

A tight lower bound for the online bounded space hypercube bin packing problem*

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In the d -dimensional hypercube bin packing problem, a given list of d -dimensional hypercubes must be packed into the smallest number of hypercube bins. Epstein and van Stee [SIAM J. Comput. 35 (2005)] showed that the asymptotic performance ratio ρ of the online bounded space variant is $\Omega(\log d)$ and $O(d/\log d)$, and conjectured that it is $\Theta(\log d)$. We show that ρ is in fact $\Theta(d/\log d)$, using probabilistic arguments.

Keywords: Hypercube packing, online bin packing, asymptotic performance ratio, online bounded space packing

1 Introduction

The bin packing problem is an iconic problem in combinatorial optimization, which has been investigated intensively from many different viewpoints. In particular, it has served as a proving ground for new approaches to the development and analysis of approximation and online algorithms, as well as for the development of average case analysis techniques (see Coffman Jr. et al. (1997, 2013)).

We prove a lower bound for a variant of the bin packing problem, in which the items to be packed are d -dimensional hypercubes, also referred to as d -hypercubes or simply hypercubes, when the dimension is clear. More precisely, we prove a tight lower bound for the *online bounded space d -hypercube bin packing problem*, settling an open problem raised by Epstein and van Stee (2005). Before we state our result (Theorem 5), we introduce the required concepts and definitions and discuss briefly the relevant literature.

The *d -hypercube bin packing problem (d -CPP)* is defined as follows. We are given a list L of *items*, where each item h in L is a d -hypercube of side length $s(h) \leq 1$, and an unlimited number of *bins*, each of which is a unit d -hypercube (that is, a d -hypercube of side length 1). The goal is to find a packing \mathcal{P} of

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the items in L into the smallest possible number of bins. More precisely, we have to assign each item h to a bin, and specify its position in that bin. We require that the items be placed parallel to the axes of the bin and, crucially, we require that the items in a bin should not overlap. The *size* $|\mathcal{P}|$ of the packing \mathcal{P} is the number of *used* bins (those with assigned items).

The d -hypercube bin packing problem (d -CPP) is in fact a special case of the d -dimensional bin packing problem (d -BPP), in which one has to pack d -dimensional parallelepipeds into d -dimensional unit bins. For $d = 1$, both problems reduce to the well known *bin packing problem*.

In the *online* variant of d -CPP, the items arrive sequentially and each item must be placed in some bin as soon as it arrives, without knowledge of the next items. The *online bounded space* variant of d -CPP is a restricted variant of online d -CPP. Whenever a new empty bin is used in the packing process, it is considered an *open* bin and it remains so until it is considered *closed*, after which point it is not allowed to accept other items. In this variant, regardless of the instance I , at every point of the process, not more than M bins should be open, where M is some constant that does not depend on I .

As usual for bin packing problems, we consider the asymptotic performance ratio to measure the quality of algorithms. For an algorithm \mathcal{A} and an input list L , let $\mathcal{A}(L)$ be the number of bins used by the solution produced by \mathcal{A} for the list L . Furthermore, let $\text{OPT}(L) = \min |\mathcal{P}|$, where the minimum is taken over all possible packings \mathcal{P} of L into unit bins. The *asymptotic performance ratio* of \mathcal{A} is

$$\mathcal{R}_{\mathcal{A}}^{\infty} = \limsup_{n \rightarrow \infty} \sup_L \left\{ \frac{\mathcal{A}(L)}{\text{OPT}(L)} : \text{OPT}(L) = n \right\}. \quad (1)$$

Given a packing problem Π , the *optimal asymptotic performance ratio* for Π is

$$\mathcal{R}_{\Pi}^{\infty} = \inf \{ \mathcal{R}_{\mathcal{A}}^{\infty} : \mathcal{A} \text{ is an algorithm for } \Pi \}. \quad (2)$$

Many results have been obtained for online d -BPP and d -CPP (see, *e.g.*, Balogh et al. (2019, 2012); Blitz et al. (2017); Christensen et al. (2017); Heydrich and van Stee (2016); Seiden (2002); van Vliet (1992)). In our brief discussion of the literature below, we restrict ourselves to the online bounded space versions of d -BPP and d -CPP.

For online bounded space 1-BPP, Lee and Lee (1985) gave an algorithm called HARMONIC_M with asymptotic performance ratio at most $(1 + \varepsilon)\Pi_{\infty}$, where $\varepsilon \rightarrow 0$ as $M \rightarrow \infty$, and $\Pi_{\infty} \approx 1.69103$ is a certain explicitly defined constant. These authors also proved that no algorithm for online bounded space 1-BPP can have asymptotic performance ratio smaller than Π_{∞} . For online bounded space d -BPP for general d , a lower bound of Π_{∞}^d was implicitly proved by Csirik and van Vliet (1993), and Epstein and van Stee (2005) proved an asymptotically matching upper bound.

