

Destroying Multicolored Paths and Cycles in Edge-Colored Graphs

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We study the computational complexity of c -COLORED P_ℓ DELETION and c -COLORED C_ℓ DELETION. In these problems, one is given a c -edge-colored graph and wants to destroy all induced c -colored paths or cycles, respectively, on ℓ vertices by deleting at most k edges. Herein, a path or cycle is c -colored if it contains edges of c distinct colors. We show that c -COLORED P_ℓ DELETION and c -COLORED C_ℓ DELETION are NP-hard for each non-trivial combination of c and ℓ . We then analyze the parameterized complexity of these problems. We extend the notion of neighborhood diversity to edge-colored graphs and show that both problems are fixed-parameter tractable with respect to the *colored neighborhood diversity* of the input graph. We also provide hardness results to outline the limits of parameterization by the standard parameter solution size k . Finally, we consider bicolored input graphs and show a special case of 2-COLORED P_4 DELETION that can be solved in polynomial time.

Keywords: NP-hard problem, graph modification, edge-colored graphs, parameterized complexity

1 Introduction

A classic type of graph problems are *edge-deletion* problems, where one wants to modify a given graph such that it fulfills some graph property Π using a minimum number of edge deletions. While edge-deletion problems are well-studied on simple, uncolored graphs [ASS17, BBD06, CDFG20, EMC88, Yan81], there is not much work on such problems on edge-colored graphs. At the same time, edge-colored graphs have an increasing number of applications, for example as a formal model of multilayer networks, an important tool for describing complex systems with multiple types of relations [BCG⁺13, BBC⁺14, SK20]. Motivated by this, we analyze the computational complexity of the following fundamental edge-deletion problem on edge-colored graphs.

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c -COLORED P_ℓ DELETION ($cP_\ell D$)

Input: A c -edge-colored graph $G = (V, E = E_1 \sqcup \dots \sqcup E_c)$ ⁽ⁱ⁾, an integer k .

Question: Is there a set S of at most k edges such that deleting S from G results in a graph that contains no c -colored P_ℓ as induced subgraph?

Herein, a path is c -colored if its edge set is colored by exactly c distinct colors. The set S is called a *solution*. We also consider c -COLORED C_ℓ DELETION ($cC_\ell D$) where we aim to destroy all c -colored induced cycles of ℓ vertices for a fixed $c \in [\ell]$. Moreover, several of our results hold also for the variant where we aim to destroy non-induced c -colored paths or cycles. Let us remark that we are not aware of a direct application of these problems. It seems, however, very plausible that algorithmic knowledge about these fundamental edge-deletion problems will be useful in more applied settings. They may, for example, serve as a starting point of showing hardness of other edge-modification problems in edge-colored graphs.

Related Work. The uncolored case of $cP_\ell D$ and $cC_\ell D$ is NP-hard for any $\ell \geq 3$ [EMC88, Yan81]. In fact, it is known that H DELETION is NP-complete for any graph H if and only if H has at least two edges [ASS17]. Recently, it was shown that $2P_3 D$ is NP-hard as well [GKSS21]. If the input is restricted to graphs G where each bicolored P_3 of G is an induced subgraph of G , then $2P_3 D$ is polynomial-time solvable [GKSS21]. Furthermore, the problem of destroying not only induced, but also non-induced bicolored paths containing three vertices is polynomial-time solvable on bicolored graphs but NP-hard on 3-colored graphs [CL18]. The problem of detecting paths and cycles with certain edge-colorings has also received a considerable amount of attention [ADF⁺08, BLWZ05, GLM⁺09, GdLMM12]. Considering vertex-colored graphs instead of edge-colored graphs, Bruckner et al. [BHK⁺12] studied the parameterized complexity of an edge-deletion problem, where one aims to obtain a graph in which no connected component contains two vertices of the same color.

Our Results. First, we consider the classical complexity of $cP_\ell D$ and $cC_\ell D$ in Section 3. We show that $cP_\ell D$ is NP-hard for each $\ell \geq 3$ and each $c \in [\ell - 1]$ and that $cC_\ell D$ is NP-hard for each $\ell \geq 3$ and each $c \in [\ell]$. Since $cP_\ell D$ is trivially solvable if $\ell < 3$ or $c \geq \ell$, and $cC_\ell D$ is not properly defined if $\ell < 3$ and trivially solvable if $c > \ell$, this implies that $cP_\ell D$ and $cC_\ell D$ are NP-hard for each non-trivial combination of c and ℓ . An aspect that complicates the analysis of edge-deletion problems is that an edge deletion may create a new forbidden induced subgraph. To formalize this effect, we introduce the following notion. An input graph G of an instance (G, k) of $cP_\ell D$ ($cC_\ell D$) is *strictly non-cascading* if each subgraph of G that is a c -colored path (a c -colored cycle) is an *induced* subgraph. Using these terms, $2P_3 D$ is polynomial-time solvable on strictly non-cascading graphs [GKSS21]. We show that $cP_\ell D$ is NP-hard on strictly non-cascading graphs for each $\ell \geq 4$ and each $c \in [2, \ell - 2]$ and that $cC_\ell D$ is NP-hard on strictly non-cascading graphs for each $\ell \geq 3$ and each $c \in [\ell]$.

Second, we consider the parameterized complexity of $cP_\ell D$ in Section 4. We consider a new parameter that we call the *colored neighborhood diversity*. This parameter measures the number of sets of vertices that have different colored neighborhoods in the edge-colored graph. We show that $cP_\ell D$ and $cC_\ell D$ are fixed-parameter tractable with respect to this parameter. We believe that colored neighborhood diversity may be of broader interest in the study of computational problems on edge-colored input graphs. Furthermore, we study parameterization by the standard parameter solution size k . By using standard search tree techniques for uncolored graphs [Cai96], it is easy to see that $cP_\ell D$ and $cC_\ell D$ are fixed-parameter

⁽ⁱ⁾ The \sqcup -operator denotes the disjoint union. Hence, $E_i \cap E_j = \emptyset$ for $i, j \in [c], i \neq j$.

tractable with respect to k for any fixed $\ell \geq 1$ and any $c \in [\ell - 1]$ or $c \in [\ell]$, respectively. We prove that for $cP_\ell D$ (or $cC_\ell D$, respectively) this cannot be improved to a running time where the exponential factor is independent of ℓ unless $\text{FPT}=\text{W}[2]$ even if forbidden subgraphs can be detected in polynomial time.

Finally, we study $2P_4 D$ on subclasses of bicolored input graphs. We show that $2P_4 D$ remains NP-hard even when each edge color induces a cluster graph. If, additionally, the input graph contains no induced bicolored P_4 that starts and ends with edges of the same color, then $2P_4 D$ can be solved in polynomial time. The algorithm is based on a characterization of such graphs that may be of independent interest.

2 Preliminaries

For integers $a \in \mathbb{N}$ and $b \in \mathbb{N}$ we define $[a, b] := \{a, \dots, b\}$ if $a \leq b$ and $[a, b] = \emptyset$ if $a > b$. Furthermore, we define $[b] := [1, b]$. We denote the vertex set of a graph G by $V(G)$ and its edge set by $E(G)$. Throughout this work let $n := |V(G)|$ and $m := |E(G)|$. By $|G| := n + m$ we denote the *size* of G . In contrast to a graph, the edges of a *multigraph* are a multiset. Hence, in a multigraph two vertices can have multiple edges between them. In an *edge-colored graph*, the edge set is partitioned into c disjoint, non-empty subsets E_1, \dots, E_c . We call such a graph *c-colored* and if $c = 1$ we call the graph *uncolored*. For sake of illustration we define $E_b := E_1$ as the set of *blue* edges, $E_r := E_2$ as the set of *red* edges, $E_y := E_3$ as the set of *yellow* edges, $E_g := E_4$ as the set of *green* edges, and draw them in the figures accordingly. For a given graph G we denote the set of all edges with color $\alpha \in [c]$ by $E_\alpha(G)$.

For two vertex sets $V' \subseteq V(G)$ and $V'' \subseteq V(G)$ we define $E_G(V', V'') := \{\{u, v\} \in E \mid u \in V' \text{ and } v \in V''\}$ and $E_G(V') := E_G(V', V')$. We call $G' := (V', E')$ a *subgraph* of G if $V' \subseteq V(G)$ and $E' \subseteq E_G(V')$. If $E' = E_G(V')$ we call (V', E') an *induced subgraph* and write $G[V']$. We denote the graph formed by deleting the edges of an edge set E' from G by $G - E' := (V, E \setminus E')$. For a vertex $v \in V(G)$, we denote the *open neighborhood* of v in G by $N_G(v) := \{u \in V(G) \mid \{u, v\} \in E(G)\}$ and the *closed neighborhood* by $N_G[v] := N_G(v) \cup \{v\}$. We denote the *degree* of v in G by $\deg_G(v) := |N_G(v)|$. For every color $i \in [c]$ we define the *i-neighborhood* by $N_G^i(v) := \{u \in V(G) \mid \{u, v\} \in E_i\}$. We may drop the subscript \cdot_G when it is clear from context.

We say that a vertex set $V' \subseteq V(G)$ is a *vertex cover* for G if at least one of u and v is in V' for each edge $\{u, v\} \in E(G)$. We say that a vertex set $V' \subseteq V(G)$ is an *independent set* if $\{u, v\} \notin E(G)$ for each pair of vertices $u, v \in V'$. A graph G is called *tripartite* if $V(G)$ can be partitioned into 3 (possibly empty) independent sets.

A graph G is a *path* if it is possible to index its vertices with numbers from $[n]$, in such a way that $\{v_i, v_j\} \in E(G)$ if and only if $i + 1 = j$. A path with the additional edge $\{v_n, v_1\}$ is called a *cycle*. We denote a path (cycle) consisting of ℓ vertices by P_ℓ (C_ℓ). The length of a path (cycle) G is the number of its edges, $|E(G)|$. For any graph G , the *2-subdivision* of G is the graph we get from inserting two new vertices on every edge, that is, from replacing each edge $\{u, v\} \in E(G)$ by a P_4 with vertices $\{u, x, y, v\}$. A graph G is a *2-subdivision graph* if it is the 2-subdivision of some graph H . The *girth* of G is the length of a shortest cycle in G , for acyclic graphs the girth is infinite.

For two edge-colored graphs G and F , we say that G is *isomorphic* to F , and we write $G \cong F$ if there is a bijective function $\varphi : V(G) \rightarrow V(F)$ such that $\{u, v\} \in E(G)$ is an edge with color c if and only if $\{\varphi(u), \varphi(v)\} \in E(F)$ is an edge with color c . Furthermore, we say that G is *F-free* if $G[V'] \not\cong F$ for each vertex set $V' \subseteq V(G)$.

In this work we study the computational problems $cP_\ell D$ and $cC_\ell D$. We sometimes use the following problem formulation that generalizes $cP_\ell D$ and $cC_\ell D$. Herein, we let \mathcal{F} is a set of c -colored graphs.

\mathcal{F} -DELETION

Input: An edge-colored graph $G = (V, E = E_1 \sqcup \dots \sqcup E_c)$, an integer k .

Question: Is there a set S of at most k edges such that $G - S$ is F -free for every $F \in \mathcal{F}$?

Observe that, for given c and ℓ we can define \mathcal{F} as the set of c -colored P_ℓ (or c -colored C_ℓ , respectively). Thus, \mathcal{F} -DELETION generalizes the problems we consider in this work.

For the relevant notions of parameterized complexity refer to the standard textbook [CFK⁺15].

3 Classical Complexity

We first prove the NP-hardness of cP_ℓ D and cC_ℓ D. In our analysis we focus on the impact of cascading effects on the complexity: The NP-hardness of $2P_3$ D relies on the fact that edge deletions may create new bicolored P_3 s, as $2P_3$ D is polynomial-time solvable on graphs that do not provide this cascading effect [GKSS21]. Here, we show that cP_ℓ D with $\ell \geq 4$ and $c \in [2, \ell - 2]$ is NP-hard even when limited to instances where G is non-cascading, defined as follows.

Definition 1. • *Let \mathcal{F} be a set of forbidden subgraphs. An edge e of a graph G is a conflict edge if there is some $F \in \mathcal{F}$ such that e is contained in an induced F in G .*

- *Let X be the set of all conflict edges. The graph G is non-cascading if every non-induced subgraph $F \in \mathcal{F}$ in G is not an induced F in $G - X'$ for every subset $X' \subseteq X$.*
- *The graph G is strictly non-cascading if there is no non-induced subgraph $F \in \mathcal{F}$ in G .*

It is easy to see that a graph is non-cascading if it is strictly non-cascading. We call an edge *conflict-free* if it is not a conflict edge. Observe that in case of cP_ℓ D it can be checked in polynomial time if a given graph G is non-cascading since ℓ is a constant: First, determine all conflict edges of G in $\mathcal{O}(n^\ell)$ time. Second, iterate over all $\mathcal{O}(n^\ell)$ non-induced c -colored P_ℓ subgraphs of G . Let v_1, \dots, v_ℓ be the vertices of one such subgraph. Then, check if there is a conflict-free edge $\{v_i, v_j\}$ with $i, j \in [\ell]$ and $j \neq i + 1$. If this is the case for all non-induced c -colored P_ℓ subgraphs of G return *yes*. Otherwise, return *no*. The main idea of non-cascading graphs is that it is sufficient to hit all initial conflicts in the input graph and that conflict-free edges are never part of a solution. The next proposition formalizes this idea.

Proposition 1. *Let (G, k) be an instance of \mathcal{F} -DELETION where G is non-cascading, and let X be the set of conflict edges of G . If \tilde{S} is an edge-deletion set such that every induced $F \in \mathcal{F}$ in G is not an induced F in $G - \tilde{S}$, then $S := \tilde{S} \cap X$ is an edge-deletion set such that $G - S$ has no induced $F \in \mathcal{F}$.*

Proof: Assume towards a contradiction that $G - S$ contains an induced $F \in \mathcal{F}$. Let e_1, \dots, e_ℓ be the edges of an induced F in $G - S$. Since $G - \tilde{S}$ contains no induced F one of the edges e_i with $i \in [\ell]$ belongs to $\tilde{S} \setminus X$ and therefore e_i is conflict-free. Consequently, e_1, \dots, e_ℓ form a non-induced F in G . Then, since G is non-cascading and $S \subseteq X$ it follows that e_1, \dots, e_ℓ do not form an induced F in $G - S$ which contradicts our assumption. \square

Observe that Proposition 1 implies that a minimal solution only consists of conflict edges if the input graph is non-cascading.

3.1 c -colored P_ℓ Deletion

First, we show the hardness of $cP_\ell D$. The following lemma is useful for showing the correctness of our reduction.

Lemma 1. *Let $G = (V = \{v_1, \dots, v_{d \cdot (\ell-1)}\}, E)$ be a path where any ℓ consecutive vertices form a c -colored P_ℓ and let $S \subseteq E$ be an edge set of size $d - 1$. Then, $G - S$ is c -colored P_ℓ -free if and only if $S = \{\{v_i, v_{i+1}\} \mid i \bmod (\ell - 1) = 0\}$.*

Proof: Let $S = \{\{v_i, v_{i+1}\} \mid i \bmod (\ell - 1) = 0\}$. Since $|V| = d \cdot (\ell - 1)$ we conclude that $|S| = d - 1$. We will show that $G' := G - S$ is P_ℓ -free and hence, G' is c -colored P_ℓ -free. Let $V' \subseteq V$ such that the induced subgraph $G[V']$ is a c -colored P_ℓ . Since G is a path, $V' = \{v_i, \dots, v_{i+\ell-1}\}$ for some $i \in [(d-1) \cdot (\ell-1)]$. Since $|\llbracket i, i + \ell - 2 \rrbracket| = \ell - 1$ there is a $\hat{i} \in \llbracket i, i + \ell - 2 \rrbracket$ such that $\hat{i} \bmod (\ell - 1) = 0$. Thus, we conclude that there is an edge $e \in E_G(V')$ such that $e \in S$. Hence, G' is c -colored P_ℓ -free.

Conversely, let S be an edge-deletion set of size $d - 1$ such that $G - S$ is c -colored P_ℓ -free. Let $\{v_i, v_{i+1}\}$ be the j -th edge in S . We denote $V_1 := \{v_1, \dots, v_i\}$ and $V_2 := \{v_{i+1}, \dots, v_{d \cdot (\ell-1)}\}$. Since $\{v_i, v_{i+1}\}$ is the j -th edge in S , we know that $|S \cap E_G(V_1)| = j - 1$.

