Upper paired domination versus upper domination

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A paired dominating set P is a dominating set with the additional property that P has a perfect matching. While the maximum cardinality of a minimal dominating set in a graph G is called the upper domination number of G, denoted by $\Gamma(G)$, the maximum cardinality of a minimal paired dominating set in G is called the upper paired domination number of G, denoted by $\Gamma_{pr}(G)$. By Henning and Pradhan (2019), we know that $\Gamma_{pr}(G) \leq 2\Gamma(G)$ for any graph G without isolated vertices. We focus on the graphs satisfying the equality $\Gamma_{pr}(G) = 2\Gamma(G)$. We give characterizations for two special graph classes: bipartite and unicyclic graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ by using the results of Ulatowski (2015). Besides, we study the graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and a restricted girth. In this context, we provide two characterizations: one for graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and girth at least 6 and the other for C_3-free cactus graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. We also pose the characterization of the general case of C_3-free graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ as an open question.

Keywords: Paired dominating set, upper paired domination, upper domination

1 Introduction

In a graph $G = (V, E)$, a set $D \subseteq V(G)$ is called a *dominating set* if every vertex of G is either in D or adjacent to a vertex in D. We say that a dominating set D is *minimal* if no proper subset of D is a dominating set in G. The *domination number* of G, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set in G, whereas the *upper domination number* of G, denoted by $\Gamma(G)$, is the maximum cardinality of a minimal dominating set in G.

Different variants of domination concept exist in the literature. One of these variants is *paired domination*, which was first put forward by Haynes and Slater (1998). A *matching M* in a graph G is a set of pairwise non-adjacent edges. If a matching M matches all vertices of a graph G, we call M a *perfect matching*. A *paired dominating set (PDS)* of a graph G is a dominating set D of G with the additional property that the subgraph $G[D]$ induced by D contains a perfect matching M. In a similar way, the *paired domination number* of a graph G, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a PDS in G. In addition, the *upper paired domination number* of a graph G, denoted by $\Gamma_{pr}(G)$, is the maximum cardinality of a minimal PDS in G.

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Paired domination is a well-studied subject in the literature. The existing literature on paired domination can be grouped into three major categories. One category includes research works focusing on investigating paired domination number in different graph classes such as trees (Chellali and Haynes (2004)), claw-free cubic graphs (Favaron and Henning (2004)), generalized claw-free graphs (Dorbec et al. (2007a)), chordal bipartite graphs (Panda and Pradhan (2013)), and strongly orderable graphs (Pradhan and Panda (2019)). Another category contains research works presenting lower bounds for paired domination number (Lemanska (2004), Delavina et al. (2010), Hajian et al. (2019)) and characterization results with respect to a specific relationship between the paired domination number and the graph order (Henning (2007), Goddard and Henning (2009), Ulatowski (2013)). For further information about lower bounds for paired domination number, the reader is referred to a comprehensive survey by (Desormeaux et al. (2020)). Furthermore, studies on the relationship between paired domination number and other domination variants such as total domination number (Shan et al. (2004), Schaudt (2012b), Cyman et al. (2018)), induced paired-domination (Schaudt (2012a)), and double domination (Dorbec et al. (2014)) form the third category.

In this paper, we restrict our attention to the concept of upper paired domination, which is a relatively unexplored area of the literature on paired domination. Following the notation by Henning and Pradhan (2019), we denote by *Upper-PDS*, the problem of finding a Γ_{pr}-set in a graph G. Henning and Pradhan (2019) studied the concept of upper paired domination from an algorithmic perspective. They showed that while the decision version of *Upper-PDS* problem is NP-complete for general graphs, for some special graph classes including threshold graphs, chain graphs, and proper interval graphs, *Upper-PDS* is solvable in polynomial time.

There exist few research works with structural results regarding upper paired domination in the literature. Dorbec et al. (2007b) studied the relationship between the upper paired domination number and the upper total domination number of a graph. They showed that for a graph G with no isolated vertex it holds that $\Gamma_t \geq 1/2(\Gamma_{pr} + 2)$. In addition, they gave a characterization for the trees achieving the equality in this relationship. Restricted to the case of connected claw-free graphs G, Dorbec and Henning (2011) established upper bounds on $\Gamma_{pr}(G)$ with respect to the graph order n and minimum degree $\delta(G)$. Another available result is due to Ulatowski (2015) where the author provides characterizations for graphs with $\Gamma_{pr}(G) = n$ and $\Gamma_{pr}(G) = n - 1$.

Due to a result by Henning and Pradhan (2019), we know that $\Gamma_{pr}(G) \leq 2\Gamma(G)$ for any graph G without isolated vertices. In this paper we focus on graphs which have the property $\Gamma_{pr}(G) = 2\Gamma(G)$. By using the results of Ulatowski (2015), we give characterizations for two special graph classes: *bipartite* and *unicyclic* graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. Besides, we study the graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ with a restricted girth. In this context, we give a complete characterization for graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and girth at least 6. Furthermore, for the case of girth at least 4, we characterize C_3-free cactus graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and leave the characterization of the general case of C_3-free graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ as an open question.

In Section 2, after introducing some graph-theoretic notations and definitions, we provide some known results in the literature regarding upper paired domination and upper domination. We then proceed to the graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ in Section 3, where we focus particularly on bipartite graphs, unicyclic graphs, and graphs with restricted girth.

2 Preliminaries

A *graph G* is an ordered pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges each connecting a pair of vertices. Throughout this paper we assume that G is a *simple* graph, that is, a finite, undirected, and loopless graph without multiple edges. The number of vertices of a graph G is called the *order* of G. We mean by *neighborhood* of a vertex v, denoted by $N(v)$, the set of all vertices that are adjacent to that vertex. The cardinality of $N(v)$ is called the *degree* of vertex v and it is denoted by $deg(v)$. Furthermore, by $\delta(G)$ (resp. $\Delta(G)$), we refer to the *minimum* (resp. *maximum*) degree of G.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Furthermore, a subgraph of G *induced* by a set $S \subseteq V(G)$, denoted by $G[S]$, is a graph formed from the vertices of S and all edges connecting the pairs of vertices in S. We denote by P_n, C_n, and K_n a path, a cycle, and a complete graph on n vertices, respectively. The *girth* of a graph is the length of the shortest cycle of that graph. We say that a graph G is *unicyclic* if G is a connected graph containing exactly one cycle. A *cactus* graph is a connected graph in which every edge lies on at most one cycle.