For online bounded space d -CPP, Epstein and van Stee (2005) proved that its asymptotic performance ratio is $\Omega(\log d)$ and $O(d/\log d)$, and conjectured that it is $\Theta(\log d)$. They also gave an optimal algorithm for this problem, but left as an interesting open problem to determine its asymptotic performance ratio. Later, Epstein and van Stee (2007) gave lower and upper bounds for $d \in \{2, \dots, 7\}$.

Our main contribution is an $\Omega(d/\log d)$ lower bound for online bounded space d -CPP. In view of previous results by Epstein and van Stee (2005), we obtain that the asymptotic performance ratio of this problem is $\Theta(d/\log d)$, settling an open problem posed by those authors. To prove our lower bound, we follow a well known approach (see Lee and Lee (1985) and Yao (1980)), which requires the proof of the existence of a packing with a suitably large ‘weight’, for a certain definition of weight. The novelty here is that we prove the existence of such a packing with the probabilistic method.

To conclude this section, we mention that the technique that we present here may also be used to obtain lower bounds for the prices of anarchy of a game theoretic version of d -CPP, called *selfish d -hypercube bin packing game*. As this topic requires the introduction of a number of concepts, we just mention the main results for readers familiar with this line of research: *for every large enough d , the asymptotic price of anarchy (respectively, strong price of anarchy) of the selfish d -hypercube bin packing game is $\Omega(d/\log d)$ (respectively, $\Omega(\log d)$)*. The proof of one of the results can be found in Kohayakawa et al. (2017). A preliminary version of this work (Kohayakawa et al. (2018)) appeared in the proceedings of LATIN 2018.

2 Notation and homogeneous packings

The d -hypercubes $Q_k^+ = Q_k^d(\varepsilon)$ defined below will be important in what follows.

Definition 1. Let $d \geq 2$ be an integer. For every integer $k \geq 2$ and $\varepsilon > 0$, let

$$Q_k^+ = Q_k^d(\varepsilon) = \left(0, \frac{1+\varepsilon}{k}\right)^d = \left\{x \in \mathbb{R}^d : 0 < x < \frac{1+\varepsilon}{k}\right\}^d \subset \mathbb{R}^d \quad (3)$$

be the open d -hypercube of side length $(1+\varepsilon)/k$ ‘based’ at the origin.

2.1 Homogeneous packings

We shall be interested in certain types of packings of hypercubes into a unit bin.

Definition 2 (Homogeneous packings $\mathcal{H}_k^+ = \mathcal{H}_k^d(\varepsilon)$). Let $d \geq 2$ be fixed. For any integer $k \geq 2$ and $0 < \varepsilon \leq 1/(k-1)$, a packing of $(k-1)^d$ copies of $Q_k^+ = Q_k^d(\varepsilon)$ into a unit bin is said to be a *packing of type $\mathcal{H}_k^+ = \mathcal{H}_k^d(\varepsilon)$* . Packings of type \mathcal{H}_k^+ will be called *homogeneous packings*.

In the definition above, the upper bound on ε guarantees that $(k-1)^d$ copies of Q_k^+ can be packed into a unit bin (and hence \mathcal{H}_k^+ exists): it suffices to note that, under that assumption on ε , we have $(k-1)(1+\varepsilon)/k \leq 1$. Homogeneous packings are important because they can be used to create instances for which any bounded space algorithm performs badly (see Epstein and van Stee (2005, 2007)).

3 The central lemma and the main theorem

The key result used in the proof of our main theorem (Theorem 5) is the existence of a certain packing, stated in Lemma 4 below. Since this lemma is somewhat technical, we first informally describe a related result.

Consider the $S-1$ homogeneous packings \mathcal{H}_k^+ ($k = 2, \dots, S$), where $S = \lceil cd/\log d \rceil$ for a small positive constant c . Suppose also that $0 < \varepsilon \leq \varepsilon_0(d)$ for some small $\varepsilon_0(d)$. Suppose we assemble a list \mathcal{I} of d -hypercubes from these $S-1$ homogeneous packings \mathcal{H}_k^+ by selecting 90% of the members of each such \mathcal{H}_k^+ . The following holds: (*) *there is a packing of \mathcal{I} into a **single** unit bin as long as d is sufficiently large*. This fact is behind the proof of our central lemma, Lemma 4, stated in what follows. Fact (*) might look surprising at first sight, as the homogeneous packings \mathcal{H}_k^+ appear to have reasonably high occupancy.

We now give some definitions needed for the statement of Lemma 4.

