt{1 - 4x - 4x^2}}{4x}. \end{aligned}$$

This theorem settles Conjecture 10.1 from [29]. Before proving it, we show how to deduce from it the following corollary, part of which settles Hossain’s conjecture.

Corollary 4.1. *We have*

$$\begin{aligned} \sum_{n \geq 1} |\text{Av}_n(2341, 1342, \underline{3}241, \underline{3}142)|x^n &= \sum_{n \geq 1} |\text{Av}_n(1342, 3142, 3412, \underline{3}421)|x^n \\ &= \frac{1 - 2x - \sqrt{1 - 4x - 4x^2}}{4x}. \end{aligned}$$

Proof: This corollary will follow immediately from Theorem 4.1 if we can show that

$$s^{-1}(\text{Av}(132, 231)) = \text{Av}(2341, 1342, \underline{3241}, \underline{3142})$$

and

$$s^{-1}(\text{Av}(132, 312)) = \text{Av}(1342, 3142, 3412, \underline{3421}).$$

We prove the first of these equalities, the proof of the second is similar. If σ contains 2341 or $\underline{3241}$, then it follows from West's characterization of 2-stack-sortable permutations (mentioned at the start of Section 3) that $s(\sigma)$ contains 231. It is also straightforward to check that $s(\sigma)$ contains 132 if σ contains 1342 or $\underline{3142}$. This proves the containment

$$s^{-1}(\text{Av}(132, 231)) \subseteq \text{Av}(2341, 1342, \underline{3241}, \underline{3142}).$$

Now suppose $s(\sigma)$ contains either 132 or 231. We want to prove that σ contains one of the vincular patterns 2341, 1342, $\underline{3241}$, $\underline{3142}$. We will assume that σ avoids 2341, 1342, $\underline{3241}$ and prove that it contains $\underline{3142}$. Let us first assume $s(\sigma)$ contains 231. Using West's characterization of 2-stack-sortable permutations and the assumption that σ avoids 2341, we see that σ contains a $\underline{3241}$ pattern that is not part of a $\underline{35241}$ pattern. Therefore, there are indices $i_1 < i_2 < i_3 < i_4$ such that $\sigma_{i_4} < \sigma_{i_2} < \sigma_{i_1} < \sigma_{i_3}$ and such that no entry lying between σ_{i_1} and σ_{i_2} is larger than σ_{i_3} . We may assume that among all such choices for i_1, i_2, i_3, i_4 , we have made a choice that minimizes $i_2 - i_1$. Because σ avoids $\underline{3241}$, $i_2 \geq i_1 + 2$. Our choice of i_1, i_2, i_3, i_4 guarantees that $\sigma_{i_1+1} < \sigma_{i_3}$. One can check that the minimality of $i_2 - i_1$ forces $\sigma_{i_1+1} < \sigma_{i_4}$. However, this means that the standardization of $\sigma_{i_1+1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}$ is 1342, contradicting our assumption that σ avoids 1342. This shows that $s(\sigma)$ avoids 231, so it must contain 132.

One can show [21, 29] that a permutation is in $s^{-1}(\text{Av}(132))$ if and only if it avoids the pattern 1342 and avoids any 3142 pattern that is not part of either a 34152 pattern or a 35142 pattern. Since we are assuming that $s(\sigma)$ contains 132 and that σ avoids 1342, σ must contain a 3142 pattern that is not part of either a 34152 pattern or a 35142 pattern. This means that there are indices $i_1 < i_2 < i_3 < i_4$ such that $\sigma_{i_2} < \sigma_{i_4} < \sigma_{i_1} < \sigma_{i_3}$ and such that no entry lying between σ_{i_1} and σ_{i_2} is larger than σ_{i_1} . We may assume that among all such choices for i_1, i_2, i_3, i_4 , we have made a choice that minimizes $i_2 - i_1$. In fact, one can show that this minimality assumption forces $i_2 - i_1 = 1$. Hence, $\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}$ is an occurrence of the vincular pattern $\underline{3142}$ in σ . \square

In the following proof, we recycle our notation from the proof of Theorem 3.1 in Section 3. This is meant to elucidate the parallels between the proofs.