The *distance* between two vertices in a graph is the number of edges in a shortest path connecting those vertices. We show by $N_i(v)$ the set of all vertices at distance i from v. Notice that with this notation, $N_1(v)$ corresponds to the neighborhood of v which we simply denote by $N(v)$. The *private neighborhood* of a vertex $v \in S$, denoted by $pn(v, S)$, is defined as: $pn(v, S) = \{u \in V(G) \,|\, N(u) \cap S = \{v\}\}$. We call each vertex in $pn(v, S)$ a *private neighbor* of v with respect to set S. Furthermore, the *external private neighborhood* of a vertex v with respect to a set S, denoted by $epn(v, S)$, is a set containing the private neighbors of v which are not in S, that is, $epn(v, S) = pn(v, S) \setminus S$.

A *matching M* in a graph G is a set of pairwise non-adjacent edges. If a matching M matches all vertices of a graph G, we call M a *perfect matching*. Two vertices are said to be *partners* if they are joined by an edge of a perfect matching M.

A set I of vertices in a graph G is an *independent* set if no two vertices in I are adjacent. An independent set is said to be *maximal* if no other independent set properly contains it. The maximum size of an independent set in a graph G, denoted by $\alpha(G)$, is called the *independence number* of G.

Notice that any independent set S in a graph G can be extended to a maximal independent set I in G. In addition, every maximal independent set is a minimal dominating set. We will frequently use these two arguments in our forthcoming proofs.

The most related results in the literature which provide useful tools for our work is due to Ulatowski (2015). We state the first result of Ulatowski in Lemma 1, which describes the graphs with upper domination number equal to their order.

Lemma 1 *(Ulatowski (2015)) For a graph G of order n, $\Gamma_{pr}(G) = n$ if and only if G is isomorphic to mK_2.*

Here, mK_2 denotes a graph with $m \geq 1$ copies of disjoint K_2. The result in Lemma 1 implies that K_2 is the only connected graph satisfying $\Gamma_{pr}(G) = n$. The next result establishes an upper bound for the upper domination number of a connected graph.

Lemma 2 *(Ulatowski (2015)) If G is a connected graph of order $n \geq 3$, then $\Gamma_{pr}(G) \leq n - 1$.*

In the same work, Ulatowski characterized the graphs satisfying the equality in the bound of Lemma 2. However, before stating this result, we recall some definitions and notations. The *subdivided star* $K_{1,t}^*$ is the graph obtained from a star $K_{1,t}$ by subdividing every edge once. Let $K_{1,t}^{*\Delta}$ for $\Delta \geq 0$ be a family of graphs obtained by attaching Δ triangles to the central vertex of a $K_{1,t}^*$ (see Figure 1).
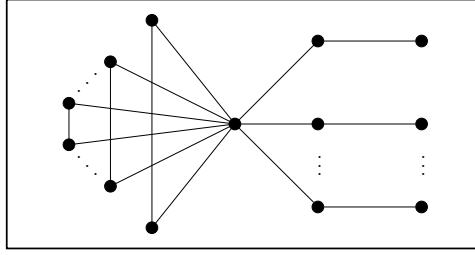
Fig. 1: A graph in the family $K_{1,t}^{*\Delta}$

Lemma 3 states the second result of Ulatowski regarding the graphs with upper domination number equal to one less than their order.

Lemma 3 *(Ulatowski (2015)) Let G be a connected graph of order $n \geq 3$. Then $\Gamma_{pr}(G) = n - 1$ if and only if $G \in \{C_3, C_5, K_{1,t}^{*\Delta}\}$.*

The following two lemmas give necessary conditions for the minimality of a paired dominating set. Notice that the notation $epn(u, v; S)$ which is used by Dorbec and Henning (2011) is defined as follows:
$\forall u, v \in S, epn(u, v; S) = \{w \in N(u) \cup N(v) \setminus S \mid N(w) \cap S \subseteq \{u, v\}\}$
In other words, for a vertex $w \in epn(u, v; S)$ it holds that either $w \in epn(u, S)$, or $w \in epn(v, S)$, or w is adjacent to both u and v and no other vertex in $S \setminus \{u, v\}$.

Lemma 4 *(Dorbec and Henning (2011)) Let S be a minimal PDS in a connected graph G of order at least 3 and let $\{u, v\} \subset S$ and $S' = S \setminus \{u, v\}$. If S' dominates both u and v and $G[S']$ contains a perfect matching, then $|epn(u, v; S)| \geq 1$.*

Lemma 5 *(Dorbec and Henning (2011)) Let S be a minimal PDS in a connected graph G of order at least 3 and let M be a perfect matching in $G[S]$. If $uv \in M$ and both u and v have degree at least 2 in $G[S]$, then $|epn(u, v; S)| \geq 1$.*

The result in Lemma 6 is useful in determining the relationship between the upper domination number and the upper paired domination number.

Lemma 6 *(Henning and Pradhan (2019)) Every minimal paired dominating set P in G contains a minimal dominating set S such that $|S| \geq |P|/2$.*

Here we state the relationship between the upper domination number and the upper paired domination number in Corollary 6.1, which is an immediate result of Lemma 6.

Corollary 6.1 *(Henning and Pradhan (2019)) For any graph G without isolated vertices, $\Gamma_{pr}(G) \leq 2\Gamma(G)$.*

In the remainder of this paper, we will investigate the properties of the graphs satisfying $\Gamma_{pr}(G) = 2\Gamma(G)$.
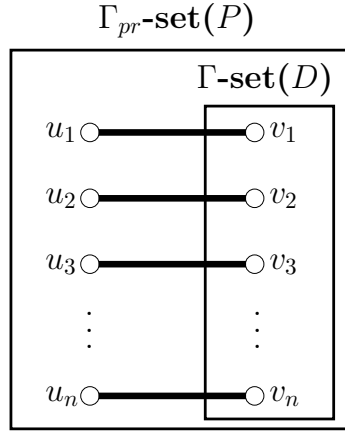
Γ_{pr}-**set**(P)

Fig. 2: A Γ_{pr}-set P in a graph with the property $\Gamma_{pr}(G) = 2\Gamma(G)$

3 Graphs with the property $\Gamma_{pr}(G) = 2\Gamma(G)$

The following result for the graphs with the property $\Gamma_{pr}(G) = 2\Gamma(G)$ is obtained from Lemma 6.

Lemma 7 *Let G be a graph with the property $\Gamma_{pr}(G) = 2\Gamma(G)$. Then every Γ_{pr}-set of G contains an independent Γ-set (see Figure 2).*

Proof: Let P be a Γ_{pr}-set of G with a perfect matching M. By Lemma 6, P contains a minimal dominating set D such that $|D| \geq \Gamma_{pr}/2$. Since $\Gamma_{pr}(G) = 2\Gamma(G)$, it follows that $|D| \geq \Gamma(G)$. Thus, D is a Γ-set. Now it suffices to prove that D is an independent set. Suppose to the contrary that D has two adjacent vertices v_1 and v_2. Suppose further that $u_1v_1, u_2v_2 \in M$. Let $M' = M \setminus \{u_1v_1, u_2v_2\} \cup \{v_1v_2\}$ and $P' = P \setminus \{u_1, u_2\}$. Notice that P' is a dominating set in G since it contains D. Furthermore, it has a perfect matching M'; therefore, P' is a paired dominating set, a contradiction to the minimality of P. Thus, D is an independent set. □

In the following, we first state our results for two special graph classes with the property $\Gamma_{pr}(G) = 2\Gamma(G)$, namely bipartite and unicyclic graphs. We then investigate special graph classes with $\Gamma_{pr}(G) = 2\Gamma(G)$ and restricted girth.