Definition 3 (ε -packings). A packing \mathcal{U} of d -hypercubes into a unit bin is called an ε -packing if, for every member Q of \mathcal{U} , there is some integer $k \geq 2$ such that Q is a copy of $Q_k^+ = Q_k^d(\varepsilon)$.

Let \mathcal{U} be an ε -packing for some $\varepsilon > 0$. Let

$$K(\mathcal{U}) = \{k \geq 2: \mathcal{U} \text{ contains a copy of } Q_k^+\}. \quad (4)$$

For every $k \in K(\mathcal{U})$, let

$$\nu_k(\mathcal{U}) \text{ be the total number of copies of } Q_k^+ \text{ in } \mathcal{U}. \quad (5)$$

Clearly, we have $0 \leq \nu_k(\mathcal{U}) \leq (k-1)^d$ for every k (recall that $\varepsilon > 0$). Finally, we define the *weight* of \mathcal{U} as

$$w(\mathcal{U}) = \sum_{k \in K(\mathcal{U})} \frac{\nu_k(\mathcal{U})}{(k-1)^d}. \quad (6)$$

We shall be interested in ε -packings \mathcal{U} with large weight. Our main lemma is as follows.

Lemma 4 (Central lemma). *There is an absolute constant d_0 for which the following holds for any $d \geq d_0$. For any $0 < \varepsilon \leq d^{-2}$, the unit bin admits an ε -packing \mathcal{U} with*

$$w(\mathcal{U}) \geq \frac{d}{5 \log d}. \quad (7)$$

In (7) and in what follows, $\log x$ stands for the natural logarithm of x . The proof of Lemma 4 is postponed to Section 4. We now deduce our main result, Theorem 5, from Lemma 4, following the approach used by Lee and Lee (1985). For experts in the area, given Lemma 4, the proof of Theorem 5 is routine. The short proof below is included for the benefit of non-experts.

Theorem 5 (Main Theorem). *There is an absolute constant d_0 such that, for any $d \geq d_0$, the asymptotic performance ratio of the online bounded space d -hypercube bin packing problem is at least $d/10 \log d$.*

Proof: Let \mathcal{A} be any algorithm for the online bounded space d -hypercube bin packing problem. Let M be the maximum number of bins that \mathcal{A} leaves open during its execution. To prove that \mathcal{A} has asymptotic performance ratio at least $d/10 \log d$ if d is large enough, we construct a suitable instance \mathcal{I} for \mathcal{A} .

Let d_0 be as in Lemma 4 and suppose $d \geq d_0$. Fix any ε with $0 < \varepsilon \leq d^{-2}$ and let \mathcal{U} be a packing as in the statement of Lemma 4. The instance \mathcal{I} will be constructed as follows. First, we choose a suitable integer N and take $2MN$ copies of \mathcal{U} . We then construct \mathcal{I} by arranging the hypercubes in these copies in a linear order, with all the hypercubes of the same size appearing together. Let us now formally describe \mathcal{I} .

Let $N = \prod_{k \in K(\mathcal{U})} (k-1)^d$. Recall that \mathcal{U} contains $\nu_k(\mathcal{U})$ copies of Q_k^+ for every $k \in K(\mathcal{U})$. Let $K = |K(\mathcal{U})|$ and suppose $K(\mathcal{U}) = \{k_1, \dots, k_K\}$. The instance \mathcal{I} that we shall construct is the concatenation of K segments, say $\mathcal{I} = \mathcal{I}_1 \dots \mathcal{I}_K$, with each segment \mathcal{I}_ℓ ($1 \leq \ell \leq K$) composed of a sequence of $f(\ell) = 2MN\nu_{k_\ell}(\mathcal{U})$ copies of $Q_{k_\ell}^+$. This completes the definition of our instance \mathcal{I} .

The following assertion, to be used later, concerning the offline packing of the hypercubes in \mathcal{I} is clear, as we obtained \mathcal{I} by rearranging the hypercubes in $2MN$ copies of \mathcal{U} .

$$\text{The hypercubes in } \mathcal{I} \text{ can be packed into at most } 2MN \text{ unit bins.} \quad (8)$$

We now prove that, when \mathcal{A} is given the instance \mathcal{I} above, it will have performance ratio at least as bad as $w(\mathcal{U})/2$. In view of (7) in Lemma 4, this will complete the proof of Theorem 5.

Let us examine the behaviour of \mathcal{A} when it is given input \mathcal{I} . Fix $1 \leq \ell \leq K$ and suppose that \mathcal{A} has already seen the hypercubes in $\mathcal{I}_1 \dots \mathcal{I}_{\ell-1}$ and it has already packed them somehow. We now consider what happens when \mathcal{A} examines the $f(\ell)$ hypercubes in \mathcal{I}_ℓ , which are all copies of $Q_{k_\ell}^+$.