First, assume towards a contradiction that $i \geq j \cdot (\ell - 1) + 1$. Then,

$$|V_1| = i \geq j \cdot (\ell - 1) + 1.$$

Since any ℓ consecutive vertices form a c -colored P_ℓ , we conclude that $G[V_1]$ contains at least j edge-disjoint c -colored P_ℓ s. Since $|S \cap E_G(V_1)| = j - 1$, this is a contradiction to the condition that $G - S$ is c -colored P_ℓ -free. Thus, $i \leq j \cdot (\ell - 1)$.

Next, assume towards a contradiction that $i \leq j \cdot (\ell - 1) - 1$. Then,

$$\begin{aligned} |V_2| &= d \cdot (\ell - 1) - i \\ &\geq (d - j) \cdot (\ell - 1) + 1. \end{aligned}$$

Since any ℓ consecutive vertices form a c -colored P_ℓ , we conclude that $G[V_2]$ contains at least $(d - j)$ edge-disjoint c -colored P_ℓ s. Hence, $|S \cap E_G(V_2)| \geq (d - j)$. We conclude that $|S| \geq d$, since $|S \cap E_G(V_1)| = j - 1$ and $\{v_i, v_{i+1}\} \in S$. This is a contradiction to the condition that $|S| = d - 1$. Thus, $i \geq j \cdot (\ell - 1)$.

Hence, we know that $i = j \cdot (\ell - 1)$. Thus, the j -th edge in S is $\{v_{j \cdot (\ell-1)}, v_{j \cdot (\ell-1)+1}\}$ and therefore $S = \{\{v_i, v_{i+1}\} \mid i \bmod (\ell - 1) = 0\}$. \square

Now we will show that $cP_\ell D$ is NP-hard for $\ell \geq 4$ even if the input graph has a somewhat simple structure. From this result we will be able to prove that $cP_\ell D$ remains NP-hard on non-cascading input graphs if $\ell \geq 4$ and $c \in [2, \ell - 2]$.

Theorem 1. *$cP_\ell D$ is NP-hard for each $\ell \geq 4$ and each $c \in [2, \ell - 2]$ even if the maximum degree of G is three, and the girth of G is greater than $2 \cdot d \cdot \ell$ for any constant $d \geq 1$.*

Proof: We give a polynomial-time reduction from the NP-complete (3,B2)-SAT problem [BKS03], a version of 3SAT where one is given a CNF formula Φ on variables x_1, \dots, x_η where every clause contains exactly three literals and each literal x_i and $\neg x_i$ occurs exactly twice in Φ .

Construction: Let Φ be a (3,B2)-SAT formula with clauses $\mathcal{C} = \{c_1, \dots, c_\mu\}$ and variables $\mathcal{X} = \{x_1, \dots, x_\eta\}$. We use the following *gadgets* to construct an equivalent instance $(G = (V, E), k)$ of $cP_\ell D$ from Φ .

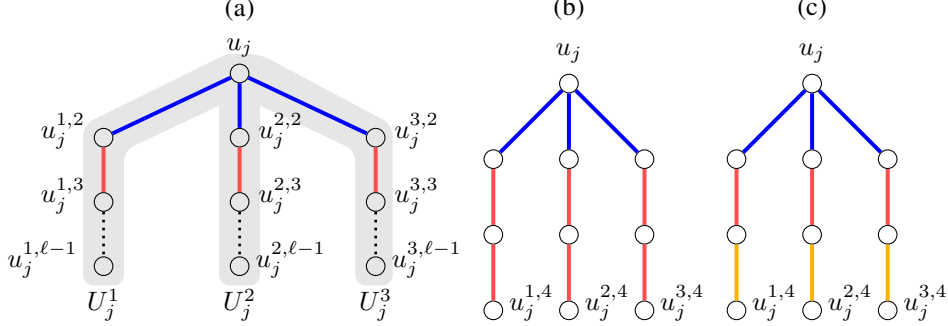


Fig. 1: (a) General structure of a clause gadget Z_j . The black dotted line represents a path containing $\ell - 3$ vertices in total. (b) Clause gadget for $\ell = 5$, $c = 2$. (c) Clause gadget for $\ell = 5$, $c = 3$.

For each clause $c_j \in \mathcal{C}$ we construct a *clause gadget* Z_j consisting of three vertex sets U_j^1, U_j^2 and U_j^3 , each containing $\ell - 1$ vertices. For each $p \in [3]$, we denote the vertices in U_j^p by $u_j^{p,1}, \dots, u_j^{p,\ell-1}$. For $s \in [\ell - 2]$ we add edges $\{u_j^{p,s}, u_j^{p,s+1}\}$. If $s < c$ we add an edge of color s . Else we add an edge of color c . In other words $\{u_j^{p,1}, u_j^{p,2}\}$ is blue, $\{u_j^{p,2}, u_j^{p,3}\}$ is red and the color of the next edges depends on c . Observe that $G[U_j^p]$ is a c -colored $P_{\ell-1}$. We connect the three $P_{\ell-1}$ s by identifying $u_j^{1,1} = u_j^{2,1} = u_j^{3,1} =: u_j$ (see Fig. 1 for an example).

For each variable $x_i \in \mathcal{X}$ we construct a *variable gadget* X_i as follows. First, let $z := d \cdot (\ell - 1) + 1$. Note that z is the minimum number of vertices on a path that contains d edge-disjoint P_ℓ s. The variable gadget X_i consists of four vertex sets of z vertices $T_i^1, T_i^2, F_i^1, F_i^2$ and a vertex set of $\ell - 4$ vertices $W_i := \{w_i^1, \dots, w_i^{\ell-4}\}$. Note that $W_i = \emptyset$ for $\ell = 4$. For $q \in [2]$ the vertices in T_i^q are denoted by $t_i^{q,1}, \dots, t_i^{q,z}$ and the vertices in F_i^q are denoted by $f_i^{q,1}, \dots, f_i^{q,z}$. For $s \in [z - 1]$ we add edges $\{t_i^{q,s}, t_i^{q,s+1}\}$ and $\{f_i^{q,s}, f_i^{q,s+1}\}$. If $0 < s \bmod(\ell - 1) < c$ we add an edge with color $s \bmod(\ell - 1)$ and else we add an edge with color c . So if $s \bmod(\ell - 1) = 1$ we add a blue edge, if $s \bmod(\ell - 1) = 2$ we add a red edge, and otherwise the edge-color depends on c .

We connect T_i^1 and T_i^2 by identifying $t_i^{1,1} = t_i^{2,1} =: t_i$ and analogously we connect F_i^1 and F_i^2 by identifying $f_i^{1,1} = f_i^{2,1} =: f_i$. If $\ell = 4$ we add a red edge $\{t_i, f_i\}$. If $\ell > 4$ we connect t_i and f_i by a $(c - 1)$ -colored path with vertices in $\{t_i\} \cup W_i \cup \{f_i\}$ that does not contain a blue edge (see Fig. 2). Since $|\{t_i\} \cup W_i \cup \{f_i\}| = \ell - 2 \geq c$, this path always has at least $(c - 1)$ edges. Hence, we can always connect t_i and f_i by a $(c - 1)$ -colored path.

Note that any ℓ consecutive vertices in $G[T_i^1]$, $G[T_i^2]$, $G[F_i^1]$ and $G[F_i^2]$ form a c -colored P_ℓ . Hence, $G[T_i^1]$, $G[T_i^2]$, $G[F_i^1]$ and $G[F_i^2]$ are four paths, each containing d edge-disjoint c -colored P_ℓ s. And since $\{t_i^{q,2}, t_i\} \in E_b(G)$ and t_i and f_i are connected by $(c - 1)$ -colored $P_{\ell-1}$ with no blue edge, the induced subgraphs $G[\{t_i^{q,2}, t_i, w_i^1, \dots, w_i^{\ell-4}, f_i, f_i^{q',2}\}]$ are also c -colored P_ℓ s for $q, q' \in [2]$.

Then, we denote the following edge sets (see Fig. 2).

$$T_i^b := \{\{t_i^{q,s}, t_i^{q,s+1}\} \in E \mid s \bmod(\ell - 1) = 1, q \in [2]\}.$$

$$F_i^b := \{\{f_i^{q,s}, f_i^{q,s+1}\} \in E \mid s \bmod(\ell - 1) = 1, q \in [2]\}.$$

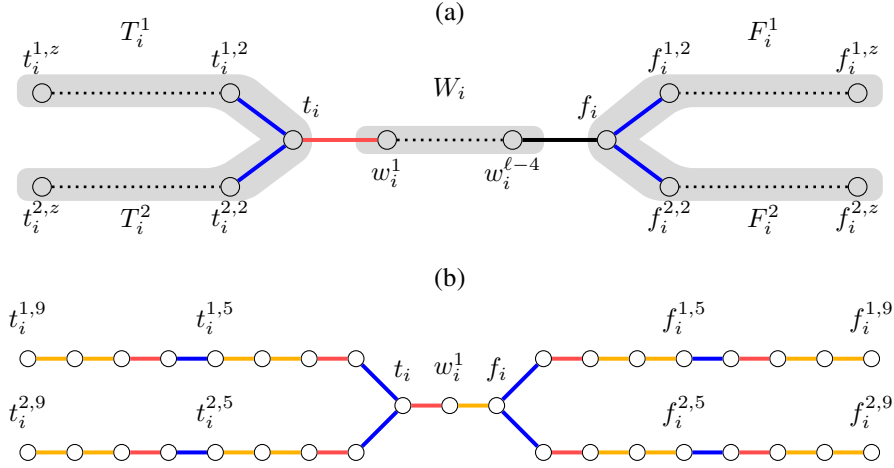


Fig. 2: (a) The generalized structure of a variable gadget X_i . Note that $W_i = \emptyset$ if $\ell = 4$. The edges of color c are black. (b) An exemplary variable gadget for $\ell = 5$, $c = 3$, $d = 2$.

$$T_i^r := \{\{t_i^{q,s}, t_i^{q,s+1}\} \in E \mid s \bmod(\ell - 1) = 2, q \in [2]\}.$$

$$F_i^r := \{\{f_i^{q,s}, f_i^{q,s+1}\} \in E \mid s \bmod(\ell - 1) = 2, q \in [2]\}.$$

Note that $|T_i^r| = |F_i^r| = |T_i^b| = |F_i^b| = 2 \cdot d$.

To connect the variable and clause gadgets we identify vertices as follows (see Fig. 3). For $p \in [3]$, $q \in [2]$, any variable $x_i \in \mathcal{X}$ and any clause $c_j \in \mathcal{C}$ we set

$$u_j^{p,2} = \begin{cases} t_i^{q,z} & \text{if the literal } x_i \text{ has its } q\text{-th occurrence as the } p\text{-th literal in } c_j \\ f_i^{q,z} & \text{if the literal } \neg x_i \text{ has its } q\text{-th occurrence as the } p\text{-th literal in } c_j. \end{cases}$$

Note that for each clause gadget Z_j and $p \in [3]$ the vertex $u_j^{p,2}$ is identified with exactly one vertex from a variable gadget and for each variable gadget X_i and $q \in [2]$ the vertices $t_i^{q,z}$ and $f_i^{q,z}$ are each identified with exactly one vertex from a clause gadget, since each literal occurs exactly twice in Φ .

It is easy to see that the maximum degree of G is three. Furthermore, the girth of G is at least $2 \cdot d \cdot \ell$, since the smallest possible cycle in G contains the vertices from $T_i^1 \cup T_i^2 \cup \{u_j\}$ or $F_i^1 \cup F_i^2 \cup \{u_j\}$, respectively. Such a cycle will be constructed, when there is a clause $c_j = (x_i \vee x_i \vee \dots)$ or $c_j = (\neg x_i \vee \neg x_i \vee \dots)$ in Φ . We complete the construction by setting $k := 4 \cdot d \cdot \eta + 2 \cdot \mu$.

Intuition: Before we prove the correctness of the reduction, we informally describe its idea.

Each variable gadget contains $4 \cdot d$ edge-disjoint c -colored P_ℓ s. So we have to delete at least $4 \cdot d$ edges per variable gadget. We can make a variable gadget c -colored P_ℓ -free with $4 \cdot d$ edge deletions by deleting the edges in T_i^r and F_i^b , or the edges in F_i^r and T_i^b . The former models the assignment $\mathcal{A}(x_i) = \text{true}$, and the latter models the assignment $\mathcal{A}(x_i) = \text{false}$. If we delete T_i^r and F_i^b the vertices $f_i^{1,z}$ and $f_i^{2,z}$ are part of a $(c-1)$ -colored $P_{\ell-1}$ with no blue edges, while the vertices $t_i^{1,z}$ and $t_i^{2,z}$ are part of a $(c-1)$ -colored $P_{\ell-2}$. If we delete F_i^r and T_i^b , it is the other way around. We will be able to make a clause gadget c -colored P_ℓ -free with two edge deletions if and only if there is at least one vertex $u_j^{p,2}$ that is not part

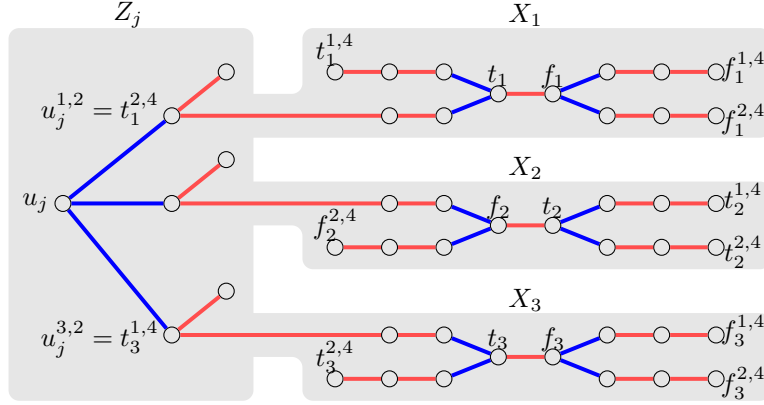


Fig. 3: A part of the constructed graph for $\ell = 4, d = 1$. Left: the clause gadget Z_j for a clause $c_j = (x_1 \vee \neg x_2 \vee x_3)$. Right: the variable gadgets X_1, X_2 and X_3 . Note that x_1 has its second occurrence as a positive literal in c_j , x_2 has its first occurrence as a negative literal in c_j and x_3 has its first occurrence as a positive literal in c_j . The clause gadgets where the variables have their other occurrences are not shown.

of a $(c-1)$ -colored $P_{\ell-1}$ from the connected variable gadget. Thus, we can make the constructed graph c -colored P_ℓ -free with exactly k edge deletions if and only if each clause is satisfied.

Before we give the correctness proof, we observe the following.

Claim 1. Let $G' := G - (F_i^b \cup T_i^r)$ and $G'' := G - (T_i^b \cup F_i^r)$ for any variable gadget X_i . Then:

- I. No blue edge $e_b \in E_{G'}(X_i)$ is part of a P_ℓ in G' .
- II. No blue edge $e_b \in E_{G''}(X_i)$ is part of a P_ℓ in G'' .

Proof. We only prove **I**, since the proof for **II** works analogously. Let $e_b \in E_{G'}(X_i)$ be a blue edge. Since $F_i^b \cap E(G') = \emptyset$, we conclude that $e_b \in T_i^b$. Hence, we know that $e_b = \{t_i^{q,s}, t_i^{q,s+1}\}$ for some s such that $s \bmod(\ell-1) = 1$ and $q \in [2]$. We consider two cases.

Case 1: $s = 1$. By definition of T_i^r we know that $\{t_i^{q,2}, t_i^{q,3}\} \in T_i^r$. We can conclude that $\deg_{G'}(t_i^{q,2}) = 1$, since $T_i^r \cap E(G') = \emptyset$. Furthermore, since $F_i^b \cap E(G') = \emptyset$, we conclude that $\deg_{G'}(f_i) = 1$. Thus, the longest path in G' containing e_b is the induced subgraph $G'[\{t_i^{q,2}, t_i\} \cup W_i \cup \{f_i\}]$. By construction, it is easy to see that $|\{t_i^{q,2}, t_i\} \cup W_i \cup \{f_i\}| = \ell - 1$. Thus, e_b is not part of a P_ℓ in G' .