3.1 Bipartite graphs with the property $\Gamma_{pr(G)} = 2\Gamma(G)$

In this section, we characterize bipartite graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. We state our obtained result for bipartite graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ in Theorem 8.

Theorem 8 *Let G be a connected bipartite graph. Then $\Gamma_{pr}(G) = 2\Gamma(G)$ if and only if G is isomorphic to K_2.*

Proof: Let G be a connected bipartite graph with $\Gamma_{pr}(G) = 2\Gamma(G)$ and order n. Note that G has at least one partite of size at least $n/2$, which implies a minimal dominating set of size at least $n/2$. Hence

$\Gamma(G) \geq n/2$. Then we have $\Gamma_{pr}(G) \geq n$ which implies that $\Gamma_{pr}(G) = n$. By Lemma 1, G is isomorphic to mK_2. Since G is connected, it is isomorphic to K_2.

For the converse direction, it is easy to see that $\Gamma(K_2) = 1$ and $\Gamma_{pr}(K_2) = 2$ and hence $\Gamma_{pr}(K_2) = 2\Gamma(K_2)$. $\square$

3.2 Unicyclic graphs with the property $\Gamma_{pr}(G) = 2\Gamma(G)$

The aim of this section is to describe unicyclic graphs with the property $\Gamma_{pr}(G) = 2\Gamma(G)$. Before stating our result on unicyclic graphs with the property $\Gamma_{pr}(G) = 2\Gamma(G)$ in Theorem 10, we mention the following lemma which establishes lower bounds for upper domination number in unicyclic graphs:

Lemma 9 *Let G be a connected unicyclic graph of order n. Then the following hold:*

- *For even n, $\Gamma(G) \geq n/2$*

- *For odd n, $\Gamma(G) \geq (n-1)/2$*

Proof: Let G be a connected unicyclic graph of order n. Note that G has a single cycle, say C. Let x and y be two adjacent vertices of G on C. Let further G' be a graph obtained by removing the edge between x and y; that is, $V(G') = V(G)$ and $E(G') = E(G) - \{xy\}$. Since G' has no cycles, it is a tree and consequently a bipartite graph. If n is even, then G' has either two partites of size $n/2$ or at least one partite, say A' of size at least $n/2+1$. In the former case, at least one of the partites of size $n/2$ in G' is also an independent set in G and hence $\Gamma(G) \geq n/2$. In the latter case, one possibility is that x and y reside in different partites, in which case A' is also an independent set in G of size at least $n/2+1$. However, the other possibility is that x and y reside in the same partite A', in which case $A' - \{x\}$ is an independent set of size at least $n/2$ in G. Both possibilities imply that $\Gamma(G) \geq n/2$ for even n.

On the other hand, if n is odd, then G' has a partite, say A', of size at least $(n+1)/2$. Here two possibilities exist. One is that x and y are in different partites, in which case A' is also an independent set in G and thus, $\Gamma(G) \geq (n+1)/2$. The other possibility is that x and y reside in A', in which case $A' - \{x\}$ is an independent set of size at least $(n+1)/2 - 1 = (n-1)/2$ in G and hence, $\Gamma(G) \geq (n-1)/2$. Both possibilities yield $\Gamma(G) \geq (n-1)/2$ for odd n. $\square$

Now we are ready to state the main result of this section in Theorem 10.

Theorem 10 *Let G be a connected unicyclic graph. Then $\Gamma_{pr}(G) = 2\Gamma(G)$ if and only if $G \in \{C_3, C_5, K_{1,t}^{*1}\}$.*

Proof: Let G be a connected unicyclic graph with $\Gamma_{pr}(G) = 2\Gamma(G)$ and order n. In the case of even n, by Lemma 9, we have $\Gamma(G) \geq n/2$, which yields $\Gamma_{pr}(G) = n$. Then, by Lemma 1, G is isomorphic to mK_2 which is not unicyclic, contradiction. Thus, the case of even n does not lead to a unicyclic graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. However, for odd n, it follows from Lemma 9 that $\Gamma(G) \geq (n-1)/2$. This, in turn, leads to $\Gamma_{pr}(G) \geq (n-1)$. Since n is odd we have $\Gamma_{pr}(G) = (n-1)$. Then, it follows from Lemma 3 that G is isomorphic to $\{C_3, C_5, K_{1,t}^{*\Delta}\}$. Obviously, C_3 and C_5 are unicyclic graphs and $K_{1,t}^{*1}$ (for $\Delta = 1$) is the only unicyclic graph in the family $K_{1,t}^{*\Delta}$. Therefore, $G \in \{C_3, C_5, K_{1,t}^{*1}\}$.

For the converse direction, we show that if $G \in \{C_3, C_5, K_{1,t}^{*1}\}$, then $\Gamma_{pr}(G) = 2\Gamma(G)$. For the case of C_3 and C_5, it is easy to verify that $\Gamma(C_3) = 1$, $\Gamma_{pr}(C_3) = 2$ and $\Gamma(C_5) = 2$, $\Gamma_{pr}(C_5) = 4$. Furthermore, $\Gamma(K_{1,t}^{*1}) = t + 1$ and $\Gamma_{pr}(K_{1,t}^{*1}) = 2(t + 1)$.

$\square$

3.3 Graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ *and restricted girth*

In this section, we address the problem of describing graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ from a girth point of view. We begin with the case of graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and girth at least 6.

We first give some definitions and notations that we frequently use in the forthcoming proofs. Let P be any Γ_{pr}-set of a graph G with $\Gamma_{pr}(G) = 2\Gamma(G)$. By $G[P]$, we refer to the subgraph induced by the set P. Furthermore, if a vertex in P is only adjacent to a single vertex in P, we name it a *leaf* in $G[P]$.

Theorem 11 *Let G be a graph of girth at least 6. Then $\Gamma_{pr}(G) = 2\Gamma(G)$ if and only if G is isomorphic to mK_2 (for $m \geq 1$).*

Proof: Let G be a graph of girth at least 6 with $\Gamma_{pr}(G) = 2\Gamma(G)$. If $\Gamma_{pr}(G) = n$, by Lemma 1, G is isomorphic to mK_2 (for $m \geq 1$) and we are done.