Clearly, since $\varepsilon > 0$, the $f(\ell)$ copies of $Q_{k_\ell}^+$ in \mathcal{I}_ℓ cannot be packed into fewer than

$$\frac{f(\ell)}{(k_\ell - 1)^d} = \frac{2MN\nu_{k_\ell}(\mathcal{U})}{(k_\ell - 1)^d} \geq \frac{MN\nu_{k_\ell}(\mathcal{U})}{(k_\ell - 1)^d} + M \quad (9)$$

unit bins. Therefore, even if some hypercubes in \mathcal{I}_ℓ are placed in bins still left open after the processing of $\mathcal{I}_1 \dots \mathcal{I}_{\ell-1}$, the hypercubes in \mathcal{I}_ℓ will add at least $MN\nu_{k_\ell}(\mathcal{U})/(k_\ell - 1)^d$ new bins to the current output of \mathcal{A} . Thus, the total number of bins that \mathcal{A} will use when processing \mathcal{I} is at least

$$\sum_{k \in K(\mathcal{U})} \frac{MN\nu_k(\mathcal{U})}{(k - 1)^d} = MN \sum_{k \in K(\mathcal{U})} \frac{\nu_k(\mathcal{U})}{(k - 1)^d} = MN w(\mathcal{U}). \quad (10)$$

In view of (8), it follows that the asymptotic performance ratio of \mathcal{A} is at least

$$\frac{MN w(\mathcal{U})}{2MN} = \frac{1}{2} w(\mathcal{U}), \quad (11)$$

as claimed. This completes the proof of Theorem 5. \square

4 Proof of Lemma 4

The ε -packing \mathcal{U} whose existence is asserted in our central lemma, Lemma 4, will be described in terms of certain ‘codes’, that is, sets of ‘codewords’ or simply ‘words’. We shall use such codes to ‘place’ copies of certain $Q_k^+ = Q_k^d(\varepsilon)$ in the packing \mathcal{U} . We make this precise in Section 4.1. The proof of the existence of appropriate codes will be given in Section 4.2. The proof of Lemma 4 is given in Section 4.3.

4.1 Placing hypercubes according to codewords

Let d and $k \geq 2$ be fixed. Let a d -letter word $w \in [k]^d$ from the *alphabet* $[k] = \{1, \dots, k\}$ be given. In what follows, we shall fix some $0 < \varepsilon \leq \varepsilon_0(d)$ and we shall consider translations $Q(w) = Q^{(k)}(w)$ of the hypercube Q_k^+ specified by such words w in a certain way (for the definition of $Q_k^+ = Q_k^d(\varepsilon)$, recall (3)). Furthermore, later, we shall consider certain *sets* $L_k \subset [k]^d$ of such words and we shall define packings of the form $\mathcal{P}_{L_k} = \{Q(w) : w \in L_k\}$. Note that \mathcal{P}_{L_k} is composed of copies of Q_k^+ . To obtain the packing \mathcal{U} whose existence is asserted in Lemma 4, we shall consider the union of such packings \mathcal{P}_{L_k} for $k = 2, \dots, S$, with $S = \lceil 2d/9 \log d \rceil$ and certain families $\mathcal{L} = \{L_k : 2 \leq k \leq S\}$ (see Lemma 13).

Let us now define $Q(w) = Q^{(k)}(w)$, the translation of Q_k^+ specified by $w = (w_1, \dots, w_d) \in [k]^d$. For $w = (w_i)_{1 \leq i \leq d}$ with $w_i < k$ for every i , we let $Q(w)$ be the translation

$$x[w] + Q_k^+ = \{x[w] + z : z \in Q_k^+\} \quad (12)$$

of Q_k^+ , where

$$x[w] = \frac{1 + \varepsilon}{k} (w_1 - 1, \dots, w_d - 1). \quad (13)$$

Thus, while Q_k^+ has its ‘base point’ at the origin, $Q(w)$ has its base point at $x[w]$ (see $Q(w)$ and $Q(w'')$ in Figure 1).

In what follows, we shall always have $0 < \varepsilon < 1/(k-1)$. Therefore, if $w_i < k$ for every i , then $Q(w)$ is contained in the unit hypercube $[0, 1]^d$, whereas if $w_i = k$ for some i , then $x[w] + Q_k^+ = \{x[w] + z : z \in Q_k^+\}$ with $x[w]$ as defined in (13) is *not* contained in $[0, 1]^d$ (see Q' in Figure 1). Since we want $Q(w)$ to be contained in $[0, 1]^d$ for every $w \in [k]^d$, we actually define $x[w]$ as in (15) below.