Proof of Theorem 4.1: The first and second equalities in Theorem 4.1 were proven in [29] and [26], respectively, so we need only prove the last equality. Let

$$\mathcal{D}_\ell(n) = \{\pi \in \text{Av}_{n+\ell}(132, 231) : \text{tl}(\pi) = \ell\} \quad \text{and} \quad \mathcal{D}_{\geq \ell}(n) = \{\pi \in \text{Av}_{n+\ell}(132, 231) : \text{tl}(\pi) \geq \ell\}.$$

Let $B_\ell(n) = |s^{-1}(\mathcal{D}_\ell(n))|$ and $B_{\geq \ell}(n) = |s^{-1}(\mathcal{D}_{\geq \ell}(n))|$.

Suppose $n \geq 1$ and $\pi \in \mathcal{D}_\ell(n+1)$ (see Figure 3). Using the fact that π avoids 132 and 231 and has tail length $\ell < n + \ell + 1$, we easily find that $\pi_1 = n + 1$ and that 1 is a tail-bound descent of π . The Decomposition Lemma tells us that $|s^{-1}(\pi)|$ is equal to the number of triples (H, μ, λ) , where $H \in \text{SW}_1(\pi)$, $\mu \in s^{-1}(\pi_V^H)$, and $\lambda \in s^{-1}(\pi_S^H)$. Choosing H amounts to choosing the number $j \in \{1, \dots, \ell\}$ such that the northeast endpoint of H is $(n + 1 + j, n + 1 + j)$. The permutation π and the choice of

H determine the permutations π_U^H and π_S^H . On the other hand, the choices of H and the permutations π_U^H and π_S^H uniquely determine π . It follows that $B_\ell(n+1)$, which is the number of ways to choose an element of $s^{-1}(\mathcal{D}_\ell(n+1))$, is also the number of ways to choose j , the permutations π_U^H and π_S^H , and the permutations μ and λ . Let us fix a choice of j .

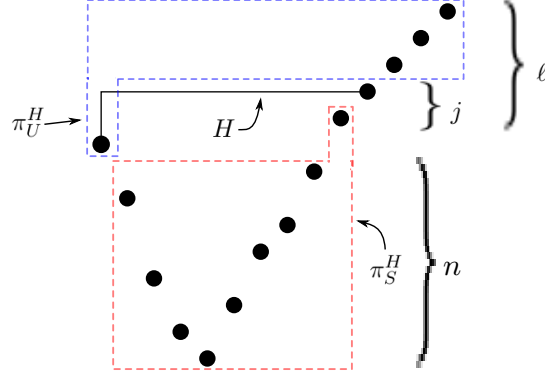


Fig. 3: An example of a permutation $\pi \in D_5(9)$ and a hook H with $j = 2$ (in the notation of the proof of Theorem 4.1).

The permutation π_U^H must be the increasing permutation $(n+1)(n+2+j)\cdots(n+\ell+1)$ of length $\ell-j+1$. By Theorem 1.1, there are $C_{\ell-j+1}$ choices for μ . Now, π_S^H is a permutation of the $(n+j-1)$ -element set $\{1, \dots, n+j\} \setminus \{n+1\}$, so choosing π_S^H is equivalent to choosing its standardization. This standardization is in $\mathcal{D}_{\geq j-1}(n)$. Any element of $\mathcal{D}_{\geq j-1}(n)$ can be chosen as the standardization of π_S^H . Also, π_S^H has the same fertility as its standardization. Combining these facts, we find that the number of choices for π_S^H and λ is $|s^{-1}(\mathcal{D}_{\geq j-1}(n))| = B_{\geq j-1}(n)$. We obtain the recurrence relation

$$B_\ell(n+1) = \sum_{j=1}^{\ell} C_{\ell-j+1} B_{\geq j-1}(n) \quad \text{for } n \geq 1. \quad (7)$$

Let

$$G_\ell(x) = \sum_{n \geq 0} B_{\geq \ell}(n) x^n \quad \text{and} \quad I(x, y) = \sum_{\ell \geq 0} G_\ell(x) y^\ell.$$

Note that

$$G_\ell(0) = B_{\geq \ell}(0) = |s^{-1}(123 \cdots \ell)| = C_\ell$$

by Theorem 1.1. Let $C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}$ be the generating function of the sequence of Catalan numbers. Since $B_{\geq 0}(n) = |s^{-1}(\text{Av}_n(132, 231))|$, our goal is to understand the generating function

$$I(x, 0) - 1 = G_0(x) - 1 = \sum_{n \geq 0} B_{\geq 0}(n) x^n - 1 = \sum_{n \geq 1} |s^{-1}(\text{Av}_n(132, 231))| x^n.$$