We will now show that the case $\Gamma_{pr}(G) \leq (n-1)$ does not lead to a graph with $\Gamma_{pr}(G) = 2\Gamma(G)$ and complete the proof. Suppose that $\Gamma_{pr}(G) \leq (n-1)$. By Lemma 7, G has a Γ_{pr}-set P, which has an independent Γ-set B inside it. We further define set $A = P \setminus B$ as the set of partners of the vertices in B. Let $A = \{a_i\}$ and $B = \{b_i\}$ for $1 \leq i \leq \Gamma(G)$. Note that P has a perfect matching including pairs of matched vertices (a_i, b_i) for $a_i \in A$, $b_i \in B$, and $1 \leq i \leq \Gamma(G)$. Since $\Gamma_{pr}(G) \leq (n-1)$, it implies that there exists at least one vertex x in $V(G) \setminus P$.

We first show that $G[P] \neq mK_2$ where $m = \Gamma(G)$. Suppose to the contrary that $G[P] = \Gamma(G)K_2$. Note that the vertex x is adjacent to at most one vertex of each pair of matched vertices (a_i, b_i) since the girth is at least 6. Let Z be a set including one vertex from each pair of vertices (a_i, b_i) in P which is not adjacent to x. Since $G[P] = \Gamma(G)K_2$, the set Z is an independent set. Thus, $Z \cup x$ forms an independent set of size $\Gamma(G) + 1$, a contradiction to B being a Γ-set. Therefore, $G[P] \neq mK_2$ and without loss of generality, we assume that there exist at least two pairs of matched vertices, say (a_1, b_1) and (a_2, b_2), which have two adjacent endpoints, say $a_1a_2 \in E(G)$. Now by Lemma 4, it holds that $|epn(b_1, b_2; P)| \geq 1$. Let y be a vertex in $epn(b_1, b_2; P)$. Due to the girth restriction, y is not adjacent to both of b_1 and b_2. Thus, y is adjacent to exactly one of b_1 and b_2, say b_1. Notice that for each pair of matched vertices (a_i, b_i) for $2 \leq i \leq \Gamma(G)$, one of the following three cases holds:

Case 1: $b_ia_1 \notin E(G)$.
Case 2: $b_ia_1 \in E(G)$ and a_i is a leaf in $G[P]$.
Case 3: $b_ia_1 \in E(G)$ and a_i is not a leaf in $G[P]$.

Note that in Case 3 a vertex b_i for $2 \leq i \leq \Gamma(G)$ has a neighbor a_1 which is different from its partner a_i. Hence b_i has degree at least two in $G[P]$. Besides, the partner of b_i, namely a_i, has degree at least two in $G[P]$ since it is not a leaf in $G[P]$. Therefore, it follows by Lemma 5 that $|epn(a_i, b_i; P)| \geq 1$. This in turn implies that there exists at least one vertex c_i in $V(G) \setminus P$ which is a private neighbor of a_i and b_i. Due to girth at least 6 restriction, c_i is adjacent to exactly one of a_i and b_i. Since b_i is a vertex in the Γ-set B, c_i is only adjacent to b_i (see Figure 3).

Now let us define the three sets A', B', and C as follows: for each pair of matched vertices (a_i, b_i) for $2 \leq i \leq \Gamma(G)$ if Case 1 holds, then put b_i in B'; if Case 2 holds, then put a_i in A', and if Case 3 holds, then put c_i in C. It is easy to see that B' is an independent set since $B' \subseteq B$. Furthermore, $I = A' \cup C \cup \{y\}$ is an independent set since $I \subset N_2(a_1)$ and the girth of G is at least 6. The vertices in A' are leaves in $G[P]$; that is, they are only adjacent their partners b_i in $B \setminus B'$. Thus, no vertex in A' is adjacent to a vertex in B'. Besides, the vertices in C are private neighbors which are only adjacent to a vertex b_i in $B \setminus B'$. Hence, no vertex in C is adjacent to a vertex in B'. By definition no vertex in B' is adjacent to a_1
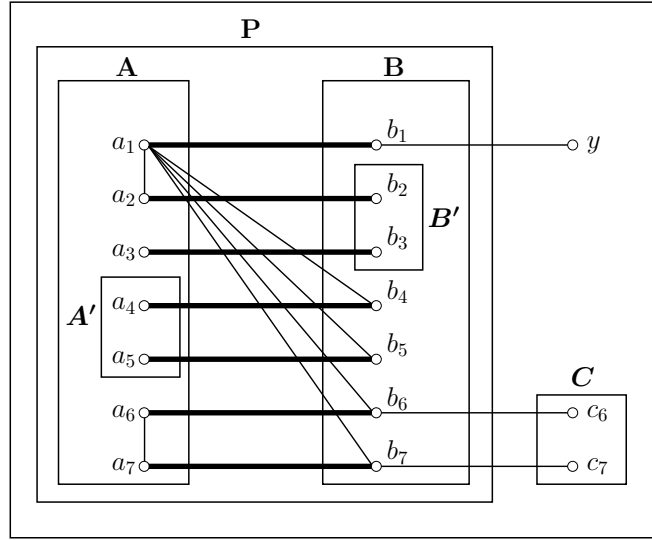
Fig. 3: The sets A', B', and C

and the vertex y is adjacent only to b_1. Hence we have that $\{y, a_1\} \cup A' \cup B' \cup C$ is an independent set. Note that the sets A', B', and C are mutually disjoint sets since for each pair of matched vertices (a_i, b_i) for $2 \le i \le \Gamma(G)$, exactly one of the three aforementioned cases holds. Thus, $|A' \cup B' \cup C| = \Gamma(G) - 1$. Hence, the set $\{y, a_1\} \cup A' \cup B' \cup C$ is an independent set of size $\Gamma(G) + 1$, a contradiction to B being a Γ-set. Therefore, there exists no graph G with $\Gamma_{pr}(G) \le (n-1)$, $\Gamma_{pr}(G) = 2\Gamma(G)$, and girth at least 6. Hence, G is isomorphic to mK_2 (for $m \ge 1$) and we are done.

For the converse direction, it can easily be verified that $\Gamma(mK_2) = m$ and $\Gamma_{pr}(mK_2) = 2m$. Therefore, $\Gamma_{pr}(mK_2) = \Gamma(mK_2)$.   □

In what follows, we proceed to the graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and girth smaller than 6. We focus on graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$ and girth at least 4 and provide a characterization for a special family of graphs with the mentioned properties, that is, C_3-free cactus graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$.

From here onward, we assume that G is a C_3-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Furthermore, recall that by Lemma 7, G has a Γ_{pr}-set P with an independent Γ-set inside it. Let further $P = A \cup B$, where B is an independent Γ-set and A is the set of partners of the vertices in B. Let $A = \{a_i\}$ and $B = \{b_i\}$ for $1 \le i \le \Gamma(G)$. Note that P has a perfect matching including pairs of matched vertices (a_i, b_i) for $a_i \in A$, $b_i \in B$, and $1 \le i \le \Gamma(G)$. We now continue with presenting a number of lemmas which provide essential tools for the characterization of C_3-free cactus graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$.