Definition 6 (Base point coordinates of $Q(w)$). For every $k \geq 2$ and $0 < \varepsilon < 1/(k-1)$, let

$$x^{(k)}(v) = x_\varepsilon^{(k)}(v) = \begin{cases} \frac{(1+\varepsilon)(v-1)}{k}, & \text{if } 1 \leq v < k, \\ 1 - \frac{1+\varepsilon}{k}, & \text{if } v = k. \end{cases} \quad (14)$$

For $w = (w_1, \dots, w_d) \in [k]^d$, let

$$x[w] = (x^{(k)}(w_1), \dots, x^{(k)}(w_d)). \quad (15)$$

Finally, for convenience, for $1 \leq v \leq k$, let

$$y^{(k)}(v) = x^{(k)}(v) + \frac{1+\varepsilon}{k}. \quad (16)$$

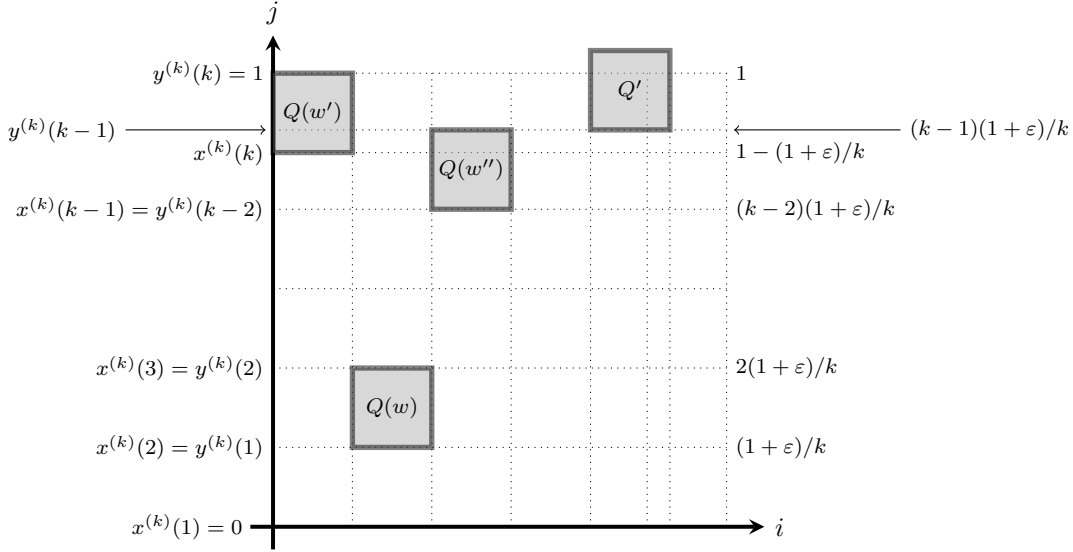


Fig. 1: Projections on the (i, j) -plane of hypercubes $Q(w)$, $Q(w')$ and $Q(w'')$ with $w_i = w_j = 2$, $w'_i = 1$ and $w'_j = k$, and $w''_i = 3$ and $w''_j = k-1$. The hypercube Q' is not contained in $[0, 1]^d$.

We now state three simple facts that the reader may find useful to check on their own to get used to the definitions above. First, note that $\mathcal{P} = \{Q(w) : w \in [k-1]^d\}$ is a packing of $(k-1)^d$ copies of Q_k^+ into

the unit bin $[0, 1]^d$; that is, \mathcal{P} is a packing of type \mathcal{H}_k^+ (recall Definition 2). Secondly, $\{Q(w) : w \in [k]^d\}$ is *not* a packing. Finally, $\{Q(w) : w \in [k]^d \text{ with } w_i \neq k - 1 \text{ for every } i\}$ is a packing (and is also a packing of type \mathcal{H}_k^+).

Note that, because $\varepsilon < 1/(k - 1)$, for every $k \geq 2$, we have

$$\begin{aligned} 0 = x^{(k)}(1) < y^{(k)}(1) = x^{(k)}(2) < y^{(k)}(2) = x^{(k)}(3) < \dots < y^{(k)}(k - 2) \\ &= x^{(k)}(k - 1) < x^{(k)}(k) < y^{(k)}(k - 1) < y^{(k)}(k) = 1 \end{aligned} \quad (17)$$

(see Figure 1). For every $k \geq 2$ and $1 \leq v \leq k$, let

$$I^{(k)}(v) = (x^{(k)}(v), y^{(k)}(v)) \subset [0, 1]. \quad (18)$$

Finally, note that

$$Q(w) = Q^{(k)}(w) = x[w] + Q_k^+ = I^{(k)}(w_1) \times \dots \times I^{(k)}(w_d) \subset [0, 1]^d. \quad (19)$$

We close this section observing the following.