Note that $B_\ell(1) = 0$ because there are no permutations of length $\ell + 1$ with tail length ℓ . Therefore, the recurrence (7) tells us that

$$\sum_{n \geq 0} B_\ell(n+1)x^n = \sum_{n \geq 1} B_\ell(n+1)x^n = \sum_{j=1}^{\ell} C_{\ell-j+1} \sum_{n \geq 1} B_{\geq j-1}(n)x^n = \sum_{j=1}^{\ell} C_{\ell-j+1}(G_{j-1}(x) - C_{j-1}).$$

Consequently,

$$\sum_{\ell \geq 0} \sum_{n \geq 0} B_\ell(n+1)x^n y^\ell = \sum_{\ell \geq 0} \sum_{j=1}^{\ell} C_{\ell-j+1}(G_{j-1}(x) - C_{j-1})y^\ell = (C(y) - 1)(I(x, y) - C(y)). \quad (8)$$

On the other hand,

$$\begin{aligned} \sum_{\ell \geq 0} \sum_{n \geq 0} B_\ell(n+1)x^n y^\ell &= \sum_{\ell \geq 0} \sum_{n \geq 0} B_{\geq \ell}(n+1)x^n y^\ell - \sum_{\ell \geq 0} \sum_{n \geq 0} B_{\geq \ell+1}(n)x^n y^\ell \\ &= \frac{1}{x} \sum_{\ell \geq 0} (G_\ell(x) - C_\ell)y^\ell - \frac{1}{y} \sum_{\ell \geq 0} G_{\ell+1}(x)y^{\ell+1} = \frac{I(x, y) - C(y)}{x} - \frac{I(x, y) - I(x, 0)}{y}. \end{aligned} \quad (9)$$

Combining (8) and (9) and rearranging terms, we get the equation

$$I(x, y)(xy(C(y) - 1) - y + x) = -yC(y) + xI(x, 0) + xyC(y)(C(y) - 1). \quad (10)$$

As in the previous section, we use the kernel method. Let

$$Y = Y(x) = \frac{3x + 2x^2 - x\sqrt{1 - 4x - 4x^2}}{2(1 + x)^2}.$$

After verifying that $xY(C(Y) - 1) - Y + x = 0$, we use (10) to find that

$$I(x, 0) - 1 = -1 + \frac{1}{x}(YC(Y)) - xYC(Y)(C(Y) - 1).$$

After some elementary algebraic manipulations, this expression simplifies to the generating function

$$\frac{1 - 2x - \sqrt{1 - 4x - 4x^2}}{4x},$$

as desired. \square

Remark 4.1. Now that we have determined $I(x, 0)$, one could use (10) to find the bivariate generating function $I(x, y)$, which counts the permutations in the set

$$s^{-1}(\text{Av}(132, 231)) = \text{Av}(2341, 1342, \underline{3}241, \underline{3}142)$$

according to length and an additional statistic.

5 Counting $s^{-1}(\text{Av}_{n,k}(231, 312, 321))$

A *composition of b into a parts* is an a -tuple of positive integers that sum to b . Let $\text{Comp}_a(b)$ denote the set of all compositions of b into a parts. There is a natural partial order \preceq on $\text{Comp}_a(b)$ defined by declaring that $(x_1, \dots, x_a) \preceq (y_1, \dots, y_a)$ if $\sum_{i=1}^{\ell} x_i \leq \sum_{i=1}^{\ell} y_i$ for all $\ell \in \{1, \dots, a\}$. For $x = (x_1, \dots, x_a) \in \text{Comp}_a(b)$, let

$$D_x = |\{y \in \text{Comp}_a(b) : y \preceq x\}|.$$

Let $\psi(x)$ be the partition (composition with nonincreasing parts) that has $x_i - 1$ parts of size $a - i$ for all $i \in \{1, \dots, a - 1\}$ (and has no parts of size at least a). The quantity D_x can also be interpreted as the number of partitions whose Young diagrams fit inside the Young diagram of $\psi(x)$. Said differently, D_x is the size of the order ideal in Young's lattice generated by $\psi(x)$ (see [29] for more details). Let us write $C_x = \prod_{t=1}^a C_{x_t}$, where C_j is the j^{th} Catalan number. Put together, Theorems 6.1 and 6.2 in [29] state that

$$|s^{-1}(\text{Av}_n(231, 312, 321))| = \sum_{k=0}^{n-1} \frac{1}{n+1} \binom{n-k-1}{k} \binom{2n-2k}{n} = \sum_{k=0}^{n-1} \sum_{q \in \text{Comp}_{k+1}(n-k)} C_q D_q. \quad (11)$$