Lemma 12 *Let G be C_3-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further P be any Γ_{pr}-set of G. Then, for $1 \le i \le \Gamma(G)$, at least one vertex of each pair of matched vertices (a_i, b_i) is a leaf in $G[P]$.*

Proof: Suppose to the contrary that there exist k pairs of matched vertices (a_i, b_i) in P such that both a_i and b_i have degree at least 2 in $G[P]$ for $1 \le k \le \Gamma(G)$. We first look at the case $k = 1$, where there exists a single pair of matched vertices, say (a_1, b_1), in P such that both a_1 and b_1 have degree at least 2
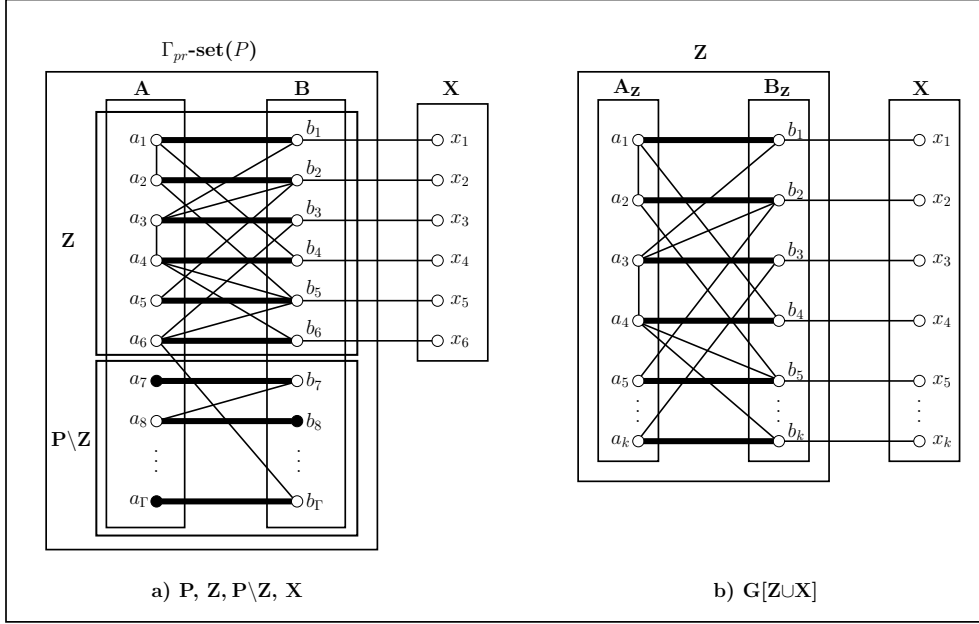
Fig. 4: The sets $P, Z, P \setminus Z$ and X in G and the subgraph $G[Z \cup X]$

in $G[P]$. By Lemma 5, $|epn(a_1, b_1; P)| \geq 1$, which implies that a_1 and b_1 have a private neighbor x_1 in $V(G) \setminus P$. Since G is a C_3-free graph, x_1 is not adjacent to both a_1 and b_1. Thus, x_1 is adjacent to exactly one of a_1 and b_1. Since B is a Γ-set, x_1 is adjacent to b_1. We define I_L as a set containing one leaf in $G[P]$ from each pair of matched vertices (a_i, b_i) for $2 \leq i \leq \Gamma(G)$ in. It is clear that I_L is an independent set. Thus, $\{x_1, a_1\} \cup I_L$ is an independent set of size $\Gamma(G) + 1$, which implies a minimal dominating set of size at least $\Gamma(G) + 1$, a contradiction to B being a Γ-set of G. Hence, we are done with the case $k = 1$.

Then we proceed to the case with k pairs of matched vertices (a_i, b_i) in P such that both a_i and b_i have degree at least 2 in $G[P]$ for $2 \leq k \leq \Gamma(G)$. Let Z be the set containing pairs of matched vertices (a_i, b_i) in P such that both a_i and b_i have degree at least 2 in $G[P]$. We further assume that $Z = A_z \cup B_z$ where $A_z \subseteq A$ and $B_z \subseteq B$ (see Figure 4). By Lemma 5, for each pair of (a_i, b_i) in Z, we have that $|epn(a_i, b_i; P)| \geq 1$ for $1 \leq i \leq k$. This implies that each pair of vertices a_i and b_i in Z have a private neighbor x_i in $V(G) \setminus P$. The vertex x_i is not adjacent to both a_i and b_i since G is a C_3-free graph. Thus, each x_i is adjacent to exactly one of a_i and b_i. Since B is a Γ-set, x_i is adjacent only to b_i. We define X as a set containing x_i for $1 \leq i \leq k$ (see Figure 4). Notice that from each pair of matched vertices (a_i, b_i) in $P \setminus Z$ at least one vertex is a leaf in $G[P]$. The leaves in $G[P]$ are shown with filled circles in Figure 4. Let I_L be a set containing one vertex from each pair of matched vertices (a_i, b_i) in $P \setminus Z$ which is a leaf in $G[P]$. Therefore, $|I_L| = \Gamma(G) - |Z|$. We continue with the following claims.

Claim 1: Each $a_i \in A_z$ has at least one neighbor in B_z different from its partner b_i.

Proof of Claim 1: Suppose to the contrary that a vertex $a_i \in A_z$, say a_1, is adjacent only to its partner b_1 in $G[Z]$. Then $\{a_1, x_1\} \cup (B_z \setminus \{b_1\}) \cup I_L$ is an independent set of size $2 + \Gamma(G) - 1 = \Gamma(G) + 1$, a
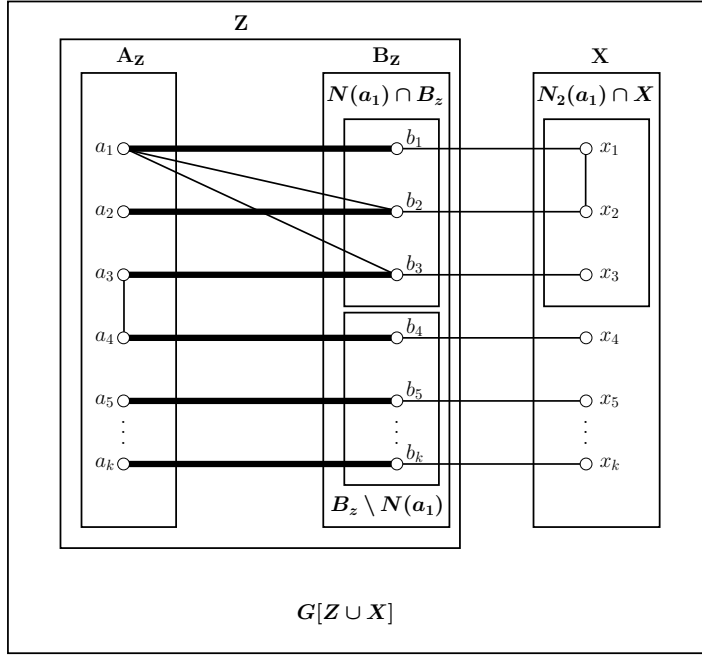
Fig. 5: The sets $N(a_1) \cap B_z$ and $N_2(a_1) \cap X$ in $G[Z \cup X]$

contradiction to B being a Γ-set. □

Claim 2: For any $a_i \in A_Z$, it holds that at least two vertices in $N_2(a_i) \cap X$ are adjacent.