Fact 7. *The following assertions hold for any positive S .*

(i) *Suppose $2 \leq k < k' \leq S$ and $0 < \varepsilon \leq S^{-2}$. Then*

$$y^{(k)}(k - 1) < x^{(k')}(k'). \quad (20)$$

In particular, the intervals $I^{(k)}(v)$ ($1 \leq v < k$) are disjoint from $I^{(k')}(k')$.

(ii) *For any $2 \leq k \leq S$, the intervals $I^{(k)}(v)$ ($1 \leq v \leq k$) are pairwise disjoint, except for the single pair formed by $I^{(k)}(k - 1)$ and $I^{(k)}(k)$.*

Proof: Assertion (ii) is clear (recall (17)). The second assertion in (i) follows from inequality (20), and therefore it suffices to verify that inequality. We have $y^{(k)}(k - 1) = x^{(k)}(k - 1) + (1 + \varepsilon)/k = (k - 1)(1 + \varepsilon)/k = 1 + \varepsilon - (1 + \varepsilon)/k$. Moreover, $x^{(k')}(k') = 1 - (1 + \varepsilon)/k'$. Therefore, (20) is equivalent to

$$\varepsilon < (1 + \varepsilon) \left(\frac{1}{k} - \frac{1}{k'} \right). \quad (21)$$

Since $k + 1 \leq k' \leq S$ and $\varepsilon \leq S^{-2}$, inequality (21) does hold. \square

4.2 Separated families of gapped codes

Let an integer $d \geq 2$ be fixed. We shall consider sets of words $L_k \subset [k]^d = \{1, \dots, k\}^d$ for $k \geq 2$. We refer to such L_k as *codes* or *k-codes*. As discussed in the beginning of Section 4.1, we shall design such L_k to specify packings $\mathcal{P}_{L_k} = \{Q(w) : w \in L_k\}$.

We start with the following definition.

Definition 8 (Gapped codes). Suppose $k \geq 2$ and let a k -code $L_k \subset [k]^d$ be given. We say that L_k *misses* j at coordinate i_0 if every word $w = (w_i)_{1 \leq i \leq d}$ in L_k is such that $w_{i_0} \neq j$. Furthermore, L_k is said to be *gapped* if, for each $1 \leq i \leq d$, either L_k misses $k - 1$ at i or L_k misses k at i .

Suppose L_k is a gapped code, and suppose $w = (w_i)_{1 \leq i \leq d}$ and $w' = (w'_i)_{1 \leq i \leq d}$ are distinct words in L_k . Then $Q(w)$ and $Q(w')$ do not overlap: this can be checked from (19) and Fact 7(ii). Thus, if L_k is gapped, then

$$\mathcal{P}_{L_k} = \{Q(w) : w \in L_k\} \quad (22)$$

is a packing.

We now introduce a certain notion of ‘compatibility’ between two codes L_k and $L_{k'}$, so that \mathcal{P}_{L_k} and $\mathcal{P}_{L_{k'}}$ can be put together to obtain a packing if they come from ‘compatible’ codes L_k and $L_{k'}$.

Definition 9 (Separated codes). Suppose $2 \leq k < k' \leq S$ and $L_k \subset [k]^d$ and $L_{k'} \subset [k']^d$ are given. We say that L_k and $L_{k'}$ are *separated* if, for any $w = (w_i)_{1 \leq i \leq d} \in L_k$ and any $w' = (w'_i)_{1 \leq i \leq d} \in L_{k'}$, there is some i such that $w_i < k < k' = w'_i$.

Suppose L_k and $L_{k'}$ are gapped and separated and suppose $k < k' \leq S$ and $\varepsilon \leq S^{-2}$ for some S (we shall later set S to be a certain value $S(d)$). Consider the packings \mathcal{P}_{L_k} and $\mathcal{P}_{L_{k'}}$ as defined in (22). Fact 7(i) and (19) imply that $\mathcal{P}_{L_k} \cup \mathcal{P}_{L_{k'}}$ is a packing. Indeed, let $w = (w_i)_{1 \leq i \leq d} \in L_k$ and any $w' = (w'_i)_{1 \leq i \leq d} \in L_{k'}$ be given. Then, by definition, there is some i such that $w_i < k < k' = w'_i$. This implies that $Q(w) = Q^{(k)}(w)$ and $Q(w') = Q^{(k')}(w')$ are disjoint ‘in the i th dimension’ (see Fact 7(i)).

Definition 10 (Separated families). Let $\mathcal{L} = (L_k)_{2 \leq k \leq S}$ be a family of k -codes $L_k \subset [k]^d$. If, for every $2 \leq k < k' \leq S$, the codes L_k and $L_{k'}$ are separated, then we say that \mathcal{L} is a *separated family* of codes.