From this equation, it is natural to conjecture that

$$\sum_{q \in \text{Comp}_{k+1}(n-k)} C_q D_q = \frac{1}{n+1} \binom{n-k-1}{k} \binom{2n-2k}{n} \quad (12)$$

for all nonnegative integers n and k ; this is the content of Conjecture 6.1 in [29]. In that article, the current author showed that this conjecture is equivalent to the following theorem. Let $\text{des}(\pi)$ denote the number of descents of a permutation π .

Theorem 5.1. *For all $n \geq 1$ and $k \geq 0$, we have*

$$|s^{-1}(\text{Av}_{n,k}(231, 312, 321))| = \frac{1}{n+1} \binom{n-k-1}{k} \binom{2n-2k}{n},$$

where $\text{Av}_{n,k}(231, 312, 321) = \{\pi \in \text{Av}_n(231, 312, 321) : \text{des}(\pi) = k\}$.

In this section, we use the Decomposition Lemma to prove this theorem, thereby settling the aforementioned conjecture. Our result implies the first equality in (11), so the proof given in this section yields a completely new proof of Theorem 6.2 from [29] (see that article for more details).

Proof Proof of Theorem 5.1: Let

$$\mathcal{D}_{\ell}(n, k) = \{\pi \in \text{Av}_{n+\ell,k}(231, 312, 321) : \text{tl}(\pi) = \ell\}$$

and

$$\mathcal{D}_{\geq \ell}(n, k) = \{\pi \in \text{Av}_{n+\ell,k}(231, 312, 321) : \text{tl}(\pi) \geq \ell\}.$$

Let $B_{\ell}(n, k) = |s^{-1}(\mathcal{D}_{\ell}(n, k))|$ and $B_{\geq \ell}(n, k) = |s^{-1}(\mathcal{D}_{\geq \ell}(n, k))|$.

Suppose $n \geq 0$ and $\pi \in \mathcal{D}_\ell(n+1, k)$ (see Figure 4). Using the fact that π avoids 312 and 321, we easily find that $\pi_n = n+1$. Because π avoids 231, we must have $\pi_{n+1} = n$. Note that n is a tail-bound descent of π . The Decomposition Lemma tells us that $|s^{-1}(\pi)|$ is equal to the number of triples (H, μ, λ) , where $H \in \text{SW}_n(\pi)$, $\mu \in s^{-1}(\pi_U^H)$, and $\lambda \in s^{-1}(\pi_S^H)$. Choosing H amounts to choosing the number $j \in \{1, \dots, \ell\}$ such that the northeast endpoint of H is $(n+1+j, n+1+j)$. Let us fix such a j . The permutation $\pi_S^H = n(n+2)(n+3) \cdots (n+j)$ is an increasing permutation of length j . Therefore, there are C_j choices for λ by Theorem 1.1.

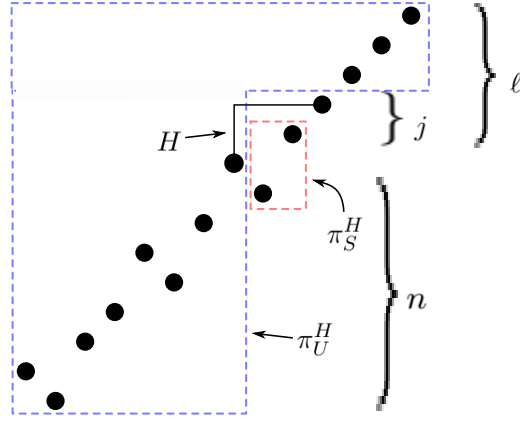


Fig. 4: An example of a permutation $\pi \in D_5(9, 3)$ and a hook H with $j = 2$ (in the notation of the proof of Theorem 4.1).