Proof of Claim 2: Suppose to the contrary that there exists a vertex in A_Z, say a_1, such that $(N_2(a_1) \cap X)$ is an independent set (see Figure 5). Then, $a_1 \cup (N_2(a_1) \cap X) \cup (B_Z \setminus N(a_1)) \cup I_L$ is an independent set of size $\Gamma(G) + 1$, a contradiction to B being a Γ-set. Thus, at least two vertices in $N_2(a_i) \cap X$ are adjacent. □

The argument in Claim 2 implies that for each vertex $a_i \in A_Z$, there is at least one pair of adjacent vertices (x_k, x_l) in X. Since $|A_Z| = k$, there must exist at least k pairs of adjacent vertices in X. However, since $|X| = k$, there exist at most $k/2$ pairs with disjoint vertices in X. Therefore, there exist at least two vertices in A_z, say a_1 and a_2, whose corresponding pairs of adjacent vertices in X are not disjoint; that is, these pairs have either one or two vertices in common. Recall that each vertex $x_i \in X$ is a private neighbor of a vertex $b_i \in B_z$; that is, each x_i is adjacent to a single vertex b_i in B_z. Now let x_1 and x_2 be the corresponding pair of adjacent vertices for a_1 in X. This implies that b_2 is also adjacent to a_1 and we have a 5-cycle $C_1 = (a_1, b_1, x_1, x_2, b_2)$. Note that if x_1 and x_2 are also the corresponding pair of adjacent vertices for a_2, then b_1 is also adjacent to a_2 and we have a 5-cycle $C_2 = (a_2, b_1, x_1, x_2, b_2)$. However, C_1 and C_2 are two cycles with a common edge $x_1 x_2$, a contradiction to G being a cactus graph. In the other case, if the corresponding pair of adjacent vertices for a_2 has only one vertex, say x_2, in common with that of a_1, then x_2 is adjacent to another vertex in X, say x_3. This in turn implies that b_3 is also adjacent to a_2 and we have a 5-cycle $C_3 = (a_2, b_2, x_2, x_3, b_3)$. In this case, we have two cycles C_1

and C_3 with a common edge b_2x_2, a contradiction to G being a cactus graph. Therefore, there are no pairs of matched vertices (a_i, b_i) in P such that both a_i and b_i have degree at least 2 in $G[P]$ for $1 \leq k \leq \Gamma(G)$. □

Lemma 12 implies that at least one vertex of each pair of matched vertices (a_i, b_i) in P is a leaf in $G[P]$. We define the set L_p as a set containing one leaf from each pair of matched vertices (a_i, b_i) in $G[P]$ for $1 \leq i \leq \Gamma(G)$. It is clear that L_p is an independent set in $G[P]$ and $|L_p| = \Gamma(G)$. In the following lemmas, we obtain some other properties of C_3-free cactus graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$.

Lemma 13 *Let G be a C_3-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further P be any Γ_{pr}-set of G. If there exists a vertex x in $V(G) \setminus P$, it has exactly two neighbors in P.*

Proof: We first prove that if there exists a vertex x in $V(G) \setminus P$, it has at least two neighbors in P.

Claim 13.1 *Every vertex x in $V(G) \setminus P$ has at least two neighbors in P.*

Proof of Claim 13.1: Suppose to the contrary that there exists a vertex x in $V(G) \setminus P$ which has exactly one neighbor in P, say b_1. By Lemma 12, one vertex from each pair of matched vertices (a_i, b_i) in P is a leaf in $G[P]$. Let further L'_p be a set containing one leaf in $G[P]$ from each pair of matched vertices (a_i, b_i) for $2 \leq i \leq \Gamma(G)$. Thus, $|L'_p| = \Gamma(G) - 1$. Then, $\{a_1, x\} \cup L'_p$ is a minimal dominating set of size $2 + \Gamma(G) - 1 = \Gamma(G) + 1$, a contradiction to B being a Γ-set of G. Therefore, each vertex x in $V(G) \setminus P$ has at least two neighbors in P. □

Now we proceed by showing that the case of a vertex x in $V(G) \setminus P$ with at least three neighbors in P leads to a contradiction and complete the proof of Lemma 13. Suppose to the contrary that x is a vertex in $V(G) \setminus P$ with at least three neighbors in P. We define a set Z as follows: for each pair of matched vertices (a_i, b_i) in P, if $a_i \in N(x)$, put b_i in Z; otherwise, if $b_i \in N(x)$, put a_i in Z.

We first show that Z is an independent set. Suppose to the contrary that two vertices in Z, say a_1 and a_2 are adjacent. By definition of Z, the partners of these vertices, namely b_1 and b_2 are neighbors of x. Moreover, since a_1 and a_2 are adjacent, by Lemma 4, the vertices b_1 and b_2 have a private neighbor, say y, in $V(G) \setminus P$. Definitely, the vertex y is different from x since x has at least three neighbors in P and cannot be a private neighbor for b_1 and b_2. However, y is adjacent to exactly one of b_1 or b_2 since otherwise we have two cycles $(yb_1a_1a_2b_2)$ and $(xb_1a_1a_2b_2)$ with a common edge a_1a_2, a contradiction to G being a cactus graph. Thus, y is adjacent to one of b_1 or b_2, say b_1. Then, y is a vertex in $V(G) \setminus P$ with exactly one neighbor b_1 in P, a contradiction to Claim 13.1. Therefore, Z is an independent set.