Case 2: $s > 1$. It is easy to see that

$$s \bmod(\ell-1) = 1 \Leftrightarrow (s+1) \bmod(\ell-1) = 2 \quad (1)$$

$$\Leftrightarrow (s+2-\ell) \bmod(\ell-1) = 2. \quad (2)$$

From equation (1) we conclude that $\{t_i^{q,s+1}, t_i^{q,s+2}\} \in T_i^r$ and from equation (2) we conclude that $\{t_i^{q,s+2-\ell}, t_i^{q,s+3-\ell}\} \in T_i^r$. Since $T_i^r \cap E(G') = \emptyset$, we can conclude that $\deg_{G'}(t_i^{q,s+1}) = 1$ and $\deg_{G'}(t_i^{q,s+3-\ell}) = 1$. Thus, the longest path in G' containing e_b is $G'[\{t_i^{q,s+3-\ell}, \dots, t_i^{q,s+1}\}]$. It is easy to see that $|\{t_i^{q,s+3-\ell}, \dots, t_i^{q,s+1}\}| = \ell - 1$. Thus, e_b is not part of a P_ℓ in G' .

Hence, no blue edge from $E_{G'}(X_i)$ is part of a P_ℓ in G' . \diamond

Correctness: We show the correctness of the reduction by proving that there is a satisfying assignment for Φ if and only if (G, k) is a yes-instance of $cP_\ell D$.

(\Rightarrow) Let $\mathcal{A} : \mathcal{X} \rightarrow \{\text{true}, \text{false}\}$ be a satisfying assignment for Φ . We will prove that (G, k) is a yes-instance of $cP_\ell D$ by constructing an edge-deletion set S of size k such that $G - S$ is c -colored P_ℓ -free.

For each variable $x_i \in \mathcal{X}$ we add $4 \cdot d$ edges to S . If $\mathcal{A}(x_i) = \text{true}$, then we add T_i^r and F_i^b to S . If $\mathcal{A}(x_i) = \text{false}$, then we add F_i^r and T_i^b to S . Since \mathcal{A} satisfies Φ , there is at least one variable $x_i \in \mathcal{X}$ such that $\mathcal{A}(x_i)$ satisfies c_j for each clause $c_j \in \mathcal{C}$. Let $p \in [3]$ such that the p -th literal of c_j satisfies c_j . Let $\{\alpha, \beta\} := [3] \setminus \{p\}$. We add $\{u_j, u_j^{\alpha,2}\}$ and $\{u_j, u_j^{\beta,2}\}$ to S . Note that we added exactly two edges per clause. Hence, $|S| = 4 \cdot d \cdot \eta + 2 \cdot \mu = k$.

Next, we show that $G' := G - S$ is c -colored P_ℓ -free. Since G is a c -colored graph, it is sufficient to prove that no blue edge is part of a c -colored P_ℓ in G' . First, let $e_b \in E_{G'}(X_i)$ be a blue edge from any variable gadget X_i . Since either $T_i^r, F_i^b \subseteq S$ or $F_i^r, T_i^b \subseteq S$, we can conclude from Claim 1 that e_b is not part of a P_ℓ in G' . Note that the proof for Claim 1 shows that e_b can neither be part of a c -colored P_ℓ with edges from X_i , nor be part of a c -colored P_ℓ with edges from a connected clause gadget Z_j . Next, let $\{u_j, u_j^{p,2}\} \in E'_{G'}(Z_j)$ be the blue edge in G' from any clause gadget Z_j . By construction of S we know that $\deg_{G'}(u_j) = 1$. This implies that $\{u_j, u_j^{p,2}\}$ is only part of a $P_{\ell-1}$ in $G'[Z_j]$. Hence, we can conclude that any c -colored P_ℓ containing $\{u_j, u_j^{p,2}\}$ has to contain edges from a variable gadget.

We know by construction that $u_j^{p,2}$ is identified with a vertex from a variable gadget X_i such that $\mathcal{A}(x_i)$ satisfies c_j . Without loss of generality, assume $\mathcal{A}(x_i) = \text{true}$. Then, $u_j^{p,2}$ is identified with a vertex $t_i^{q,z}$ for some $q \in [2]$ and $T_i^r, F_i^b \subseteq S$. Assume towards a contradiction that there is a vertex set $V' \subseteq V$ such that $\{u_j, u_j^{p,2}\} \subseteq V'$ and $G'[V']$ is a c -colored P_ℓ . Since $\deg_{G'}(u_j) = 1$, we conclude that $\{t_i^{q,z-\ell+2}, \dots, t_i^{q,z}\} \subseteq V'$. Since $(z - \ell + 2) \bmod (\ell - 1) = 2$ we know by definition of T_i^r that $\{t_i^{q,z-\ell+2}, t_i^{q,z-\ell+3}\} \in T_i^r$. Since $T_i^r \subseteq S$, we conclude that $\{t_i^{q,z-\ell+2}, t_i^{q,z-\ell+3}\} \in S$. Hence, $G'[V']$ is not a c -colored P_ℓ , a contradiction. Hence, $\{u_j, u_j^{p,2}\}$ is not part of a c -colored P_ℓ in G' . Thus, no blue edge is part of a c -colored P_ℓ in G' . Hence, G' is c -colored P_ℓ -free.

(\Leftarrow) Let S be an edge-deletion set with $|S| \leq k$ such that $G - S$ is c -colored P_ℓ -free. Before we define a satisfying assignment $\mathcal{A} : \mathcal{X} \rightarrow \{\text{true}, \text{false}\}$ for Φ , we show two Claims. First, we show how many edges from each variable gadget and clause gadget have to be in S .

Claim 2. $|S \cap E_G(X_i)| = 4 \cdot d$ for any variable gadget X_i and $|S \cap E_G(Z_j) \cap E_b| = 2$ for any clause gadget Z_j .

Proof. First, we will show that $|S \cap E_G(Z_j)| \geq 2$. Assume towards a contradiction that $|S \cap E_G(Z_j)| < 2$. First, consider the vertex sets $V_1 := U_j^1 \cup \{u_j^{2,2}\}$ and $V_2 := U_j^1 \cup \{u_j^{3,2}\}$. Note that $G[V_1]$ and $G[V_2]$ are two different c -colored P_ℓ s and that $V_1 \cap V_2 = U_j^1$. Hence, we conclude that $|S \cap U_j^1| = 1$. Next, consider the vertex set $V_3 := U_j^2 \cup \{u_j^{3,2}\}$. The induced subgraph $G[V_3]$ is a c -colored P_ℓ and $V_3 \cap U_j^1 = \emptyset$. This is a contradiction, since $G - S$ is c -colored P_ℓ -free. Hence, $|S \cap E_G(Z_j)| \geq 2$.

By construction, $G[X_i]$ contains $4 \cdot d$ edge-disjoint c -colored P_ℓ s. Hence, we know that $|S \cap E_G(X_i)| \geq 4 \cdot d$. Since $|S| \leq k = 4 \cdot d \cdot \eta + 2 \cdot \mu$, we can conclude that $|S \cap E_G(X_i)| = 4 \cdot d$ and $|S \cap E_G(Z_j)| = 2$.

Second, we will show that this implies that $|S \cap E_G(Z_j) \cap E_b| = 2$. Assume towards a contradiction that $|S \cap E_G(Z_j) \cap E_b| < 2$. Without loss of generality we can assume that $\{u_j, u_j^{1,2}\}, \{u_j, u_j^{2,2}\} \notin S$. Let $V_1 := U_j^1 \cup \{u_j^{2,2}\}$, $V_2 := U_j^2 \cup \{u_j^{1,2}\}$ and $V_3 := U_j^3 \cup \{u_j^{1,2}\}$. The induced subgraphs $G[V_1], G[V_2]$

and $G[V_3]$ are three c -colored P_ℓ s. This is a contradiction, since $S \cap (E_G(V_\alpha) \cap E_G(V_\beta)) = \emptyset$ for $\alpha, \beta \in [3]$ with $\alpha \neq \beta$ and $|S \cap E_G(Z_j)| = 2$. Hence, $|S \cap E_G(Z_j) \cap E_b| = 2$. \diamond

Second, we specify which edges from variable gadgets have to be deleted.

Claim 3. For each variable gadget X_i we have $S \cap E_G(T_i^1 \cup T_i^2) = T_i^b$ or $S \cap E_G(F_i^1 \cup F_i^2) = F_i^b$.

Proof. First, we show that $E_G(\{t_i, t_i^{1,2}, t_i^{2,2}\}) \subseteq S$ or $E_G(\{f_i, f_i^{1,2}, f_i^{2,2}\}) \subseteq S$. Consider the four vertex sets in $\tilde{X}_i := \{T_i^1, T_i^2, F_i^1, F_i^2\}$. By construction, for each set $A \in \tilde{X}_i$ the induced subgraph $G[A]$ contains d edge-disjoint c -colored P_ℓ s, and for each $A, B \in \tilde{X}_i$ with $A \neq B$ we know $E_G(A) \cap E_G(B) = \emptyset$. From Claim 2 we know that $|S \cap E_G(X_i)| = 4 \cdot d$. Hence, we conclude that $|S \cap E_G(A)| = d$. Furthermore, we know by construction that $E_G(A) \cap E_G(W_i \cup \{t_i, f_i\}) = \emptyset$. Hence, we conclude that $S \cap E_G(W_i \cup \{t_i, f_i\}) = \emptyset$. This implies $E_G(\{t_i, t_i^{1,2}, t_i^{2,2}\}) \subseteq S$ or $E_G(\{f_i, f_i^{1,2}, f_i^{2,2}\}) \subseteq S$, since the induced subgraphs $G[W_i \cup \{t_i, f_i, t_i^{q,2}, f_i^{q',2}\}]$ are c -colored P_ℓ s for each $q, q' \in [2]$.

Without loss of generality, assume $E_G(\{t_i, t_i^{1,2}, t_i^{2,2}\}) \subseteq S$. We can conclude that $|S \cap (T_i^1 \setminus \{t_i\})| = d - 1 = |S \cap (T_i^2 \setminus \{t_i\})|$, since $|S \cap T_i^1| = d = |S \cap T_i^2|$, as shown above. Furthermore, the induced subgraphs $G[T_i^1 \setminus \{t_i\}]$ and $G[T_i^2 \setminus \{t_i\}]$ are paths of $d \cdot (\ell - 1)$ vertices where any ℓ consecutive vertices form a c -colored P_ℓ . Thus, from Lemma 1 we conclude $T_i^b \subseteq S$. Since $|T_i^b| = 2 \cdot d$ and the induced subgraph $G[F_i^1 \cup F_i^2]$ contains $2 \cdot d$ edge-disjoint c -colored P_ℓ s, we conclude that $S \cap E_G(T_i^1 \cup T_i^2) = T_i^b$ from Claim 2. Thus, $S \cap E_G(T_i^1 \cup T_i^2) = T_i^b$ or $S \cap E_G(F_i^1 \cup F_i^2) = F_i^b$. \diamond

We show how to construct an equivalent solution S' such that either $S' \cap E_G(X_i) = F_i^b \cup T_i^r$ or $S' \cap E_G(X_i) = T_i^b \cup F_i^r$ for each variable gadget X_i .

If $S \cap E_G(T_i^1 \cup T_i^2) = T_i^b$, then we can conclude that $S \cap E_G(X_i) = T_i^b \cup F$ such that $F \subseteq E_G(F_i^1 \cup F_i^2)$ from Claim 3. We will show that $S' := (S \setminus F) \cup F_i^r$ is an equivalent solution. Since $|T_i^b| = 2 \cdot d$, we can conclude that $|F| = 2 \cdot d$ from Claim 2. Hence, it is easy to see that $|S'| = |S|$. Now, we show that $G' := G - S'$ is c -colored P_ℓ -free. Since G is constructed from a (3,B2)-SAT formula, we know that for each clause gadget Z_j and each $p \in [3]$, the vertex $u_j^{p,2}$ is identified with exactly one vertex from a variable gadget. Furthermore, we know that the three edges $\{u_j, u_j^{p,2}\}$ are the only blue edges in $E_G(Z_j)$ and that G is a c -colored graph. Hence, it is sufficient to show that for any variable gadget X_i there is no blue edge $e_b \in E_{G'}(X_i)$ that is part of a c -colored P_ℓ in G' and for each clause gadget with $u_j^{p,2} = t_i^{q,z}$ or $u_j^{p,2} = f_i^{q,z}$ for some $p \in [3], q \in [2]$, the blue edge $\{u_j, u_j^{p,2}\}$ is not part of a c -colored P_ℓ in G' . Since $S' \cap E_G(X_i) = F_i^b \cup T_i^r$, it follows by Claim 1 that no blue edge from $E_{G'}(X_i)$ is part of a c -colored P_ℓ in G' . Let Z_j be a clause gadget such that $u_j^{p,2} = t_i^{q,z}$ or $u_j^{p,2} = f_i^{q,z}$ for some $p \in [3], q \in [2]$. From Claim 2 we conclude that S' contains exactly two blue edges from $E_G(Z_j)$. Hence, $\deg_{G'}(u_j) = 1$. We consider two cases.

Case 1: $u_j^{p,2} = t_i^{q,z}$ for some $p \in [3], q \in [2]$. We will show that $\{u_j, u_j^{p,2}\} \in S'$. Assume towards a contradiction that $\{u_j, u_j^{p,2}\} \notin S'$. Since $S' \cap E_G(Z_j) = S \cap E_G(Z_j)$, we conclude that $\{u_j, u_j^{p,2}\} \notin S$. And since $S \cap E_G(T_i^1 \cup T_i^2) = T_i^b$, we conclude that the induced subgraph $G[\{u_j, t_i^{q,z}, \dots, t_i^{q,z-\ell+2}\}] - S$ is a c -colored P_ℓ . This is a contradiction, since $G - S$ is c -colored P_ℓ free. Hence, $\{u_j, u_j^{p,2}\} \in S'$ and thus, $\{u_j, u_j^{p,2}\}$ is not part of a c -colored P_ℓ in G' .

Case 2: $u_j^{p,2} = f_i^{q,z}$ for some $p \in [3], q \in [2]$. Since it is obvious that $\{u_j, u_j^{p,2}\}$ is not part of a c -colored P_ℓ in G' if $\{u_j, u_j^{p,2}\} \in S'$, we only consider the case in which $\{u_j, u_j^{p,2}\} \notin S'$. Since

$(z - \ell + 2) \bmod (\ell - 1) = 2$, we know by definition of F_i^r that $\{f_i^{q, z-\ell+2}, f_i^{q, z-\ell+3}\} \in F_i^r$. This implies that $\deg_{G'}(f_i^{q, z-\ell+3}) = 1$, since $F_i^r \subseteq S'$. Thus, the longest path in G' that contains $\{u_j, u_j^{p,2}\}$ is the induced subgraph $G'[V' := \{u_j, f_i^{q, z}, \dots, f_i^{q, z-\ell+3}\}]$. It is not hard to see that $|V'| = \ell - 1$. Thus, $\{u_j, u_j^{p,2}\}$ is not part of a c -colored P_ℓ in G' .

Hence, no blue edge is part of a c -colored P_ℓ in G' and therefore, S' is a solution such that $S' \cap E_G(X_i) = T_i^b \cup F_i^r$.

If $S \cap E_G(T_i^1 \cup T_i^2) \neq T_i^b$, then we conclude that $S \cap E_G(F_i^1 \cup F_i^2) = F_i^b$ from Claim 3. Hence, $S \cap E_G(X_i) = F_i^b \cup T$ such that $T \subseteq E_G(T_i^1 \cup T_i^2)$. With an analogous argument, we can show that $S' := (S \setminus T) \cup T_i^r$ is an equivalent solution. Thus, we have shown how to construct a solution S' from S such that either $S' \cap E_G(X_i) = F_i^b \cup T_i^r$ or $S' \cap E_G(X_i) = T_i^b \cup F_i^r$.

Now we can define a satisfying assignment $\mathcal{A} : \mathcal{X} \rightarrow \{\text{true}, \text{false}\}$ for Φ as:

$$\mathcal{A}(x_i) := \begin{cases} \text{true} & \text{if } S' \cap E_G(X_i) = F_i^b \cup T_i^r \\ \text{false} & \text{if } S' \cap E_G(X_i) = T_i^b \cup F_i^r. \end{cases}$$

It remains to show that \mathcal{A} satisfies Φ . Let $c_j \in \mathcal{C}$. We know that there is exactly one $p \in [3]$ such that $\{u_j, u_j^{p,2}\} \notin S$. Let $x_i \in \mathcal{X}$ be the variable that occurs as the p -th literal in c_j . If the p -th literal in c_j is a positive literal, we know that the vertex $u_j^{p,2}$ is identified with the vertex $t_i^{q,z}$ from the variable gadget X_i for $q \in [2]$. Since S' is a solution and $\{u_j, u_j^{p,2}\} \notin S'$, we conclude that $T_i^r \subseteq S'$. Hence, $S' \cap E_G(X_i) = F_i^b \cup T_i^r$. Thus, $\mathcal{A}(x_i) = \text{true}$ and therefore $\mathcal{A}(x_i)$ satisfies c_j . If the p -th literal in c_j is a negative literal, the argument works analogously. Hence, each clause c_j is satisfied by \mathcal{A} and thus Φ is satisfied as well. \square

If the girth of a graph G is greater than ℓ , then each subgraph of G that is isomorphic to a c -colored P_ℓ is an induced subgraph. This implies the following.