Now, π_U^H is a permutation of the set $\{1, \dots, n-1\} \cup \{n+1\} \cup \{n+j+2, \dots, n+\ell+1\}$, which has $n+\ell-j$ elements. Therefore, choosing π_U^H is equivalent to choosing its standardization. The descents of π_U^H are simply the descents of π other than n , so $\text{des}(\pi_U^H) = \text{des}(\pi) - 1 = k-1$. It follows that the standardization of π_U^H is in $\mathcal{D}_{\geq \ell-j+1}(n-1, k-1)$. Any permutation in $\mathcal{D}_{\geq \ell-j+1}(n-1, k-1)$ can be chosen as the standardization of π_U^H . Since π_U^H has the same fertility as its standardization, the number of choices for π_U^H and μ is $|s^{-1}(\mathcal{D}_{\geq \ell-j+1}(n-1, k-1))| = B_{\geq \ell-j+1}(n-1, k-1)$. We obtain the recurrence relation

$$B_\ell(n+1, k) = \sum_{j=1}^{\ell} B_{\geq \ell-j+1}(n-1, k-1) C_j. \quad (13)$$

Let

$$G_{\ell,k}(x) = \sum_{n \geq 0} B_{\geq \ell}(n, k) x^n, \quad I_k(x, y) = \sum_{\ell \geq 0} G_{\ell,k}(x) y^\ell, \quad \text{and} \quad J(x, y, z) = \sum_{k \geq 0} I_k(x, y) z^k.$$

Note that

$$G_{\ell,k}(0) = B_{\geq \ell}(0, k) = |s^{-1}(\mathcal{D}_{\geq \ell}(0, k))| = \delta_{k0} C_\ell,$$

where δ_{k0} is the Kronecker delta. Let $C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}$. Because $B_{\geq 0}(n, k) = |s^{-1}(\text{Av}_{n,k}(231, 312, 321))|$, our goal is to understand the generating function

$$J(x, 0, z) = \sum_{n \geq 0} \sum_{k \geq 0} B_{\geq 0}(n, k) x^n z^k = \sum_{n \geq 0} \sum_{k \geq 0} |s^{-1}(\text{Av}_{n,k}(231, 312, 321))| x^n z^k.$$

The recurrence (13) is equivalent to

$$\sum_{n \geq 0} B_\ell(n+1, k) x^n = x \sum_{j=1}^{\ell} G_{\ell-j+1, k-1}(x) C_j,$$

so

$$\sum_{\ell \geq 0} \sum_{n \geq 0} B_\ell(n+1, k) x^n y^\ell = x \sum_{\ell \geq 0} \sum_{j=1}^{\ell} G_{\ell-j+1, k-1}(x) C_j y^\ell = x \frac{C(y) - 1}{y} (I_{k-1}(x, y) - I_{k-1}(x, 0)). \quad (14)$$

On the other hand,

$$\begin{aligned} \sum_{\ell \geq 0} \sum_{n \geq 0} B_\ell(n+1, k) x^n y^\ell &= \sum_{\ell \geq 0} \sum_{n \geq 0} B_{\geq \ell}(n+1, k) x^n y^\ell - \sum_{\ell \geq 0} \sum_{n \geq 0} B_{\geq \ell+1}(n, k) x^n y^\ell \\ &= \frac{1}{x} \sum_{\ell \geq 0} (G_{\ell, k}(x) - G_{\ell, k}(0)) y^\ell - \frac{1}{y} \sum_{\ell \geq 0} G_{\ell+1, k}(x) y^{\ell+1} = \frac{I_k(x, y) - \delta_{k0} C(y)}{x} - \frac{I_k(x, y) - I_k(x, 0)}{y}. \end{aligned} \quad (15)$$

Combining (14) and (15) yields the equation

$$x \frac{C(y) - 1}{y} (I_{k-1}(x, y) - I_{k-1}(x, 0)) = \frac{I_k(x, y) - \delta_{k0} C(y)}{x} - \frac{I_k(x, y) - I_k(x, 0)}{y}. \quad (16)$$

If we multiply both sides of (16) by z^k and sum over $k \geq 0$, we obtain

$$xz \frac{C(y) - 1}{y} (J(x, y, z) - J(x, 0, z)) = \frac{J(x, y, z) - C(y)}{x} - \frac{J(x, y, z) - J(x, 0, z)}{y}.$$

We can rewrite this equation as

$$\frac{1}{y} (J(x, y, z) - J(x, 0, z)) (x^2 z (C(y) - 1) + x - y) = J(x, 0, z) - C(y). \quad (17)$$