Let L'_p be a set containing one leaf in $G[P]$ from each pair of matched vertices (a_i, b_i) in P such that neither a_i nor b_i is adjacent to x. It is obvious that $|L'_p| = \Gamma(G) - |Z|$. Then, $\{x\} \cup Z \cup L'_p$ is an independent set of size $\Gamma(G) + 1$, which implies a minimal dominating set of size at least $\Gamma(G) + 1$, a contradiction to B being a Γ-set. Therefore, any vertex x in $V(G) \setminus P$ has exactly two neighbors inside P. □

Lemma 14 *Let G be C_3-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further P be any Γ_{pr}-set of G. If there exists a vertex x in $V(G) \setminus P$, then the partners of the two neighbors of x in P are adjacent.*

Proof: Let x be a vertex in $V(G) \setminus P$. By Lemma 13, the vertex x has exactly two neighbors in P, say b_1 and b_2. Suppose to the contrary that the partners of b_1 and b_2, namely a_1 and a_2, are non-adjacent. By Lemma 12, we know that at least one vertex from each pair of matched vertices (a_i, b_i) in P is a leaf

in $G[P]$. Let L'_p be the set containing one leaf in $G[P]$ from each pair of matched vertices (a_i, b_i) for $3 \leq i \leq \Gamma(G)$. Note that $|L'_p| = \Gamma(G) - 2$. Thus, $\{x, a_1, a_2\} \cup L'_p$ is a minimal dominating set of size $\Gamma(G) + 1$, a contradiction to B being a Γ-set. Therefore, the partners of the neighbors of x in P, namely a_1 and a_2, are adjacent. $\square$

Lemma 15 *Let G be a C_3-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further P be any Γ_{pr}-set of G. If there exist two vertices x_1 and x_2 in $V(G) \setminus P$, then they have no common neighbor in P.*

Proof: Suppose to the contrary that x_1 and x_2 are two vertices in $V(G) \setminus P$, which have common neighbors in P. By Lemma 13, each of x_1 and x_2 has exactly two neighbors in P. Let a_1 and a_2 be the two neighbors of x_1 in P. By Lemma 14, the partners of a_1 and a_2, namely b_1 and b_2, are adjacent. If x_1 and x_2 have two common neighbors in P, that is, if x_2 is also adjacent to a_1 and a_2, then we have two cycles $(x_1 a_1 b_1 b_2 a_2)$ and $(x_2 a_1 b_1 b_2 a_2)$ with a common edge $b_1 b_2$, a contradiction to G being a cactus graph. On the other hand, if x_1 and x_2 have only one common neighbor, say a_2, then x_2 has another neighbor in P, say a_3. By Lemma 14, the partners of a_2 and a_3, namely b_2 and b_3, are adjacent. Then we have two cycles $(x_1 a_1 b_1 b_2 a_2)$ and $(x_2 a_2 b_2 b_3 a_3)$ with a common edge $a_2 b_2$, a contradiction to G being a cactus graph. Hence, x_1 and x_2 have no common neighbor in P. $\square$

Lemma 16 *Let G be a C_3-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further P be any Γ_{pr}-set of G. Then, $\Delta(G[P]) \leq 2$.*

Proof: Let G be a C_3-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further P be any Γ_{pr}-set of G which includes pairs of matched vertices (a_i, b_i) for $1 \leq i \leq \Gamma(G)$. Suppose to the contrary that a vertex in P, say a_1, has at least three neighbors in P. One of these three neighbors is the partner of a_1, namely b_1. Without loss of generality, let b_2 and b_3 be the other two neighbors of a_1 in P. Since a_1 is adjacent to b_2, by Lemma 4, we have $|epn(a_2, b_1; P)| \geq 1$, which implies that a_2 and b_1 have a private neighbor x in $V(G) \setminus P$. By Lemma 13, x is adjacent to both a_2 and b_1. In addition, since a_1 is adjacent to b_3, by Lemma 4, we have $|epn(a_3, b_1; P)| \geq 1$. This implies that a_3 and b_1 have a private neighbor y in $V(G) \setminus P$. By Lemma 13, y is adjacent to both a_3 and b_1. However, x and y are two vertices in $V(G) \setminus P$ with a common neighbor b_1 in P, a contradiction to Lemma 15. Thus, a vertex in P has at most two neighbors in P; that is, $\Delta(G[P]) \leq 2$. $\square$

Lemma 17 *Let G be a C_3-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further P be any Γ_{pr}-set of G. At most one vertex from each pair of matched vertices (a_i, b_i) in P has a neighbor in $V(G) \setminus P$.*

Proof: Suppose to the contrary that there exists a pair of matched vertices in P, say (a_1, b_1), such that both a_1 and b_1 have neighbors in $V(G) \setminus P$. Let further x_1 be the neighbor of b_1 and x_2 be the neighbor of a_1 in $V(G) \setminus P$. It is clear that $x_1 \neq x_2$ since G is a C_3-free graph. By Lemma 13, x_1 has two neighbors in P. Hence, we may assume that x_1 is adjacent to another vertex in P, say b_2. Similarly, x_2 has two neighbors in P; however, by Lemma 15, x_2 has no common neighbor with x_1 in P. Thus, we may assume that x_2 is adjacent to b_3. By Lemma 14, a_1 is adjacent to a_2 and b_1 is adjacent to a_3. Then, (a_1, b_1) is a pair of matched vertices both of which have degree at least two in $G[P]$, a contradiction to Lemma 12. Thus, at most one vertex from each pair of matched vertices (a_i, b_i) in P has a neighbor in $V(G) \setminus P$. $\square$

Lemma 18 *Let G be a C_3-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further P be any Γ_{pr}-set of G. Then any two vertices x_1 and x_2 in $V(G) \setminus P$ are non-adjacent.*

Proof: Let G be a C_3-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. Let further P be any Γ_{pr}-set of G which includes pairs of matched vertices (a_i, b_i) for $1 \leq i \leq \Gamma(G)$. Suppose to the contrary that x_1 and x_2 are two adjacent vertices in $V(G) \setminus P$. We know that by Lemma 13 and Lemma 15, x_1 is adjacent to exactly two vertices in P, say $\{b_1, b_2\}$, and x_2 is adjacent to two different vertices, say $\{b_3, b_4\}$. By Lemma 14, the partners of b_1 and b_2, namely a_1 and a_2, and the partners of b_3 and b_4, namely a_3 and a_4, are adjacent. By Lemma 16, a_1, a_2, a_3, and a_4 have no other neighbors in $G[P]$. Then there exists an independent set $I = B \setminus \{b_1, b_2, b_3, b_4\}$ in $G[P]$ such that $|I| = \Gamma - 4$. Then $\{x_1, b_1, b_2, a_3, a_4\} \cup I$ is a minimal dominating set of size $5 + |I| = 5 + \Gamma(G) - 4 = \Gamma(G) + 1$, a contradiction to B being a Γ-set. Therefore, any two vertices x_1 and x_2 in $V(G) \setminus P$ are non-adjacent. $\quad\square$

Now we are ready to give our main result in this section in Theorem 19, which describes the structure of C_3-free cactus graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. Notice that the graph $m_1 C_5 + m_2 K_2$, which is stated in Theorem 19, is a graph composed of m_1 copies of disjoint C_5 and m_2 copies of disjoint K_2.