Remark 11. For $2 \leq k \leq d$, let $L_k = \{w = (w_i)_{1 \leq i \leq d} \in [k]^d : w_k = k \text{ and } w_i < k \text{ for all } i \neq k\}$. Then $\mathcal{L} = (L_k)_{2 \leq k \leq d}$ is a separated family of gapped codes. Fix $0 < \varepsilon \leq d^{-2}$. Consider $\mathcal{P} = \bigcup_{2 \leq k \leq d} \mathcal{P}_{L_k}$ with \mathcal{P}_{L_k} as in (22). Since each L_k is gapped, the \mathcal{P}_k are packings. Also, since $\mathcal{L} = (L_k)_{2 \leq k \leq d}$ is a separated family, \mathcal{P} is a packing. Furthermore, we have $\nu_k(\mathcal{P}) = |L_k| = (k-1)^{d-1}$ (recall (5)) and $w(\mathcal{P}) = \sum_{2 \leq k \leq d} 1/(k-1) \sim \log d$ (recall (6)). The existence of \mathcal{P} implies a weak form of Theorem 5 (namely, a lower bound of $\Omega(\log d)$ instead of $\Omega(d/\log d)$).

Remark 11 above illustrates the use we wish to make of separated families of gapped codes. Our focus will now shift onto producing much ‘better’ families than the one explicitly defined in Remark 11. Indeed, we now prove Lemma 13 below, which asserts the existence of such better families. We shall need the following auxiliary lemma.

Lemma 12. *There is an absolute constant d_0 such that, for any $d \geq d_0$, there are sets $F_1, \dots, F_d \subset [d]$ such that (i) for every $1 \leq k \leq d$, we have $|F_k| = \lceil d/2 \rceil$ and (ii) for every $1 \leq k < k' \leq d$, we have $|F_k \cap F_{k'}| < 7d/26$.*

Proof: Let $r = \lceil d/2 \rceil$. We select each F_k ($1 \leq k \leq d$) among the r -element subsets of $[d]$ uniformly at random, with each choice independent of all others. Let $s = 7d/26$. Note that, for any $k \neq k'$, we have $\mathbb{E}(|F_k \cap F_{k'}|) = r^2/d$. Let $\lambda = r^2/d$. Let

$$t = s - \lambda \geq s - (d/2 + 1)^2/d \geq \frac{7d}{26} - \frac{1}{d} \left(\frac{d^2}{4} + d + 1 \right) \geq \frac{d}{52} - 2 \geq \frac{d}{53}, \quad (23)$$

as long as d is large enough. We may now apply a Chernoff bound for the hypergeometric distribution (see, e.g., Janson et al. (2000), Theorem 2.10, inequality (2.12)) to obtain that

$$\mathbb{P}(|F_k \cap F_{k'}| \geq s) = \mathbb{P}(|F_k \cap F_{k'}| \geq \lambda + t) \leq \exp\left(-\frac{2(d/53)^2}{\lceil d/2 \rceil}\right) \leq e^{-3d/53^2} \quad (24)$$

for every large enough d . Therefore, the expected number of pairs $\{k, k'\}$ with $1 \leq k < k' \leq d$ for which $|F_k \cap F_{k'}| \geq s$ is less than $d^2 \exp(-3d/53^2)$, which tends to 0 as $d \rightarrow \infty$. Therefore, for any large enough d , a family of sets F_1, \dots, F_d as required does exist. \square

We are now ready to state and prove the lemma that asserts the existence of a separated family of gapped codes that is ‘better’ than the one defined in Remark 11.

Lemma 13 (Many large, separated gapped codes). *There is an absolute constant $d_0 \geq 2$ such that, for any $d \geq d_0$, there is a separated family $\mathcal{L} = (L_k)_{2 \leq k \leq S}$ of gapped k -codes $L_k \subset [k]^d$ such that*

$$|L_k| \geq \frac{10}{11}(k-1)^d \quad (25)$$

for every $2 \leq k \leq S$, where

$$S = \left\lceil \frac{2d}{9 \log d} \right\rceil. \quad (26)$$

Proof: Let S be as in (26) and let F_1, \dots, F_d be as in Fact 12. In what follows, we only use the F_k for $2 \leq k \leq S$. For each $2 \leq k \leq S$, we construct $L_k \subset [k]^d$ in two parts. Suppose first that we have L'_k with

$$L'_k \subset ([k] \setminus \{k-1\})^{F_k} = \{w = (w_i)_{i \in F_k} : w_i \in [k] \setminus \{k-1\} \text{ for all } i \in F_k\}. \quad (27)$$