Corollary 1. $cP_\ell\text{D}$ is NP-hard for each $\ell \geq 4$ and $c \in [2, \ell - 2]$ on strictly non-cascading graphs.

Furthermore, consider the problem ALL- $cP_\ell\text{D}$ where the input consists of a graph G and an integer k and the task is to destroy all (not necessarily induced) c -colored P_ℓ s with at most k edge deletions. Theorem 1 then implies the following.

Corollary 2. ALL- $cP_\ell\text{D}$ is NP-hard for each $\ell \geq 4$ and $c \in [2, \ell - 2]$.

Next, we show that also the remaining case where $c = \ell - 1$ is NP-hard.

Theorem 2. $(\ell - 1)P_\ell\text{D}$ is NP-hard for any $\ell \geq 4$ even if the maximum degree of G is 16.

Proof: We prove this theorem by giving a polynomial time reduction from the NP-hard $2P_3\text{D}$ problem [GKSS21].

Construction: Let (G, k) be an instance of $2P_3\text{D}$. We will show how to construct an equivalent instance (H, k) of $(\ell - 1)P_\ell\text{D}$ for any $\ell \geq 4$. We use the instance (G, k) and add vertices and edges with new colors. Hence, $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. For each vertex $v \in V(G)$ we add $\deg_G(v) \cdot (\ell - 3)$ new vertices v_i^j for $i \in [\deg_G(v)]$ and $j \in [\ell - 3]$ to $V(H)$. Recall that yellow is the third color. We add yellow edges $\{v, v_i^1\}$ for each $i \in [\deg_G(v)]$. And for each $j \in [\ell - 4]$ we add an edge $\{v_i^j, v_i^{j+1}\}$ with color $j + 3$. Note that for each $i \in [\deg_G(v)]$ the induced subgraph $H[\{v, v_i^1, \dots, v_i^{\ell-3}\}]$ is an $(\ell - 3)$ -colored $P_{\ell-2}$. Hence, each vertex $v \in V(G)$ is part of $\deg_G(v)$ edge-disjoint $(\ell - 3)$ -colored $P_{\ell-2}$ in H (see Fig. 4). The budget k remains the same.

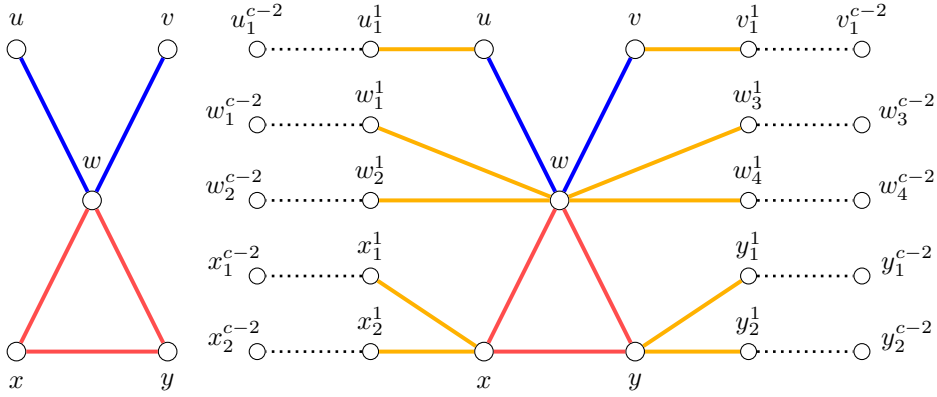


Fig. 4: (a) A bicolored graph G . (b) The graph H with the new vertices and edges. The black dotted lines each represent a $(\ell - 4)$ -colored path containing $\ell - 3$ edges that are neither blue, red, nor yellow.

By construction, we know that $\deg_H(v) = 2 \cdot \deg_G(v)$ for each $v \in V(G)$, and that $\deg_H(v_i^j) \in [2]$ for the new vertices $v_i^j \in V(H) \setminus V(G)$. Since $2P_3D$ is NP-hard even if the maximum degree of G is eight [GKSS21], the correctness of the reduction will imply the NP-hardness even if the maximum degree of H is 16.

Correctness: We will now prove the correctness of the reduction by showing that (G, k) is a yes-instance of $2P_3D$ if and only if (H, k) is a yes-instance of $(\ell - 1)P_\ell D$.

(\Rightarrow) Let S be an edge-deletion set of size at most k such that $G - S$ is bicolored P_3 -free. Since H is a $(\ell - 1)$ -colored graph, each $(\ell - 1)$ -colored P_ℓ in H has to include exactly one red edge and exactly one blue edge. By construction, we know that an edge $e \in E(H)$ is red or blue if and only if $e \in E(G)$. Since v_i^1 is the only vertex from $\{v_i^1, \dots, v_i^{\ell-3}\}$ that is adjacent to a vertex from $V(G)$ for each $i \in [\deg_G(v)]$, we conclude that each $(\ell - 1)$ -colored P_ℓ in H has to include an induced bicolored P_3 from G . Since $G - S$ is bicolored P_3 -free, $H - S$ is $(\ell - 1)$ -colored P_ℓ -free. Thus, S is a solution for (H, k) .

(\Leftarrow) Let S be an edge-deletion set of size at most k such that $H - S$ is $(\ell - 1)$ -colored P_ℓ -free.

First, we will show how to construct an equivalent solution S' with $|S'| \leq |S|$ such that S' only contains blue, red and yellow edges. Note that this is only necessary if $\ell > 4$. Let $\{v_i^j, v_i^{j+1}\} \in S$ be an edge with color $c > 3$. Let $V' \subseteq V(H)$ be a vertex set such that $v_i^j, v_i^{j+1} \in V'$ and $H[V']$ is an $(\ell - 1)$ -colored P_ℓ . Since H is an $(\ell - 1)$ -colored graph, each $(\ell - 1)$ -colored P_ℓ has to include a red edge and a blue edge. By construction, the only vertices, to which blue and red edges can be incident, are the vertices from $V(G)$. Hence, we conclude that $v \in V'$. Thus, $S' := (S \setminus \{v_i^j, v_i^{j+1}\}) \cup \{\{v, v_i^1\}\}$ is an equivalent solution that only contains blue, red, and yellow edges.

Second, we will show how to construct an equivalent solution S'' from S' that only contains blue and red edges. Recall that $E_H(\{v\}, \{v_i^1 \mid i \in [\deg_G(v)]\})$ denotes the set of all yellow edges that are incident to a vertex $v \in V(G)$. Let $\{v, v_i^1\} \in S'$ be a yellow edge. We have to consider two cases.

Case 1: $E_H(\{v\}, \{v_i^1 \mid i \in [\deg_G(v)]\}) \subseteq S'$. Since G is a $(\ell - 1)$ -colored graph, we know by construction that each $(\ell - 1)$ -colored P_ℓ that contains the vertex v , has to contain a red or blue edge that is

incident to v . Thus, we can construct an equivalent solution by swapping the $\deg_G(v)$ many yellow edges that are incident to v with the $\deg_G(v)$ blue or red edges that are incident to v .

We set $S'' := (S' \setminus E_H(\{v\}, \{v_i^1 \mid i \in [\deg_G(v)]\})) \cup E_G(\{v\}, N_G(v))$. Since there are no red or blue edges in $H - S''$ that are incident to v , we conclude that v is not part of a $(\ell - 1)$ -colored P_ℓ in $H - S''$ and therefore no edge in $E_H(\{v\}, \{v_i^1 \mid i \in [\deg_G(v)]\})$ is part of a $(\ell - 1)$ -colored P_ℓ in $H - S''$. Hence, S'' is an equivalent solution with no yellow edges.

Case 2: $E_H(\{v\}, \{v_i^1 \mid i \in [\deg_G(v)]\}) \not\subseteq S'$. Then, for some $\alpha \in [\deg_G(v)]$ there is a yellow edge $\{v, v_\alpha^1\} \notin S'$. Let $\{v, v_\beta^1\} \in S'$ such that $\beta \neq \alpha$. Since S' is a solution and $H[\{v, v_\alpha^1, \dots, v_\alpha^{c-2}\}] \cong H[\{v, v_\beta^1, \dots, v_\beta^{c-2}\}]$, we conclude that $S' \setminus \{v, v_\beta^1\}$ is a solution. We set $S'' := (S' \setminus E_H(\{v\}, \{v_i^1 \mid i \in [\deg_G(v)]\}))$ and get an equivalent solution with no yellow edges.

Thus, S'' only contains red and blue edges. It remains to show that S'' is a solution for (G, k) . Assume towards a contradiction that there is an induced bicolored P_3 in $G - S''$. Without loss of generality, this implies that there is an edge set $\{\{u, v\}\{v, w\}\} \subseteq (E(G) \setminus S'')$ so that $\{u, v\}$ is a blue edge, $\{v, w\}$ is a red edge and $\{u, w\} \notin (E(G) \setminus S'')$. Since S'' only contains blue or red edges, we can conclude that there is a vertex set $U_\alpha := \{u, u_\alpha^1, \dots, u_\alpha^{\ell-3}\}$ such that $H[U_\alpha] - S''$ is an induced $(\ell - 3)$ -colored $P_{\ell-2}$ with no blue or red edges. Hence, we conclude that $H[U_\alpha \cup \{v, w\}]$ is an induced $(\ell - 1)$ -colored P_ℓ . This is a contradiction, since S'' is a solution for (H, k) . Hence, $G - S''$ is bicolored P_3 -free. Thus, S'' is a solution for (G, k) . \square

Since $2P_3D$ is NP-hard [GKSS21] and $1P_\ell D$ is NP-hard [EMC88], we can conclude the following from Theorem 1 and Theorem 2.

Corollary 3. $cP_\ell D$ is NP-hard for each $\ell \geq 3$ and each $c \in [\ell - 1]$.

3.2 c -colored C_ℓ Deletion

Next, we analyze the complexity of $cC_\ell D$ which is known to be NP-hard for any $\ell \geq 3$ and $c = 1$ [Yan81]. We show that $cC_\ell D$ is NP-hard for any $\ell \geq 3$ and $c \in [\ell]$. The result is based on a reduction from the NP-hard problem VERTEX COVER (VC). In VC one is given a graph G and an integer k and asks if there is a subset S of vertices of G such that $|S| \leq k$ and every edge of G has at least one endpoint in S . But before we consider $cC_\ell D$, we establish an NP-hardness result for VC on C_3 -free and tripartite graphs that we will use in our reduction.

Lemma 2. VC is NP-hard even if G is C_3 -free and tripartite.

Proof: INDEPENDENT SET (IS) is NP-hard on 2-subdivision graphs [Pol74]. Since (G, k) is a yes-instance of IS if and only if $(G, n - k)$ is a yes-instance of VC [GJ79], we conclude that VC is NP-hard on 2-subdivision graphs. Since each 2-subdivision graph is C_3 -free and tripartite, VC is NP-hard on C_3 -free and tripartite graphs. \square

Now we can show the NP-hardness of $cC_\ell D$ for $c = \ell$. With this result we will be able to prove the NP-hardness for all $c \in [\ell]$.

Lemma 3. $\ell C_\ell D$ is NP-hard for any $\ell \geq 3$ even if G has girth ℓ and every C_ℓ in G is ℓ -colored.

Proof: We give a polynomial time reduction from the NP-hard VC problem on C_3 -free and tripartite graphs (see Lemma 2). Note that this reduction is very similar to the one given by Yannakakis [Yan81] to prove the NP-hardness for $c = 1$.

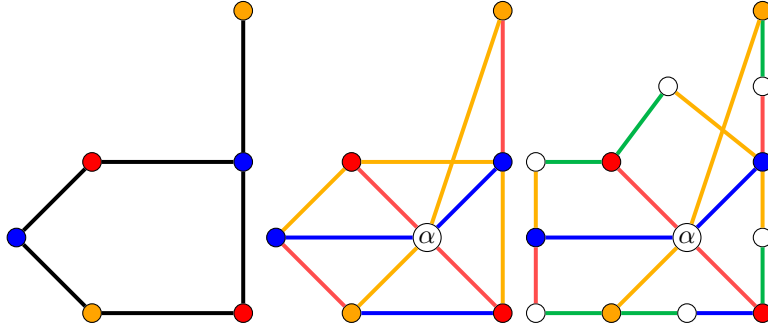


Fig. 5: (a) A tripartite, C_3 -free graph H . (b) The graph constructed on input (a) if $\ell = 3$. (c) The graph constructed on input (a) if $\ell = 4$.

Let (H, k) be an instance of VC such that H is tripartite and C_3 -free. Before we describe how to construct an equivalent instance (G, k) of $\ell C_\ell D$, we define two functions to color the vertices and edges of H . Since H is tripartite, there is a function $\varphi : V(H) \rightarrow [3]$ such that $\varphi(u) \neq \varphi(v)$ for each edge $\{u, v\} \in E(H)$. Hence, there is exactly one $\gamma \in [3] \setminus \{\varphi(u), \varphi(v)\}$. Thus, $\psi : E(H) \rightarrow [3]$ with $\psi(\{u, v\}) = \gamma$ is a well defined function.

Construction: Now we show how to construct (G, k) . If $\ell = 3$, then we assign the color $\psi(\{u, v\})$ to each edge $\{u, v\} \in E(H)$. If $\ell > 3$, then we subdivide each edge $\{u, v\} \in E(H)$ with vertices $W^{uv} := \{w_1^{uv}, \dots, w_{\ell-3}^{uv}\}$. We then color the edges as follows. We assign color $\psi(\{u, v\})$ to the edge $\{u, w_1^{uv}\}$ and color ℓ to the edge $\{w_{\ell-3}^{uv}, v\}$. For $i \in \{1, \dots, \ell-4\}$, the edges $\{w_i^{uv}, w_{i+1}^{uv}\}$ are assigned color $i+3$. Hence, the induced subgraph $G[\{u, v\} \cup W^{uv}]$ is an $(\ell-2)$ -colored $P_{\ell-1}$ with all colors except $\varphi(u)$ and $\varphi(v)$. Then, we add a vertex α . To complete the reduction we add an edge $\{v, \alpha\}$ with color $\varphi(v)$ for each $v \in V(H)$. The budget k remains the same.

Before we show the correctness of the reduction we prove the following Claim about the structure of G .

Claim 4. The girth of G is ℓ and every C_ℓ in G is ℓ -colored.

Proof. Obviously (G, k) is a trivial yes-instance if the girth of G is greater than ℓ . Since H is C_3 -free and every edge in $E(H)$ corresponds to a induced $P_{\ell-1}$ in G , we conclude that the girth of $G[V \setminus \{\alpha\}]$ is greater than ℓ . Furthermore, we know by construction that for each edge $\{u, v\} \in E(H)$ the induced subgraph $G[\{\alpha, u, v\} \cup W^{uv}]$ is an ℓ -colored C_ℓ , since u and v are connected by an $(\ell-2)$ -colored $P_{\ell-1}$ with all colors except $\varphi(u)$ and $\varphi(v)$, the edge $\{u, \alpha\}$ has color $\varphi(u)$ and the edge $\{v, \alpha\}$ has color $\varphi(v)$. Hence, we conclude that the girth of G is ℓ and every C_ℓ in G is ℓ -colored. \diamond

Correctness: To show the correctness of the reduction, we prove that (H, k) is a yes-instance of VC if and only if (G, k) is a yes-instance of $\ell C_\ell D$.

(\Rightarrow) Let $V' \subseteq V(H)$ be a vertex cover of size at most k . Consider the edge-deletion set $S := \{\{v, \alpha\} \mid v \in V'\}$. Since $|V'| = |S|$, we know that $|S| \leq k$. To show that $G' = G - S$ is C_ℓ -free, it is sufficient to show that α is not part of a C_ℓ in G' . Let $\{\alpha, v\} \in E(G')$. By construction, we know that $v \in V(H)$ and $v \notin V'$. Assume towards a contradiction that $\{v, \alpha\}$ is part of a C_ℓ . Then, there is an edge $\{u, \alpha\} \in E(G')$ such that $u \neq v$ and u is connected to v by an induced $P_{\ell-1}$. This implies that $u \in V(H)$ and $u \notin V'$. Since $u, v \in V(H)$ and u is connected to v by induced $P_{\ell-1}$ in G , we can

conclude that $\{u, v\} \in E(H)$. But that is a contradiction, since V' is a vertex cover of H and $v, u \notin V'$. Hence, G' is C_ℓ -free.