Theorem 19 *Let G be a C_3-free cactus graph. Then $\Gamma_{pr}(G) = 2\Gamma(G)$ if and only if G is isomorphic to $m_1 K_2 + m_2 C_5$ for $m_1 + m_2 \geq 1$.*

Proof: Let G be a C_3-free cactus graph with $\Gamma_{pr}(G) = 2\Gamma(G)$. By Lemma 7, G has a Γ_{pr}-set P with an independent Γ-set B inside it. Let further A be the set of partners of the vertices in B. Hence, $P = A \cup B$. Moreover, P has a perfect matching including pairs of matched vertices (a_i, b_i) for $1 \leq i \leq \Gamma(G)$. We start with the case where there exist no vertices in $V(G) \setminus P$, that is, $\Gamma_{pr}(G) = n$. By Lemma 1, G is isomorphic to $m_1 K_2$ for $m_1 \geq 1$, which is a cactus graph and we are done with this case.

Next, we proceed with the case where there exists at least one vertex x_1 in $V(G) \setminus P$, that is, $\Gamma_{pr}(G) \leq n - 1$. By Lemma 13, x_1 has two neighbors in P, say b_1 and b_2. By Lemma 14, the partners of b_1 and b_2, namely a_1 and a_2, are adjacent. Since a_1 and a_2 each has two neighbors in P, by Lemma 16, they have no other neighbors in P. By Lemma 17, a_1 and a_2 have no neighbors in $V(G) \setminus P$ since their partners, namely b_1 and b_2, have a neighbor x_1 in $V(G) \setminus P$. As a_1 and a_2 each has two neighbors in $G[P]$, by Lemma 12, their partners, namely b_1 and b_2, are only adjacent to their partners and have no other neighbors in $G[P]$. Moreover, b_1 and b_2 have no neighbors in $V(G) \setminus P$ other than x_1 by Lemma 15. The vertex x_1 has two neighbors b_1 and b_2 in P and has no other neighbors in $V(G) \setminus P$ by Lemma 18. Hence, the vertices $\{x_1, b_1, a_1, a_2, b_2\}$ form a disjoint 5-cycle in G. We can make the previous arguments for any vertex in $V(G) \setminus P$; that is, any vertex in $V(G) \setminus P$ together with four vertices from P form a disjoint 5-cycle in G. Therefore, G is composed of components which are either K_2 or C_5.

For the converse direction, it can easily be verified that if G is isomorphic to $m_1 K_2 + m_2 C_5$ for $m_1 + m_2 \geq 1$, then we have that $\Gamma(m_1 K_2 + m_2 C_5) = m_1 + 2m_2$, and $\Gamma_{pr}(m_1 K_2 + m_2 C_5) = 2m_1 + 4m_2$ and hence $\Gamma_{pr}(m_1 K_2 + m_2 C_5) = 2\Gamma(m_1 K_2 + m_2 C_5)$.

$\quad\square$

An immediate result of Theorem 19 for connected graphs is stated in Corollary 19.1.

Corollary 19.1 *Let G be a connected C_3-free cactus graph. Then $\Gamma_{pr}(G) = 2\Gamma(G)$ if and only if G is either C_5 or K_2.*

Note that some of the arguments used in Lemmas 12-18 are not restricted to cactus graphs and can be used for the general case of C_3-free graphs. Then the question that arises here is whether all lemmas mentioned above can be extended for the general case of C_3-free graphs with $\Gamma_{pr}(G) = 2\Gamma(G)$. Hence, we pose the following as an open question:

Question: Does Theorem 19 hold for C_3-free graphs?

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References

M. Chellali and T. Haynes. Trees with unique minimum paired-dominating sets. *Ars Comb.*, 73, 10 2004.

J. Cyman, M. Dettlaff, M. A. Henning, M. Lemańska, and J. Raczek. Total domination versus paired-domination in regular graphs. *Discussiones Mathematicae Graph Theory*, 38(2):573 – 586, 2018.

E. Delavina, R. Pepper, and B. Waller. Lower bounds for the domination number. *Discussiones Mathematicae Graph Theory*, 30(3):475 – 487, 2010.

W. J. Desormeaux, T. W. Haynes, and M. A. Henning. *Paired Domination in Graphs*, pages 31 – 77. Springer International Publishing, Cham, 2020.

P. Dorbec and M. Henning. Upper paired domination in claw-free graphs. *J. Comb. Optim.*, 22:235–251, 08 2011.

P. Dorbec, S. Gravier, and M. Henning. Paired-domination in generalized claw-free graphs. *J. Comb. Optim.*, 14:1–7, 07 2007a.

P. Dorbec, M. A. Henning, and J. McCoy. Upper total domination versus upper paired-domination. *Quaestiones Mathematicae*, 30(1):1–12, 2007b.

P. Dorbec, B. Hartnell, and M. A. Henning. Paired versus double domination in k 1, r-free graphs. *Journal of Combinatorial Optimization*, 27(4):688 – 694, 2014.

O. Favaron and M. Henning. Paired-domination in claw-free cubic graphs. *Graphs and Combinatorics*, 20:447–456, 11 2004. doi: 10.1007/s00373-004-0577-9.

W. Goddard and M. A. Henning. A characterization of cubic graphs with paired-domination number three-fifths their order. *Graphs and Combinatorics*, 25(5):675 – 692, 2009.

M. Hajian, M. A. Henning, and N. J. Rad. A new lower bound on the domination number of a graph. *Journal of combinatorial optimization*, 38(3):721 – 738, 2019.

T. W. Haynes and P. J. Slater. Paired-domination in graphs. *Networks*, 32(3):199–206, 1998.

M. A. Henning. Graphs with large paired-domination number. *Journal of combinatorial optimization*, 13 (1):61 – 78, 2007.

M. A. Henning and D. Pradhan. Algorithmic aspects of upper paired-domination in graphs. *Theoretical Computer Science*, 804:98 – 114, 2019.

M. Lemanska. Lower bound on the domination number of a tree. *Discussiones Mathematicae. Graph Theory*, 2, 01 2004. doi: 10.7151/dmgt.1222.

B. Panda and D. Pradhan. Minimum paired-dominating set in chordal bipartite graphs and perfect elimi-
nation bipartite graphs. *Journal of Combinatorial Optimization*, 26(4):770 – 785, 2013.

D. Pradhan and B. Panda. Computing a minimum paired-dominating set in strongly orderable graphs.
Discrete Applied Mathematics, 253:37 – 50, 2019.

O. Schaudt. Paired-and induced paired-domination in {E, net}-free graphs. *Discussiones Mathematicae
Graph Theory*, 32(3):473 – 485, 2012a.

O. Schaudt. Total domination versus paired domination. *Discussiones Mathematicae Graph Theory*, 32
(3):435–447, 2012b.

E. Shan, L. Kang, and M. A. Henning. A characterization of trees with equal total domination and paired-
domination numbers. *Australasian Journal of Combinatorics*, 30:31 – 40, 2004.

W. Ulatowski. All graphs with paired-domination number two less than their order. *Opuscula Mathemat-
ica*, 33, 01 2013.

W. Ulatowski. The paired-domination and the upper paired-domination numbers of graphs. *Opuscula
Mathematica*, 35:127, 01 2015.