As in the previous two sections, we use the kernel method to find the generating function $J(x, 0, z)$. There is a unique power series $Y = Y(x, z)$ such that $C(Y(x, z)) = x + 2x^2 + O(x^3)$ and $x^2 z (C(Y) - 1) + x - Y = 0$. Substituting this into (17) shows that $J(x, 0, z) = C(Y)$. Let $\widehat{J}(x, z) = xJ(x, 0, z) = xC(Y)$. Using the Catalan functional equation $YC(Y)^2 + 1 - C(Y) = 0$, we obtain

$$\widehat{J}(x, z) - \frac{\widehat{J}(x, z)^2}{1 - z\widehat{J}(x, z)^2} = x \left(C(Y) - \frac{x C(Y)^2}{1 - x^2 z C(Y)^2} \right) = x \left(C(Y) - \frac{x \frac{C(Y) - 1}{Y}}{1 - x^2 z \frac{C(Y) - 1}{Y}} \right). \quad (18)$$

We now use the fact that $x^2z(C(Y) - 1) + x - Y = 0$ to see that

$$C(Y) - \frac{x \frac{C(Y)-1}{Y}}{1 - x^2z \frac{C(Y)-1}{Y}} = 1 + \frac{Y-x}{x^2z} - \frac{x \frac{Y-x}{x^2zY}}{1 - x^2z \frac{Y-x}{x^2zY}} = 1.$$

Substituting this into (18) shows that

$$\widehat{J}(x, z) = x + \frac{\widehat{J}(x, z)^2}{1 - z\widehat{J}(x, z)^2}.$$

By Lagrange inversion,

$$\begin{aligned} \widehat{J}(x, z) &= x + \sum_{m \geq 0} \frac{1}{(m+1)!} \frac{\partial^m}{\partial x^m} \left(\frac{x^2}{1 - zx^2} \right)^{m+1} \\ &= x + \sum_{m \geq 0} \frac{1}{(m+1)!} \frac{\partial^m}{\partial x^m} \sum_{k \geq 0} \binom{k+m}{k} x^{2m+2k+2} z^k \\ &= x + \sum_{m \geq 0} \frac{1}{m+1} \sum_{k \geq 0} \binom{k+m}{k} \binom{2m+2k+2}{m+2k+2} x^{m+2k+2} z^k \\ &= x + x \sum_{k \geq 0} \sum_{n \geq 2k+1} \frac{1}{n+1} \binom{n-k-1}{k} \binom{2n-2k}{n} x^n z^k, \end{aligned}$$

as desired. \square

6 Further Directions

The proofs of our three main theorems relied on the Decomposition Lemma, which was proven in [24]. That article actually obtained this useful tool as a corollary of a more general result, called the ‘‘Refined Decomposition Lemma,’’ which allows one to count stack-sorting preimages of permutations according to the additional statistics *des* and *peak* (for $\pi = \pi_1 \cdots \pi_n$, $\text{peak}(\pi)$ is the number of indices $i \in \{2, \dots, n-1\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$). It could be interesting to generalize Theorems 3.1, 4.1, and 5.1 by counting the relevant preimage sets according to the statistics *des* and *peak*. One could also attempt to generalize these results to the context of ‘‘troupes,’’ which are special families of colored binary plane trees for which a version of the Decomposition Lemma holds (see [30] for more details). Of course, one could also search for other interesting sets of permutations whose stack-sorting preimages can be enumerated via the Decomposition Lemma.

The results in Sections 3 and 4 complete the enumerations of all sets of the form $s^{-1}(\text{Av}(\tau^{(1)}, \dots, \tau^{(r)}))$ for $\{\tau^{(1)}, \dots, \tau^{(r)}\} \subseteq S_3$ except the set $\{321\}$. There are currently no non-trivial results that enumerate sets of this form when one or more of the patterns $\tau^{(i)}$ has length at least 4. Obtaining such results would be a natural next step. It would be especially interesting to solve this enumerative problem using valid hook configurations (as in [29]) or the Decomposition Lemma in the cases in which the sets $s^{-1}(\text{Av}(\tau^{(1)}, \dots, \tau^{(r)}))$ have some natural alternative descriptions (i.e., if they

are characterized as the permutations avoiding some collection of patterns or vincular patterns). It is shown in [26] that

$$\sum_{n \geq 1} |s^{-1}(\text{Av}_n(132, 3412))|x^n = \sum_{n \geq 1} |s^{-1}(\text{Av}_n(231, 1423))|x^n = \sum_{n \geq 1} |s^{-1}(\text{Av}_n(312, 1342))|x^n.$$

Thus, it is natural to consider the specific problem of computing the generating function

$$\sum_{n \geq 1} |s^{-1}(\text{Av}_n(132, 3412))|x^n.$$

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