(\Leftarrow) Let S be an edge-deletion set of size at most k such that $G' := G - S$ is ℓ -colored C_ℓ -free. Let $E' := \{\{u, v\} \in E(G) \mid u, v \neq \alpha\}$ be the set of edges from G that are not incident with α . We observe that each edge $e \in E'$ is part of at most one C_ℓ in G . Since α is part of every C_ℓ in G , we know that there is a vertex $\beta \in V(G)$ such that $S' = (S \setminus \{e\}) \cup \{\{\alpha, \beta\}\}$ is an equivalent solution. Since β is adjacent to α , we can conclude that $\beta \in V(H)$.

To finish the proof, we show that $V' := \{\beta \mid \{\alpha, \beta\} \in S'\}$ is a vertex cover for H . Let $\{v_1, v_2\} \in E(H)$ be an edge from H . Assume towards a contradiction that $v_1, v_2 \notin V'$. This implies that $\{\alpha, v_1\}, \{\alpha, v_2\} \notin S'$. By construction, we know that v_1 and v_2 are connected by an induced $P_{\ell-1}$ that consists of edges from E' and contains all colors except $\varphi(v_1)$ and $\varphi(v_2)$. Since $\{\alpha, v_1\}$ has color $\varphi(v_1)$, $\{\alpha, v_2\}$ has color $\varphi(v_2)$, and $S' \cap E' = \emptyset$, we conclude that v_1 and v_2 are part of a ℓ -colored C_ℓ in G' . That is a contradiction, since G' is ℓ -colored C_ℓ -free. So $v_1 \in V'$ or $v_2 \in V'$. thus, V' is a vertex cover for H . \square

Now we will use this result to prove the NP-hardness of cC_ℓ D for all $c \in [\ell - 1]$.

Lemma 4. cC_ℓ D is NP-hard for each $c \in [\ell - 1]$ even if G has girth ℓ and each C_ℓ in G is c -colored.

Proof: We give a reduction from ℓC_ℓ D on graphs where each C_ℓ is ℓ -colored. Let (H, k) be an instance of ℓC_ℓ D where the girth of H is ℓ and each C_ℓ in H is ℓ -colored. To construct an equivalent instance (G, k) of cC_ℓ D we recolor the edges. For each $\alpha \in [c - 1]$ we set $E_\alpha(G) := E_\alpha(H)$. Next, we set $E_c(G) := E_c(H) \cup \dots \cup E_\ell(H)$. Note that the vertex set and the budget k remains the same. Since we do not add new edges, the girth remains the same, and since each C_ℓ in H is ℓ -colored, we know by construction that each C_ℓ in G is c -colored. Since every C_ℓ in H is ℓ -colored, for each vertex set $V' \subseteq V(H)$ the induced subgraph $H[V']$ is an ℓ -colored C_ℓ if and only if the induced subgraph $G[V']$ is a c -colored C_ℓ . Hence, (H, k) is a yes-instance of ℓC_ℓ D if and only if (G, k) is a yes-instance of cC_ℓ D. \square

If G has girth ℓ , then every c -colored C_ℓ in G is an induced subgraph of G . Thus, we may conclude the following from Lemmas 3 and 4.

Theorem 3. cC_ℓ D is NP-hard for each $\ell \geq 3$ and each $c \in [\ell]$ even if G has girth ℓ and each C_ℓ in G is c -colored on strictly non-cascading graphs.

4 Parameterized Complexity

Motivated by the NP-hardness of cP_ℓ D and cC_ℓ D, we now study the parameterized complexity of these problems. We first show that both problems admit an FPT-algorithm for a new parameter which we call colored neighborhood diversity. Then, we outline the limits of parameterization by the solution size k .

4.1 Parameterization by Colored Neighborhood Diversity

We extend the notion of the well-known parameter neighborhood diversity to a similar parameter for edge-colored graphs. In uncolored graphs, two vertices u and v belong to the same neighborhood class if $N[u] = N[v]$ or $N(u) = N(v)$. This defines an equivalence relation over the vertex set of the graph. The *neighborhood diversity* is then defined as the number of equivalence classes induced by this relation [Lam12]. We show that all problems cP_ℓ D with $\ell \geq 3$ and cC_ℓ D with $\ell \geq 5$ are fixed-parameter tractable for a parameter we call *colored neighborhood diversity*.

Definition 2. Let G be a c -colored graph, and let u, v be vertices of G . We say that u and v belong to the same colored neighborhood class if either

- a) $N^i[u] = N^i[v]$ for some color i , and $N^j(u) = N^j(v)$ for every other color $j \in [c] \setminus \{i\}$, or
- b) $N^j(u) = N^j(v)$ for every color $j \in [c]$.

If u and v belong to the same colored neighborhood class we write $u \sim v$.

We define the *colored neighborhood diversity* $\gamma := \gamma(G)$ as the number of equivalence classes induced by \sim . To see that this definition is sound we first show that \sim is in fact an equivalence relation.

Proposition 2. Let G be a c -colored graph. Then, \sim is an equivalence relation on the vertices of G .

Proof: By definition, \sim is reflexive and symmetric. It remains to show transitivity. Let $u \sim v$ and $v \sim w$. Consider the following cases.

Case 1: $\{u, v\} \notin E$ and $\{v, w\} \notin E$. Then, $u \sim v$ and $v \sim w$ implies $N^i(u) = N^i(v) = N^i(w)$ for each color i . Thus, we have $u \sim w$.

Case 2: $\{u, v\} \in E_i$ for some color i . Then, $v \sim w$ implies $\{u, w\} \in E_i$ and $u \sim v$ implies $\{v, w\} \in E_i$. Consequently, it holds that $N^i[u] = N^i[v] = N^i[w]$ and $N^j(u) = N^j(v) = N^j(w)$ for all $j \in [c] \setminus \{i\}$. Thus, we have $u \sim w$. \square

We refer to the equivalence classes of \sim as *colored neighborhood classes*. Observe that each colored neighborhood class K is either an independent set or a clique where all edges of $E(K)$ have the same color. Moreover, observe that the neighborhood $N(K)$ can be partitioned into non-empty vertex sets K'_1, K'_2, \dots, K'_t such that each K'_i forms a colored neighborhood class in G . Given K , we let $\mathcal{N}(K) := \{K'_1, \dots, K'_t\}$ denote the set of these colored neighborhood classes.

Throughout this section, we call a graph F *color diverse* if it holds that $|K| = 1$ for every colored neighborhood class K of F . Let \mathcal{F} be a set of c -colored graphs such that every $F \in \mathcal{F}$ is color diverse and it can be checked in polynomial time whether a graph contains some $F \in \mathcal{F}$ as induced subgraph. We show that, in this case, \mathcal{F} -DELETION is fixed-parameter tractable when parameterized by γ . For $c \geq 2$, this implies fixed-parameter tractability of $cP_\ell D$ with $\ell \geq 3$ and $cC_\ell D$ with $\ell \geq 5$ since these problems can be modeled as a special case of \mathcal{F} -Deletion since c -colored P_ℓ s with $\ell \geq 3$ and c -colored C_ℓ s with $\ell \geq 5$ are color diverse. The following definition is important for our fixed-parameter algorithm.

Definition 3. Let $G = (V, E)$ be a c -colored graph, let K be a colored neighborhood class, and let $S \subseteq E$ be an edge-deletion set. Then, S is called *consistent with K* if for every vertex $v \in N(K)$ either $E(\{v\}, K) \subseteq S$ or $E(\{v\}, K) \cap S = \emptyset$.

Intuitively, a vertex deletion set S is consistent with a colored neighborhood class all its vertices behave in the same way with respect to S . The fixed-parameter algorithm exploits that there is always a solution S that is consistent with every colored neighborhood class.

Lemma 5. Let \mathcal{F} be a set of c -colored graphs such that each colored neighborhood class of every $F \in \mathcal{F}$ has size one. Let G be a colored graph and let K be a colored neighborhood class of G . Moreover, let S be an edge-deletion set such that $G - S$ has no induced $F \in \mathcal{F}$. Then, there exists an edge-deletion set S' with $|S'| \leq |S|$ such that

- a) S' is consistent with K

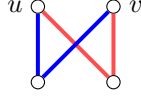


Fig. 6: An instance of $2C_4D$ that has a solution of size 1. This single edge deletion is either incident with u or incident with v .

- b) $G - S'$ is F -free for every $F \in \mathcal{F}$,
- c) for each $e \notin E(K) \cup E(K, N(K))$, we have $e \in S'$ if and only if $e \in S$.
- d) if S is consistent with a class $K' \in \mathcal{N}(K)$, then S' is consistent with K' .

Proof: We transform S into S' : For each $u \in K$ we define $S_u := \{w \in N(K) \mid \{u, w\} \in S\}$. Let $v \in K$ such that $|S_v| = \min_{u \in K} |S_u|$. The new edge-deletion set S' contains the same edges as S but for every $u \in K \setminus \{v\}$ we do the following: for every $w \in S_v$ we add the edge $\{u, w\}$ and for every $w \in N(K) \setminus S_v$ we remove the edge $\{u, w\}$. Moreover, if K is a clique, we remove all edges in $E_G(K)$.

Since $|S_v|$ is minimal, it follows that $|S'| \leq |S|$. Moreover, Statement c) holds by the construction of S' . For Statement a), let $w \in N(K)$. If $\{v, w\} \in S$ it follows that $E(\{w\}, K) \subseteq S'$. Otherwise, if $\{v, w\} \notin S$ it follows that $E(\{w\}, K) \cap S' = \emptyset$. Thus, Statement a) holds. We next show Statement b). That is, we show that $G - S'$ is F -free for every $F \in \mathcal{F}$. To this end, observe that K is a colored neighborhood class in $G - S'$ by the construction of S' . Let $e := \{u, w\} \in E(K) \cup E(K, N(K))$. Assume towards a contradiction that there exists a vertex set Z containing u and w such that $(G - S')[Z]$ is an induced $F \in \mathcal{F}$. We show that the following two cases are contradictory.

Case 1: $|Z \cap K| \geq 2$. Then, since K is a colored neighborhood class in $G - S'$, this contradicts the fact that F has only colored neighborhood classes of size one.

Case 2: $|Z \cap K| = 1$. Without loss of generality let $Z \cap K = \{u\}$. We then define $Z' := Z \setminus \{u\} \cup \{v\}$. Observe that in $G - S$, vertex v has the same colored neighbors in $N(K)$ as u in $G - S'$. We conclude that $G - S[Z' \setminus \{u\} \cup \{v\}]$ is an induced F which contradicts the fact that $G - S$ is \mathcal{F} -free.

It remains to prove d). Let $K' \in \mathcal{N}(K)$ such that S is consistent with K' . Statement c) implies that $E(\{w\}, K') \subseteq S'$ or $E(\{w\}, K') \cap S' = \emptyset$ for every $w \in N(K') \setminus K$. So, let $u \in K$. Since S is consistent with K' it holds that $E(\{v\}, K') \subseteq S$ or $E(\{v\}, K') \cap S = \emptyset$. Then, since in $G - S'$ every $u \in K$ has the same colored neighbors in $N(K)$ as v has in $G - S$, it follows that S' is consistent with K' . \square

In the proof of Lemma 5 we exploit that the neighborhood classes in every $F \in \mathcal{F}$ have size at most one. In fact, this condition is necessary since the lemma does not hold for $2C_4D$ as we can see in the example in Fig. 6.

By Lemma 5, we may assume that, for such \mathcal{F} a solution S of an instance of \mathcal{F} -DELETION is consistent with every colored neighborhood class. Then, for every pair K_1, K_2 of colored neighborhood classes either $E(K_1, K_2) \subseteq S$ or $E(K_1, K_2) \cap S = \emptyset$. This assumption can be used to obtain the following.

Theorem 4. *Let \mathcal{F} be a set of c -colored graphs such that every $F \in \mathcal{F}$ is color diverse and we can check in polynomial time whether a graph contains some $F \in \mathcal{F}$ as induced subgraph. Then, \mathcal{F} -DELETION can be solved in $2^{\gamma^2} \cdot n^{\mathcal{O}(1)}$ time, where γ denotes the colored neighborhood diversity.*

Proof: First, compute all colored neighborhood classes in polynomial time. Note that there are $\mathcal{O}(\gamma^2)$ edge sets between different colored neighborhood classes. Then, iterate over all $\mathcal{O}(2^{\gamma^2})$ possibilities to delete some of these edge sets. If one of these edge deletions leads to a solution, return *yes*. Otherwise, return *no*. \square

Corollary 4. $cP_\ell\text{D}$ with $\ell \geq 3$ and $cC_\ell\text{D}$ with $\ell \geq 5$ can be solved in $2^{\gamma^2} \cdot n^{\mathcal{O}(1)}$ time.

4.2 Parameterization by Solution Size

Observe that if ℓ is a constant, $cP_\ell\text{D}$ and $cC_\ell\text{D}$ can be solved by a naive branching algorithm with running time $\mathcal{O}(\ell^k \cdot n^\ell)$: For a given instance (G, k) check in $\mathcal{O}(n^\ell)$ time if G contains a c -colored P_ℓ (or C_ℓ , respectively). If this is not the case, then answer *yes*. Otherwise, answer *no* if $k < 1$. If $k \geq 1$, then compute a c -colored P_ℓ (or C_ℓ) with edges e_1, \dots, e_ℓ and branch into the cases $(G - e_i, k - 1)$ for $i \in \{1, \dots, \ell\}$.

To put the naive branching algorithm into context, we next study both problems parameterized by k and ℓ when ℓ is not a constant. More precisely, we study the problems COLORED PATH DELETION (CPD) and COLORED CYCLE DELETION (CCD) which are versions of $cP_\ell\text{D}$ and $cC_\ell\text{D}$ where ℓ and c are part of the input. For the parameter ℓ it is $\text{W}[1]$ -hard to decide whether a given graph has an induced P_ℓ and to decide whether a given graph has an induced C_ℓ [CF07]. Consequently, CPD and CCD are $\text{W}[1]$ -hard for ℓ even if $c = 1$ and $k = 0$. Thus, it is hopeless to obtain fixed-parameter tractability for k or even $k + \ell$.

The above might give the impression that the hardness of CPD and CCD is rooted in the problem of detecting the forbidden subgraphs. However, we show that even if the forbidden subgraphs are given, CPD and CCD are still unlikely to be fixed-parameter tractable for k . More precisely, we show that both problems are $\text{W}[2]$ -hard when parameterized by k even if the induced subgraphs can be enumerated within polynomial time.

Theorem 5. CPD is $\text{W}[2]$ -hard when parameterized by k even if

- a) all induced c -colored P_ℓ s can be enumerated in polynomial time, and
- b) the input is limited to instances where $c = 3$ and the input graph is non-cascading and has $n^{\mathcal{O}(1)}$ induced c -colored P_ℓ s.

Proof: We give a parameterized reduction from the $\text{W}[2]$ -hard problem HITTING SET (HS) parameterized by k [CFK⁺15]. In HS one is given a universe $U = \{x_1, \dots, x_\eta\}$, a family $\mathcal{F} = \{F_1, \dots, F_\mu\}$ of subsets of U , and an integer k . The question is if there is some $H \subseteq U$ with $|H| \leq k$ and $H \cap F_j \neq \emptyset$ for every $F_j \in \mathcal{F}$. Let (U, \mathcal{F}, k) be an instance of HS. We can assume that each set F_j is non-empty and that each $x_i \in U$ occurs in at least one subset $F_j \in \mathcal{F}$.

Construction: To construct an equivalent instance (G, c, ℓ, k) of CPD we first set $c = 3$ and $\ell = 1 + 3 \cdot \eta$. Then, we construct the following gadgets.

For each $x_i \in U$ we construct an *element gadget* W_i as follows: We add two vertices w_i, \tilde{w}_i to W_i and connect w_i with \tilde{w}_i by a blue edge. By W we denote the set of all element gadgets.

Next, we construct a *subset gadget* Z_j for each set $F_j \in \mathcal{F}$. We add a vertex v^j , and for each $i \in [\eta]$ a vertex u_i^j to Z_j . Then, we add a yellow edge $\{v^j, u_1^j\}$. If $x_i \in F_j$, then we connect the corresponding element gadget by adding a red edge $\{u_i^j, w_i\}$ and if $i < \eta$ a red edge $\{\tilde{w}_i, u_{i+1}^j\}$. Else if $x_i \notin F_j$, then

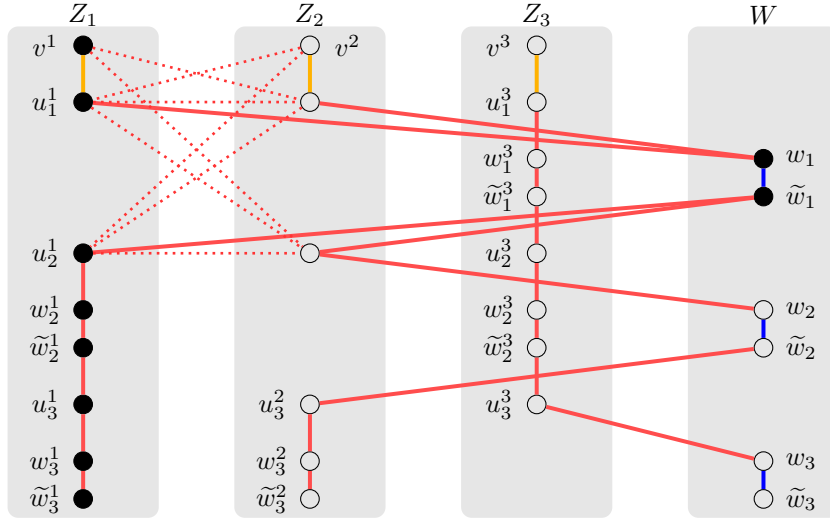


Fig. 7: The constructed graph for $\mathcal{F} = \{F_1 = \{1\}, F_2 = \{1, 2\}, F_3 = \{3\}\}$ and $U = [3]$. The dotted lines represent fixed edges, while the solid lines represent unfixed edges. The filled vertices induce the 3-colored P_{10} in $G[F_1 \cup W]$.

we add two vertices w_i^j, \tilde{w}_i^j to Z_j , and we add red edges $\{u_i^j, w_i^j\}, \{u_i^j, \tilde{w}_i^j\}$ to G and if $i < \eta$ we add a red edge $\{\tilde{w}_i^j, u_{i+1}^j\}$ to G . All edges that we have added so far are called *unfixed edges*.

Finally, we connect the subset gadgets as follows. Let F_p, F_q be subsets such that $x_i \in F_p$ and $x_i \in F_q$ for an element $x_i \in U$. We add red edges $\{u_i^p, u_i^q\}, \{v^p, u_i^q\}, \{v^q, u_i^p\}, \{u_1^p, u_i^q\}, \{u_1^q, u_i^p\}$ and if $i < \eta$ we add red edges $\{u_{i+1}^p, u_{i+1}^q\}, \{v^p, u_{i+1}^q\}, \{v^q, u_{i+1}^p\}, \{u_1^p, u_{i+1}^q\}, \{u_1^q, u_{i+1}^p\}$ (see Fig. 7). We call these edges *fixed*. Observe that for any edge $e \in E$ we call e a fixed edge if e connects two vertices from different subset gadgets and otherwise, we call e an unfixed edge.

Non-Cascading: We first argue that the constructed graph is non-cascading. By construction, every unfixed edge is a conflict edge and every fixed edge is conflict-free. Moreover, every non-induced c -colored P_ℓ contains one yellow edge $\{v^i, v_1^i\}$ from a subset gadget of Z_i and some vertex x from a subset gadget of some Z_j with $j \neq i$. Then, there are fixed edges $\{v^i, x\}$ and $\{v_1^i, x\}$. Since all fixed edges are non-conflict edges we conclude that G is non-cascading.

Enumerating c -colored P_ℓ s: By construction, for every induced c -colored P_ℓ there is one unique $v \in \{v^i \mid i \in [\mu]\}$ and one unique $w \in \{\tilde{w}_\eta\} \cup \{\tilde{w}_\eta^i \mid i \in [\mu]\}$ such that v and w are the endpoints of the induced c -colored P_ℓ . Thus, there are exactly μ induced c -colored P_ℓ in G than can be enumerated in polynomial time.

Intuition: Before we prove the correctness of the reduction, we describe its idea. We connected the subset gadgets to element gadgets such that for each subset gadget Z_j , the induced subgraph $G[Z_j \cup W]$ contains exactly one induced 3-colored P_ℓ . We then connected the subset gadgets such that there is no induced 3-colored P_ℓ in G that contains vertices from two different subset gadgets. So we can model a hitting set for a collection \mathcal{F} by deleting the edges from the corresponding element gadgets.

Correctness: To prove the correctness of the reduction, we show the following claims. First, we show that each 3-colored P_ℓ contains vertices of exactly one subset gadget if we do not delete fixed edges.

Claim 5. Let S be an edge-deletion set that does not contain fixed edges. Furthermore, let $G' := G - S$ and $V' \subseteq V(G)$ such that the induced subgraph $G'[V']$ is a 3-colored P_ℓ . Then, V' contains vertices of at most one subset gadget Z_j .

Proof. Since each 3-colored P_ℓ has to include a yellow edge, we conclude that $v^p, u_1^p \in V'$ for some $p \in [\mu]$. Assume towards a contradiction that V' contains a vertex from a clause gadget Z_q such that $q \neq p$. Without loss of generality there are adjacent vertices $\alpha, \beta \in V'$ such that $\alpha \in Z_p, \beta \in Z_q$ and $\{\alpha, \beta\} \in E(G)$, since we added the fixed edges. By construction, we know that $\{v^p, \beta\}, \{u_1^p, \beta\} \in E \setminus S$ since $\{v^p, \beta\}, \{u_1^p, \beta\}$ are fixed edges. This is a contradiction since the induced subgraph $G'[\{v^p, u_1^p, \beta\}]$ is a C_3 . \diamond

Second, we observe the following about 3-colored P_ℓ s that are induced by one subset gadget.

Claim 6. For each subset gadget Z_j , the graph $G[Z_j \cup W]$ contains exactly one 3-colored P_ℓ . This 3-colored P_ℓ contains a blue edge $\{w_i, \tilde{w}_i\}$ if and only if $x_i \in F_j$.

Proof. By construction, we know that $G[Z_j \cup W]$ contains at least one 3-colored P_ℓ , and that $\{v^j, u_1^j\}$ is the only yellow edge in $E_G(Z_j \cup W)$. Since $\deg_{G[Z_j \cup W]}(v^j) = 1$ and $\deg_{G[Z_j \cup W]}(\alpha) \leq 2$ for each $\alpha \in Z_j \cup W$, we conclude that there is at most one 3-colored P_ℓ in $G[Z_j \cup W]$.

By construction, this 3-colored P_ℓ contains a blue edge $\{w_i, \tilde{w}_i\}$ if $x_i \in F_j$. Otherwise, if $x_i \notin F_j$, then the 3-colored P_ℓ contains a red edge $\{w_i^j, \tilde{w}_i^j\}$ and does not contain the blue edge $\{w_i, \tilde{w}_i\}$. \diamond

Now we prove the correctness of the reduction by showing that (U, \mathcal{F}, k) is a yes-instance of HS if and only if (G, c, ℓ, k) is a yes-instance of CPD.

(\Rightarrow) Let $H \subseteq U$ be a hitting set of size at most k for (U, \mathcal{F}) . Consider the set $S := \{\{w_i, \tilde{w}_i\} \mid x_i \in H\}$. We will show that $G' := G - S$ is 3-colored P_ℓ -free. Since S does not contain fixed edges, we know from Claim 5 that there is no induced 3-colored P_ℓ in G' that contains vertices from two different subset gadgets. Hence, it remains to show that no subgraph $G'[Z_j \cup W]$ contains an induced 3-colored P_ℓ for any subset gadget Z_j .

Let Z_j be a subset gadget. By Claim 6 we know that there is exactly one 3-colored P_ℓ in the induced subgraph $G[Z_j \cup X]$ and that this 3-colored P_ℓ includes exactly one blue edge from each element gadget W_i where $x_i \in F_j$. Since H is a hitting set for \mathcal{F} , we conclude that S includes at least one blue edge from an element gadget W_i that is part of the 3-colored P_ℓ in $G[Z_j \cup W]$. Hence, $G'[Z_j \cup W]$ is 3-colored P_ℓ -free, which implies that G' is 3-colored P_ℓ -free.

(\Leftarrow) Conversely, let S be an edge-deletion set of size at most k such that $G - S$ is 3-colored P_ℓ -free. First, we show that $S' := S \setminus \{e \in S \mid e \text{ is fixed}\}$ is an equivalent solution. Assume towards a contradiction that $G' := G - S'$ contains an induced 3-colored P_ℓ . Let $V' \subseteq V(G)$ such that $G'[V']$ is a 3-colored P_ℓ . Since S' does not contain fixed edges, we know by Claim 5 that V' contains vertices of at most one subset gadget Z_j . Hence, $G'[V']$ only contains unfixed edges. This implies that $G'[V']$ does not contain an edge from S . Hence, the induced subgraph $G[V']$ is a 3-colored P_ℓ in G' . This is a contradiction, since $G - S$ is 3-colored P_ℓ -free. Hence, S' is an equivalent solution.

Next, we will show how to construct another equivalent solution $S'' \subseteq E_G(W)$ from S' . Let $\{\alpha, \beta\} \in S'$ such that $\{\alpha, \beta\} \notin E_G(W)$. We will show that $\{\alpha, \beta\}$ is part of at most one 3-colored P_ℓ . From Claim 5, we conclude that any 3-colored P_ℓ including $\{\alpha, \beta\}$ only includes vertices from at most one subset gadget, since S' only contains unfixed edges. Hence, $\{\alpha, \beta\}$ can only be part of a 3-colored P_ℓ in an induced subgraph $G[Z_j \cup W]$ where $\alpha \in Z_j$ or $\beta \in Z_j$. From Claim 6 we know that there is exactly one 3-colored P_ℓ in $G[Z_j \cup W]$. Hence, $\{\alpha, \beta\}$ is part of at most one 3-colored P_ℓ . This 3-colored P_ℓ has

to include a blue edge $e \in E_G(W)$, since G is 3-colored and the edges in $E_G(W)$ are the only blue edges. Hence, $(S \setminus \{\alpha, \beta\}) \cup \{e\}$ is an equivalent solution. Thus, we can construct an equivalent solution S'' that only contains edges from $E_G(W)$.

To finish the proof, we show that the set $H := \{x_i \mid \{w_i, \tilde{w}_i\} \in S''\}$ is a hitting set for \mathcal{F} . Let $F_j \in \mathcal{F}$. From Claim 6 we know that $G[Z_j \cup W]$ includes a 3-colored P_ℓ . Hence, there is at least one edge from that 3-colored P_ℓ in every solution. Since S'' is a solution that only contains edges $\{w_i, \tilde{w}_i\} \in E_G(W)$, we conclude that there is at least one $x_i \in H$ such that $x_i \in F_j$. Hence, H is a hitting set for \mathcal{F} . \square

We next extend the construction from the proof of Theorem 5 to obtain a similar W[2]-hardness for CCD.

Theorem 6. *CCD is W[2]-hard when parameterized by k even if*

- a) *all induced c -colored C_ℓ s can be enumerated in polynomial time, and*
- b) *the input is limited to instances where $c = 4$ and the input graph is non-cascading and has $n^{\mathcal{O}(1)}$ induced c -colored C_ℓ s.*

Proof: Let $I := (G, c = 3, \ell, k)$ be an instance that has been constructed from an instance of HS according to the proof of Theorem 5. Recall that G is non-cascading and that every induced c -colored P_ℓ in G there is one unique $v \in A := \{v^i \mid i \in [\mu]\}$ and one unique $w \in B := \{\tilde{w}_\eta\} \cup \{\tilde{w}_\eta^i \mid i \in [\mu]\}$ that are the endpoints of the induced c -colored P_ℓ .

We describe how to construct an instance $I' := (G', c = 4, \ell + 1, k)$ such that I' is a yes-instance of CCD if and only if I is a yes-instance of CPD.

Construction: Let 1, 2, and 3 be the colors present in G . We describe how to obtain G' that is colored with edge colors 1, 2, 3, and 4. The graph G' can be computed from G by adding a vertex set Q of size $k + 1$, and we add edges with color 4 between Q and $A \cup B$ such that $Q \cup (A \cup B)$ forms a biclique.

Analogously to Theorem 5, every non-induced 4-colored $P_{\ell+1}$ contains vertices from different subset gadgets of G and thus, G' is non-cascading. Moreover, observe that every induced 4-colored $C_{\ell+1}$ in G' consists of a 3-colored P_ℓ with one endpoint $v \in A$ and one endpoint $w \in B$ and two edges $\{v, q\}$, $\{q, w\}$ with $q \in Q$. Thus, there are $\mu \cdot (k + 1)$ induced 4-colored $C_{\ell+1}$ in G' which can be enumerated by enumerating all all 3-colored P_ℓ in G in $\text{poly}(n)$ time and combining each with one vertex from Q .

Correctness: We next show the correctness.

(\Rightarrow) Let S with $|S| \leq k$ be an edge-deletion set such that $G - S$ is 3-colored P_ℓ -free. Then, in $G' - S$ there is no induced 3-colored P_ℓ with one endpoint in A and one endpoint in B . Therefore, there is no induced 4-colored $C_{\ell+1}$ in $G' - S$. Thus, S is a solution of size at most k for I' .

(\Leftarrow) Let S' with $|S'| \leq k$ be an edge-deletion set such that $G' - S'$ is 4-colored $C_{\ell+1}$ -free.

First, assume towards a contradiction that there is some $v \in A$ and some $w \in B$ such that there is an induced 3-colored P_ℓ that starts in v , ends in w and only contains vertices from G . Note that there are $k + 1$ edge-disjoint paths (v, q, w) with $q \in Q$. Each such path has subsequent colors 4, 4 and forms an induced 4-colored $C_{\ell+2}$ with the c -colored P_ℓ connecting v and w . Consequently, for each such path (v, q, w) at least one edge on the path belongs to S' contradicting the fact that $|S'| \leq k$.

We next show that this implies that I has a solution of size at most k . Let X be the set of conflict edges in G , which is the set of all edges of G that are part of an induced c -colored P_ℓ . Furthermore, let $S \subseteq S'$ be the set of edge-deletions between the vertices of G . By the above, every induced c -colored P_ℓ in G

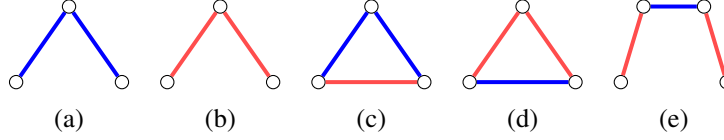


Fig. 8: The double cluster graphs are the graphs that have none of (a) to (d) as induced subgraph; \mathcal{T} is the class of graphs that do not have (a) to (e) as induced subgraph.

is not an induced c -colored P_ℓ in $G - S$. Then, since G is non-cascading, we conclude by Proposition 1 that $G - (S \cap X)$ has no induced c -colored P_ℓ . Thus, $S \cap X$ is a solution of size at most k for I . \square

5 $2P_4$ Deletion on Double Cluster Graphs

We now study $2P_4$ D on bicolored graphs where both the blue and the red subgraph are a cluster graph; we refer to such graphs as *double cluster graphs*. Equivalently, a graph is a double cluster graph if it has no graph from Fig. 8 (a)–(d) as induced subgraph. As we show, $2P_4$ D remains NP-hard on double cluster graphs. This is in sharp contrast to $2P_3$ D which is solvable in polynomial time in this graph class [GKSS21]. This hardness is tight in the following sense: In double cluster graphs, there exist only two types of induced bicolored P_4 s, the P_4 with subsequent edge colors red, blue, red and the P_4 with subsequent edge colors blue, red, blue. If the input is a double cluster graph where one of these bicolored P_4 is forbidden, say the P_4 shown in Fig. 8 (e), then $2P_4$ D can be solved in polynomial time. In the following we denote this graph class by \mathcal{T} .

Theorem 7. *$2P_4$ D remains NP-hard on bicolored graphs where both the blue and the red subgraph are a cluster graph and the maximum degree is five.*

Proof: We give a polynomial-time reduction from the NP-complete (3,B2)-SAT problem [BKS03]. Recall that (3,B2)-SAT is a version of 3SAT where one is given a CNF formula Φ on variables x_1, \dots, x_η where every clause c_1, \dots, c_μ contains exactly three literals and each literal x_i and $\neg x_i$ occurs exactly twice in Φ .

Construction: The instance (G, k) of $2P_4$ D consists of one clause gadget Z_i for each clause c_i and one variable gadget X_j for each variable x_j . The clause gadget Z_i contains a blue triangle with vertices d_i^1, d_i^2 , and d_i^3 . Furthermore, for each $z \in [3]$ we attach one leaf vertex c_i^z to vertex d_i^z in the triangle. Each edge $\{c_i^z, d_i^z\}$ is colored red.

The variable gadget X_j is constructed as follows:

- We add a red clique on four vertices r_j^1, r_j^2, r_j^3 , and r_j^4 to G .
- We add six blue triangles $\{r_j^1, p_j^1, p_j^2\}$, $\{r_j^2, q_j^1, q_j^2\}$, $\{p_j^3, p_j^5, p_j^7\}$, $\{q_j^3, q_j^5, q_j^7\}$, $\{p_j^4, p_j^6, p_j^8\}$, and $\{q_j^4, q_j^6, q_j^8\}$ to G .
- We add four blue edges $\{p_j^9, t_j^1\}$, $\{p_j^{10}, t_j^2\}$, $\{q_j^9, f_j^1\}$, and $\{q_j^{10}, f_j^2\}$ to G .
- We add the eight red edges $\{p_j^1, p_j^3\}$, $\{p_j^2, p_j^4\}$, $\{p_j^5, p_j^9\}$, $\{p_j^6, p_j^{10}\}$, $\{q_j^1, q_j^3\}$, $\{q_j^2, q_j^4\}$, $\{q_j^5, q_j^9\}$, and $\{q_j^6, q_j^{10}\}$ to G .

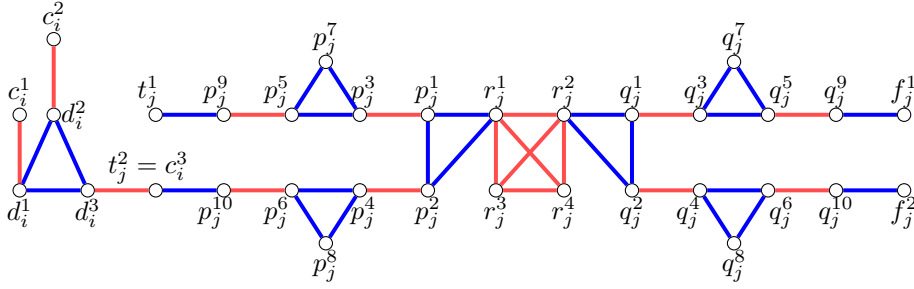


Fig. 9: Visualization of the construction for some clause c_i and a variable x_j where x_j has its second occurrence as the third literal in c_i .

To connect the variable and the clause gadgets we identify vertices as follows. For $z \in [3]$, $y \in [2]$, any clause c_i , and any variable x_j we set

$$c_i^z = \begin{cases} t_j^y & \text{if the literal } x_j \text{ has its } y\text{-th occurrence as the } z\text{-th literal in } c_i, \\ f_j^y & \text{if the literal } \neg x_j \text{ has its } y\text{-th occurrence as the } z\text{-th literal in } c_i. \end{cases}$$

Observe that since each clause contains exactly three literals and since each literal x_j and $\neg x_j$ occurs exactly twice in Φ , each vertex c_i^z is identified with exactly one vertex t_j^y or f_j^y and vice versa. For an illustration of this construction see Fig. 9.

It is easy to see that the maximum degree of G is five, that the blue subgraph of G is a cluster graph, and that the red subgraph of G is a cluster graph. We complete the construction by setting $k := 9\eta + 2\mu$.

Intuition: Before we prove the correctness, we informally describe its idea. Since the budget is tight, exactly two edges of each clause gadget and exactly nine edges of each variable gadget will be deleted. There are exactly two possibilities to delete nine edges in a variable gadget to make it conflict-free. One possibility corresponds to setting this variable to `true` and the other possibility corresponds to setting this variable to `false`. Furthermore, one remaining red edge in a clause gadget will represent the literal fulfilling that clause.

Before we prove the correctness of the reduction, we make the following observations.

Claim 7. For any edge-deletion set S of G and any clause gadget Z_i we have $|S \cap E(Z_i)| \geq 2$. Furthermore, if $|S \cap E(Z_i)| = 2$, then $|S \cap E_r(Z_i)| = 2$.

Proof. Consider the three bicolored P_4 s: $P_1 := \{c_i^1, d_i^1, d_i^2, c_i^2\}$, $P_2 := \{c_i^1, d_i^1, d_i^3, c_i^3\}$, and $P_3 := \{c_i^2, d_i^2, d_i^3, c_i^3\}$. Note that no edge is contained in all three P_4 s. Thus, $|S \cap E(Z_i)| \geq 2$.

Next, consider that case that $|S \cap E(Z_i)| = 2$. Assume towards a contradiction that S contains a blue edge of Z_i . Without loss of generality, assume that $\{d_i^1, d_i^2\} \in S$. Then, $G \setminus \{\{d_i^1, d_i^2\}\}$ contains the three bicolored P_4 s: P_2 , P_3 , and $\{d_i^1, d_i^3, d_i^2, c_i^2\}$. Note that no edge is contained in all three P_4 s. Hence, $|S \cap E(Z_i)| \geq 3$, a contradiction. Thus, $|S \cap E_r(Z_i)| = 2$. \diamond

Claim 8. For any edge-deletion set S of G and any variable gadget X_j we have $|S \cap E(X_j)| \geq 9$. Furthermore, if $|S \cap E(X_j)| = 9$, then there exists an edge-deletion set S' of G such that $|S' \cap E(X_j)| = 9$ and either

1. $S' \cap E(X_j) = \mathcal{E}_1$ which consists of the edges of the blue triangle $\{r_j^2, q_j^1, q_j^2\}$, the four red edges $\{p_j^1, p_j^3\}$, $\{p_j^2, p_j^4\}$, $\{q_j^5, q_j^9\}$, and $\{q_j^6, q_j^{10}\}$ and the two blue edges $\{p_j^9, t_j^1\}$ and $\{p_j^{10}, t_j^2\}$, or
2. $S' \cap E(X_j) = \mathcal{E}_2$ which consists of the edges of the blue triangle $\{r_j^1, p_j^1, p_j^2\}$, the four red edges $\{q_j^1, q_j^3\}$, $\{q_j^2, q_j^4\}$, $\{p_j^5, p_j^9\}$, and $\{p_j^6, p_j^{10}\}$ and the two blue edges $\{q_j^9, f_j^1\}$ and $\{q_j^{10}, f_j^2\}$.

Proof. The fact that $|S \cap E(X_j)| \geq 9$ follows from the existence of an edge-disjoint bicolored P_4 packing \mathcal{P} containing the following nine P_4 s. These are $\{t_j^1, p_j^9, p_j^5, p_j^7\}$, $\{t_j^2, p_j^{10}, p_j^6, p_j^8\}$, $\{f_j^1, q_j^9, q_j^5, q_j^7\}$, $\{f_j^2, q_j^{10}, q_j^6, q_j^8\}$, $\{p_j^7, p_j^3, p_j^1, r_j^1\}$, $\{q_j^7, q_j^3, q_j^1, r_j^2\}$, $\{p_j^8, p_j^4, p_j^2, p_j^1\}$, $\{q_j^8, q_j^4, q_j^2, q_j^1\}$ and $\{p_j^2, r_j^1, r_j^2, q_j^2\}$ in G .

Next, we consider an edge-deletion set S of G with $|S \cap E(X_j)| = 9$. First, we prove that $\{r_j^1, p_j^1\}$, $\{r_j^1, p_j^2\} \in S$ or $\{r_j^2, q_j^1\}$, $\{r_j^2, q_j^2\} \in S$. Assume towards a contradiction that this is not the case. Thus, assume without loss of generality that $\{r_j^1, p_j^1\}$, $\{r_j^2, q_j^1\} \in G - S$. Hence, $\{r_j^1, r_j^2\} \in S$. Next, observe that $G - \{\{r_j^1, r_j^2\}\}$ contains nine bicolored P_4 s. These are $\{p_j^1, r_j^1, r_j^3, r_j^2\}$, $\{q_j^1, r_j^2, r_j^4, r_j^1\}$, $\{t_j^1, p_j^9, p_j^5, p_j^7\}$, $\{t_j^2, p_j^{10}, p_j^6, p_j^8\}$, $\{f_j^1, q_j^9, q_j^5, q_j^7\}$, $\{f_j^2, q_j^{10}, q_j^6, q_j^8\}$, $\{p_j^7, p_j^3, p_j^1, r_j^1\}$, $\{p_j^8, p_j^4, p_j^2, r_j^1\}$, and finally $\{q_j^7, q_j^3, q_j^1, r_j^2\}$ only sharing the edges $\{r_j^1, p_j^1\}$, and $\{r_j^2, q_j^1\}$. Thus, $|S \cap E(X_j)| \geq 10$, a contradiction.

Hence, in the following we assume without loss of generality that $\{r_j^2, q_j^1\} \in S$ and that $\{r_j^2, q_j^2\} \in S$. Next, assume towards a contradiction that $\{q_j^5, q_j^9\} \notin S$. Since $\{q_j^3, q_j^5, q_j^9, f_j^1\}$ is a bicolored P_4 only sharing the edges $\{q_j^5, q_j^9\}$ and $\{q_j^9, f_j^1\}$ with any bicolored P_4 in \mathcal{P} , we conclude that $\{f_j^1, q_j^9\} \in S$. But then, $\mathcal{P} \setminus \{\{q_j^3, q_j^4, q_j^2, q_j^1\}, \{f_j^1, q_j^9, q_j^5, q_j^7\}\} \cup \{\{q_j^{10}, q_j^6, q_j^4, q_j^2\}, \{q_j^2, q_j^1, q_j^3, q_j^7\}, \{q_j^7, q_j^5, q_j^9, f_j^1\}\}$ is an edge-disjoint packing of size nine, a contradiction. Hence, $\{q_j^5, q_j^9\} \in S$. Analogously, we can show that $\{q_j^6, q_j^{10}\} \in S$. Since $\{q_j^7, q_j^3, q_j^1, q_j^2\}$ and $\{q_j^8, q_j^4, q_j^2, q_j^1\}$ are two bicolored P_4 s not in the packing \mathcal{P} , we conclude that $\{q_j^1, q_j^2\} \in S$.

Hence, in the graph $G[t_j^1, t_j^2, r_j^1, \dots, r_j^4, p_j^1, \dots, p_j^{10}]$ the edge-deletion set S does exactly four deletions. Furthermore, observe that

$$\mathcal{Q} := \{\{t_j^1, p_j^9, p_j^5, p_j^7\}, \{t_j^2, p_j^{10}, p_j^6, p_j^8\}, \{p_j^7, p_j^3, p_j^1, r_j^1\}, \{p_j^8, p_j^4, p_j^2, p_j^1\}\}$$

is a packing of four edge-disjoint bicolored P_4 s. Also, observe that $\{p_j^5, p_j^3, p_j^1, p_j^2\}$ is also a bicolored P_4 in G which only has the edge $\{p_j^1, p_j^3\}$ in common with each bicolored P_4 in \mathcal{Q} . Thus, $\{p_j^1, p_j^3\} \in S$. Analogously, we can show that $\{p_j^2, p_j^4\} \in S$.

Furthermore, observe that $\{t_j^1, p_j^9, p_j^5, p_j^3\}$ is also a bicolored P_4 in G . Hence, $\{t_j^1, p_j^9\} \in S$ or $\{p_j^9, p_j^5\} \in S$. Assume that $\{p_j^9, p_j^5\} \in S$. Since t_j^1 is the only vertex in that P_4 which has neighbors outside X_j , the set $S' := S - \{p_j^9, p_j^5\} \cup \{\{t_j^1, p_j^9\}\}$ is also an edge-deletion set with exactly nine deletions in X_j . Analogously, we can show that also $\{t_j^2, p_j^{10}\} \in S'$. \diamond

Now, we are ready to prove the correctness.

(\Rightarrow) Let $\mathcal{A} : \mathcal{X} \rightarrow \{\text{true}, \text{false}\}$ be a satisfying assignment for Φ . We will construct an edge-deletion set S of size k such that $G - S$ is bicolored P_4 -free.

For each variable x_j we do the following: If $\mathcal{A}(x_j) = \text{true}$, then we delete the first edge set \mathcal{E}_1 described in Claim 8. Otherwise, if $\mathcal{A}(x_j) = \text{false}$, then we delete the second edge set \mathcal{E}_2 described in Claim 8. Since \mathcal{A} satisfies Φ , there is at least one variable x_j such that $\mathcal{A}(x_j)$ satisfies clause c_i .



Fig. 10: Left: An example of an rb-fence. Right: An example of an rb-clique-star.

Let $z \in [3]$ be the z -th literal of c_i that satisfies clause c_i . Furthermore, let $\{\alpha, \beta\} := [3] \setminus \{z\}$. We delete the red edges $\{c_i^\alpha, d_i^\alpha\}$, and $\{c_i^\beta, d_i^\beta\}$. Observe that we deleted exactly nine edges per variable gadget and exactly two edges per clause gadget. Hence, $|S| = k = 9\eta + 2\mu$.

It remains to show that $G - S$ is bicolored P_4 -free. From Claims 7 and 8 we conclude that there is no bicolored P_4 whose vertex set is entirely contained in one clause or variable gadget. Recall that the vertices c_i^z for $z \in [3]$ which are identified with the vertices t_j^y , and f_j^y for $y \in [2]$ are the only vertices which connect variable and clause gadgets in G . Observe that all of these vertices have degree two. Consider a fixed vertex c_i^z . Next, we prove that in $G - S$ vertex c_i^z has degree at most one. Without loss of generality we assume that vertex c_i^z is identified with vertex t_j^y from variable x_j . If clause c_i is not satisfied by its z -th literal, then $\{c_i^z, d_i^z\} \in S$, and vertex c_i^z has degree at most one in $G - S$. Hence, in the following we assume that clause c_i is satisfied by its z -th literal which is x_j . Thus $\mathcal{A}(x_j) = \text{true}$ and according to our definition of S we observe that $\{t_j^y, p_j^{8+y}\} = \{c_i^z, p_j^{8+y}\} \in S$. We conclude that each vertex c_i^z in $G - S$ has degree at most one. Thus, there is no bicolored P_4 in at least two gadgets. Hence, $G - S$ is bicolored P_4 -free.

(\Leftarrow) Let $S \subseteq E(G)$ with $|S| \leq k$ be an edge set such that $G - S$ is bicolored P_4 -free. In the following we will construct a satisfying assignment $\mathcal{A} : \mathcal{X} \rightarrow \{\text{true}, \text{false}\}$ for Φ .

Since $|S| = 9\eta + 2\mu$, $|S \cap E(Z_i)| \geq 2$ for each clause gadget Z_i by Claim 7, and $|S \cap E(X_j)| \geq 9$ for each variable gadget X_j by Claim 8, we conclude that $|S \cap E_r(Z_i)| = 2$ and that $|S \cap E(X_j)| = 9$. Next, we construct an edge-deletion set S' with $|S'| = |S| = k$ which fulfills the conditions of Claim 8, that is for each variable X_j the set S' contains either the edge set \mathcal{E}_1 or the edge set \mathcal{E}_2 described in Claim 8. If S' contains the first edge set \mathcal{E}_1 described in Claim 8, we set $\mathcal{A}(x_j) = \text{true}$ and otherwise, if S' contains the second edge set \mathcal{E}_2 described in Claim 8, we set $\mathcal{A}(x_j) = \text{false}$.

It remains to show that \mathcal{A} satisfies Φ . Consider the clause gadget Z_i . By Claim 7 we conclude that $|S' \cap E_r(Z_i)| = 2$. Hence, there exist a $z \in [3]$ such that $\{c_i^z, d_i^z\} \notin S$. Without loss of generality we assume that vertex c_i^z is identified with vertex t_j^y for some variable x_j and some $y \in [2]$. Observe that $G[\{d_i^\alpha, d_i^z, c_i^z, p_j^{8+y}\}]$ for some $\alpha \in [3] \setminus \{z\}$ is an induced bicolored P_4 in G . Since $G - S$ is bicolored P_4 -free, we conclude that $\{c_i^z, p_j^{8+y}\} \in S$. Thus, $\mathcal{A}(x_j) = \text{true}$ and hence clause c_i is satisfied by \mathcal{A} . We conclude that Φ is satisfied by \mathcal{A} . \square

Before we present the polynomial-time algorithm for $2P_4$ D when the input graph is from \mathcal{T} , we characterize \mathcal{T} . More precisely, we show that each connected component in some $G \in \mathcal{T}$ is one of the following two graphs (see Fig. 10): An *rb-fence* is a graph that consists of exactly two blue cliques X and Y of size at least two and the red edges form a matching between vertices of X and Y . An *rb-clique-star* is a graph that consists of exactly one red clique X and up to $|X|$ non-overlapping blue cliques where each blue clique intersects with X in a unique vertex. To show this characterization of \mathcal{T} , we first prove the following.

Lemma 6. *Let $\{v, w\}$ be a blue edge in a bicolored graph $G \in \mathcal{T}$ such that v has a red neighbor u and w has a red neighbor x . Then, $u \neq x$ and $\{u, x\} \in E(G)$ with color blue. Furthermore, neither u and w nor w and x have a common red neighbor.*

Proof: Observe that u and x are distinct vertices, since otherwise $G[\{u, v, w\}]$ is a bicolored triangle, a contradiction to $G \in \mathcal{T}$. Since the vertices u, v, w , and x form a bicolored P_4 with subsequent edge colors red, blue and red, we conclude that this bicolored P_4 is not induced. Observe that $\{u, w\} \notin E(G)$ and that $\{v, x\} \notin E(G)$ since otherwise G would contain a bicolored triangle. Hence, $\{u, x\} \in E(G)$. Furthermore, $\{u, x\}$ is blue, since otherwise the red subgraph G_r is no cluster graph. Next, assume towards a contradiction that at least one of both red edges has a common red neighbor. Without loss of generality, we assume that vertices w and x have a common red neighbor y . Observe that $\{v, y\} \notin E(G)$ since otherwise G would contain a bicolored triangle. Next, consider the case that $\{u, y\} \in E(G)$. With symmetric arguments we conclude $\{v, y\} \notin E(G)$. Hence, $\{u, y\}$ is blue. But now $G[\{v, u, y, x\}]$ is an induced bicolored P_4 with subsequent edge colors red, blue, red; a contradiction. \square

We now present our alternative characterization of \mathcal{T} .

Proposition 3. *Let G be a graph. Then, $G \in \mathcal{T}$ if and only if each connected component of G is either an rb-fence or an rb-clique-star.*

Proof: Given an rb-fence or an rb-clique-star, the colored subgraphs of both colors are cluster graphs. Moreover, it is easy to see that no such graph has an induced bicolored P_4 with subsequent edge colors red, blue, red. Thus, every graph where each connected component is either an rb-fence or an rb-clique-star belongs to \mathcal{T} . We next show that each graph in \mathcal{T} is a disjoint union of rb-clique-stars and rb-fences.

First, consider a connected component Z of G which contains a maximal red clique C with at least three vertices. Observe that each vertex of C does not have a red neighbor outside C . Thus, each $v \in C$ only blue neighbors outside C . Since the blue graph is a cluster graph, these blue neighbors of v form a clique. Moreover, no two vertices in C have a common blue neighbor since otherwise G has a bicolored triangle. Finally, each blue neighbor of C has no red neighbor outside C since otherwise G has a red-blue-red P_4 . Thus, the connected component Z is an rb-clique-star.

Second, consider a connected component Z of G which does not contain red triangles. In other words, the red connected components of Z are edges. Let $\{u, v\}$ and $\{x, y\}$ be two red edges in Z . In the next step, we will prove that $G[\{u, v, x, y\}]$ is a C_4 where two subsequent edges have different colors. Assume towards a contradiction that this is not the case. By Lemma 6 we conclude that if $G[\{u, v, x, y\}]$ contains one blue edge, then $G[\{u, v, x, y\}]$ has to be a C_4 where two subsequent edges have different colors. Hence, we assume that $G[\{u, v, x, y\}]$ does not contain any blue edges. Without loss of generality assume that $P := (u = a_0, a_1, \dots, a_p = x)$ is a shortest path connecting the edges $\{u, v\}$ and $\{x, y\}$. Observe that since the induced red and the induced blue subgraph are both cluster graphs, the edges in P have alternating colors. Hence, $\{u, a_1\}$ is blue and $\{a_1, a_2\}$ is red. Furthermore, note that $a_1 \neq x$ and $a_1 \neq y$ since otherwise there is a blue edge in $G[\{u, v, x, y\}]$. Observe that $G[\{u, v, a_1, a_2\}]$ contains a bicolored P_4 with subsequent edge colors red, blue, red. By Lemma 6 we conclude that $\{v, a_2\} \in E(G)$ with color blue. Then, $P' := (v = a_1, \dots, a_p = x)$ is a shorter path connecting the edges $\{u, v\}$ and $\{x, y\}$ than P , a contradiction. We conclude that for each two red edges $\{u, v\}$ and $\{x, y\}$ in Z the induced subgraph $G[\{u, v, x, y\}]$ is a C_4 where two subsequent edges have different colors.

Let $R := \{\{u_i, v_i\} \mid i \in [t]\}$ be the set of red edges in Z . Since for every pair $\{u_i, v_i\}, \{u_j, v_j\}$ with $i \neq j$ the graph $G[\{u_i, v_i, u_j, v_j\}]$ is a C_4 we may assume without loss of generality that $B_1 :=$

$\{u_i \mid i \in [t]\}$ and $B_2 := \{v_i \mid i \in [t]\}$ are blue cliques in G . Observe that $E_G(B_1, B_2) = R$ since otherwise, if there is an edge $\{u_i, v_j\}$ with $i \neq j$, then $G[\{u_i, u_j, v_j\}]$ or $G[\{v_i, u_j, v_j\}]$ is a bicolored triangle in G .

If no vertex in $B_1 \cup B_2$ has a neighbor outside $B_1 \cup B_2$, then $Z = B_1 \cup B_2$ and therefore, $G[Z]$ is an rb-fence. So, without loss of generality, let x be a neighbor of u_1 outside $B_1 \cup B_2$. Then, $\{u_i, x\}$ is blue and since the blue graph is a cluster graph it follows that $B_1 \cup \{x\}$ is a blue clique in G . This implies that $B_1 \subseteq K_1$ and $B_2 \subseteq K_2$ for some maximal blue cliques K_1 and K_2 and that $G[Z] = G[K_1 \cup K_2]$ is an rb-fence. \square

The next goal is to show that $2P_4D$ can be solved in polynomial time on graphs in \mathcal{T} . To this end, we first consider the solution structure of rb-fences.

Lemma 7. *Let (G, k) be an instance of $2P_4D$ where G is an rb-fence and S be a minimum size solution. Then either G is a C_4 and $S = \emptyset$ or S contains exactly the red edges of G .*

Proof: Let K_1 and K_2 with $|K_1| \geq |K_2| \geq 2$ be the two blue cliques of G , and let $B_1 \subseteq K_1$ and $B_2 \subseteq K_2$ be such that there is a perfect red matching between B_1 and B_2 , and let $|B_1|$ and $|B_2|$ be maximal under this property. Furthermore, by R we denote the set of red edges in G . If $|K_1| = 2$ and $|R| = 2$, then G is a bicolored C_4 with alternating colors. Hence, G is $2P_4$ -free. Next, we assume that this is not the case. In the following, we prove that there is a minimal solution that deletes all red edges. Obviously, $G - R$ is $2P_4$ -free. We next show that there is no solution of size smaller than $|R|$.

First, consider the case $|K_2| \geq 3$ or $|R| = 1$. Then, there exists a vertex-pair disjoint packing of bicolored P_4 s of size $|R|$: Each red edge is the central edge of one of the P_4 s of this packing. Since one edge-deletion transforms only one bicolored P_4 of the packing, there have to be at least $|R|$ edge deletions in a minimal solution.

Second, consider the case $|K_2| = 2 = |R|$. Recall that $|K_1| \geq 3$. Since G contains a bicolored P_4 and deleting any of its three edges does not make G bicolored P_4 -free, we conclude that at least two edges of G have to be deleted. Hence, deleting both red edges is optimal. \square

Lemma 7 implies that connected components that are rb-fences can be solved in linear time. We next study the solution structure of rb-clique-stars.

Lemma 8. *Let (G, k) be an instance of $2P_4D$ where G is an rb-clique-star. Let C be the red clique and let B_1, \dots, B_ℓ be the blue cliques of G such that $|B_1| \geq \dots \geq |B_\ell|$. Furthermore, for every $i \in [\ell]$, let c_i denote the unique vertex in $B_i \cap C$.*

Then, there is a solution S such that $G - S$ consists of ℓ connected components and there is some $p \in [\ell]$ with

- a) $B_1 \cup C \setminus \{c_q \mid q \in [2, p]\}$ is a connected component in $G - S$,
- b) for every $q \in [2, p]$, B_q is a connected component in $G - S$, and
- c) for every $q \in [p + 1, \ell]$, $B_q \setminus \{c_q\}$ is a connected component in $G - S$.

Proof: Let S be an edge-deletion set of size at most k of G and let Z_1, \dots, Z_p be the connected components of $G - S$.

We prove the lemma in three steps.

Step 1: We first show that we may assume that for each $j \in [\ell]$ the vertex set $B_j \setminus \{c_j\}$ is completely contained in one connected component of $G - S$.

Observe that $B_j \setminus \{c_j\}$ is a colored neighborhood class by Definition 2. Moreover, $N(B_j \setminus \{c_j\}) = \{c_j\}$. Then, by Proposition 5 we may assume that no edge in $B_j \setminus \{c_j\}$ is part of a minimal solution S and that either $E(\{c_j\}, B_j \setminus \{c_j\}) \subseteq S$ or $E(\{c_j\}, B_j \setminus \{c_j\}) \cap S = \emptyset$. Consequently, we can safely assume that for each $j \in [\ell]$ the vertices in $B_j \setminus \{c_j\}$ are in one connected component Z_i for some $i \in [p]$.

Step 2: We next show that we may assume that the connected component containing $B_1 \setminus \{c_1\}$ is the only connected component in $G - S$ that might contain red edges.

Assume that there exist two connected components Z_i and Z_j which contain red edges. By R_i we denote the vertices in Z_i which are not incident with blue edges. We define $P := Z_j \cup R_i$ and $Q := Z_i \setminus R_i$. Then, $G[P]$ and $G[Q]$ are also P_4 -free. It remains to show that a solution introducing the connected components P and Q instead of Z_j and Z_i is optimal. Note that there are at least $(|R_i| + 1) \cdot (|R_j| + 1)$ edge deletions in $E(Z_i \cup Z_j)$ to obtain connected components Z_i and Z_j . In contrast, there are only $|R_i|$ edge deletions in $E(Z_i \cup Z_j)$ to obtain connected components P and Q . Since the number of edge deletions between $Z_i \cup Z_j$ and the rest of the graph is the same in both cases we conclude that this modification of the solution does not increase the number of edge deletions. Thus, in the following we can assume that there is at most one connected component which contains red edges.

We next prove that we can safely assume that if a connected component of $G - S$ contains red edges, it also contains B_1 . Assume that this is not the case. So let Z_1 be the connected component containing $B_1 \setminus \{c_1\}$ and let Z_j be the connected component containing all red edges. Let $c_q \in V(Z_j)$ for some $q \in [2, \ell]$ be the vertex incident with red and blue edges. If such a vertex c_q does not exist, then Z_j is a red clique and thus $G[Z_i \cup Z_j]$ is also bicolored P_4 -free and the statement follows. Otherwise, vertex c_j exists. Let $Y := Z_j \setminus B_j$ the vertices of Z_j which are only incident with red edges. Observe that $Z_j \setminus Y$ and $Z_i \cup Y$ are also bicolored P_4 free and the costs of this solution are also at most k since $|B_1| \geq |B_j|$. Hence, in the following we assume that the connected component of $G - S$ containing the red edges, also contains B_1 .

Step 3: We finally show that there is some $p \leq \ell$ satisfying statements a)–c). First, observe that each connected component contains exactly one blue clique $B_i \setminus \{c_i\}$, since otherwise, the component contains a bicolored P_4 . Let Z_i for $i \in [\ell]$ be the connected component that contains $B_i \setminus \{c_i\}$.

Assume that there are some i and j with $1 < i < j$ such that $Z_i = B_i \setminus \{c_j\}$ and $Z_j = B_j$. Note that $i < j$ implies $|B_i| \geq |B_j|$. We then define the sets $Z'_1 := Z_1 \setminus \{c_i\} \cup \{c_j\}$, $Z'_i := B_i$, and $Z'_j := B_j \setminus \{c_j\}$. It is easy to see that $G[Z'_1]$, $G[Z'_i]$, and $G[Z'_j]$ contain no induced bicolored P_4 . Moreover, note that, if a is the number of edges in the disjoint union of $G[Z'_1]$, $G[Z'_i]$, and $G[Z'_j]$ and b is the number of edges in the disjoint union of $G[Z_1]$, $G[Z_i]$, and $G[Z_j]$, we have $a - b = |B_i| - |B_j|$. Together with the fact that $|B_i| \geq |B_j|$ we conclude that there is an optimal solution introducing the connected components Z'_1 , Z'_i , and Z'_j instead of Z_1 , Z_i , and Z_j .

Consequently, we may assume that there is some $p \in [\ell]$ that satisfies statements b) and c). The component Z_1 then contains all vertices that are not in one of the components B_q with $q \in [2, p]$ or $B_q \setminus \{c_q\}$ with $q \in [p + 1, \ell]$. Thus, $Z_1 = B_1 \cup C \setminus \{c_q \mid q \in [2, p]\}$ and therefore, statement a) holds. \square

Lemma 8 implies that, in a $2P_4D$ -instance (G, k) , the connected components of G that are rb-clique-stars can be solved in polynomial time: Let the rb-clique-star contain ℓ blue cliques. Then, for every $p \leq \ell$, compute in $\mathcal{O}(nm)$ time the cost of an edge-deletion set S that forms a partition according Lemma 8 and keep the set with minimal cost. Altogether, this implies the following.

Theorem 8. $2P_4D$ can be solved in $\mathcal{O}(nm)$ time on graphs in \mathcal{T} .

6 Conclusion

We initiated a study of edge-deletion problems where the aim is to destroy paths or cycles containing a certain number of colors. We left many problems open for future research.

First, in our analysis of the classic complexity of $cP_\ell D$ for $\ell \geq 4$ on bounded-degree graphs, we have shown that for $c \in [2, \ell - 2]$ we obtain NP-hardness even if the maximum degree is 3, while for $c = \ell - 1$, we obtain hardness even if the maximum is 16. It is thus a natural question whether the NP-hardness for $c = \ell - 1$ also holds for a smaller maximum degree.

Furthermore, it is open whether there is a polynomial-time algorithm for $(\ell - 1)P_\ell D$ on non-cascading graphs for any $\ell \geq 4$ since we only showed the NP-hardness on non-cascading graphs for each $\ell \geq 4$ and each $c \in [\ell - 2]$. Similarly, we showed W[2]-hardness of CPD, the variant of $cP_\ell D$ where ℓ is part of the input and k is the parameter for non-cascading graphs, but it remains open whether CPD is W[2]-hard for k when every c -colored P_ℓ in the input graph is an induced c -colored P_ℓ . Concerning the parameterized complexity with respect to the colored neighborhood diversity γ it remains open whether $2C_4D$ is fixed-parameter tractable with respect to γ . It might also be interesting to analyze the parameterized complexity with regard to further structural graph parameters like the vertex cover number or the size of a minimum feedback vertex set. Moreover, since γ can be seen as a colorful version of the classic parameter *neighborhood diversity*, it can be interesting to investigate whether further well-known structural parameters have a corresponding colorful version that is useful for the problems studied in this work. It would also be interesting to study the corresponding edge-completion problems and edge-modification problems from an algorithmic point of view. Also, to which extent is it possible to adapt our results to related problems on vertex-colored graphs instead of edge-colored graphs?

More generally, it seems interesting to study further whether the fact that an input graph for an edge-modification problem is non-cascading has any impact on the problem difficulty. In other words, which edge-modification problems that are generally NP-hard become polynomial-time solvable on non-cascading input graphs? This question is also relevant for problems in uncolored graphs. Finally, developing a deeper understanding of properties of edge-colored graphs appears to be a wide open and fruitful research topic. Our work makes two contributions in this direction: characterizing one graph class via forbidden induced subgraphs and extending the neighborhood diversity parameter to edge-colored graphs. What are further interesting classes of edge-colored graphs and further graph parameters that take the edge-coloring into account?

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