We then set

$$\begin{aligned} L_k &= L'_k \times [k-1]^{[d] \setminus F_k} \\ &= \{w = (w_i)_{1 \leq i \leq d} : \exists w' = (w'_i)_{i \in F_k} \in L'_k \text{ such that } w_i = w'_i \text{ for all } i \in F_k \\ &\quad \text{and } w_i \in [k-1] \text{ for all } i \in [d] \setminus F_k\}. \end{aligned} \quad (28)$$

Note that, by (27) and (28), the k -code L_k will be gapped ($k-1$ is missed at every $i \in F_k$ and k is missed at every $i \in [d] \setminus F_k$). We shall prove that there is a suitable choice for the L'_k with $|L'_k| \geq (10/11)(k-1)^{|F_k|}$, ensuring that $\mathcal{L} = (L_k)_{2 \leq k \leq S}$ is separated. Since we shall then have

$$|L_k| = |L'_k|(k-1)^{d-|F_k|} \geq \frac{10}{11}(k-1)^d, \quad (29)$$

condition (25) will be satisfied and Lemma 13 will be proved. We now proceed with the construction of the codes L'_k ($2 \leq k \leq S$).

Fix $2 \leq k \leq S$. For $2 \leq \ell < k$, let $J(\ell, k) = F_k \setminus F_\ell$, and note that

$$|J(\ell, k)| > \left\lceil \frac{d}{2} \right\rceil - \frac{7}{26}d \geq \frac{3}{13}d. \quad (30)$$

Let $v = (v_i)_{i \in F_k}$ be an element of $([k] \setminus \{k-1\})^{F_k}$ chosen uniformly at random. For every $2 \leq \ell < k$, let us say that v is ℓ -bad if $v_i \neq k$ for every $i \in J(\ell, k)$. We have

$$\mathbb{P}(v \text{ is } \ell\text{-bad}) = \left(1 - \frac{1}{k-1}\right)^{|J(\ell, k)|} \leq e^{-|J(\ell, k)|/S} \leq \exp\left(-\frac{3d}{13 \lceil 2d/9 \log d \rceil}\right) \leq d^{-1}, \quad (31)$$

for every large enough d . Let us say that v is *bad* if it is ℓ -bad for some $2 \leq \ell < k$. It follows from (31) that

$$\mathbb{P}(v \text{ is bad}) \leq Sd^{-1} \leq \frac{1}{4 \log d} \leq \frac{1}{11} \quad (32)$$

if d is large enough. Therefore, at least $(10/11)(k-1)^{|F_k|}$ words $v \in ([k] \setminus \{k-1\})^{F_k}$ are not bad, as long as d is large enough. We let $L'_k \subset ([k] \setminus \{k-1\})^{F_k}$ be the set of such good words.

To complete the proof, it remains to show that the family $\mathcal{L} = (L_k)_{2 \leq k \leq S}$ is separated. More precisely, we show that with the above choice of L'_k ($2 \leq k \leq S$), the family $\mathcal{L} = (L_k)_{2 \leq k \leq S}$ with L_k as defined in (28) is separated.

To this end, fix $2 \leq \ell < k \leq S$. We show that L_ℓ and L_k are separated. Let $u = (u_i)_{1 \leq i \leq d} \in L_\ell$ and $w = (w_i)_{1 \leq i \leq d} \in L_k$ be given. By the definition of L_k , there is $v = (v_i)_{i \in F_k} \in L'_k$ such that $w_i = v_i$ for all $i \in F_k$. Furthermore, since $v \in L'_k$ is not a bad word, it is not ℓ -bad. Therefore, there is $i_0 \in J(\ell, k) = F_k \setminus F_\ell$ for which we have $v_{i_0} = k$. Observing that $i_0 \notin F_\ell$ and recalling the definition of L_ℓ , we see that $u_{i_0} < \ell < k = v_{i_0} = w_{i_0}$, as required.

The proof of Lemma 13 is now complete. \square

4.3 The packing \mathcal{U} in Lemma 4

Fix $\mathcal{L} = (L_k)_{2 \leq k \leq S}$, a separated family of gapped k -codes $L_k \subset [k]^d$. We now give, for every sufficiently small $\varepsilon > 0$, the construction of a packing $\mathcal{U}_\varepsilon(\mathcal{L})$ of d -hypercubes into the unit bin $[0, 1]^d$ using \mathcal{L} and prove that $\mathcal{U}_\varepsilon(\mathcal{L})$ is indeed a packing. Choosing \mathcal{L} as in Lemma 13 above, we shall deduce Lemma 4 by taking $\mathcal{U} = \mathcal{U}_\varepsilon(\mathcal{L})$.

Definition 14 (Packing $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(\mathcal{L})$). Suppose $\mathcal{L} = (L_k)_{2 \leq k \leq S}$ is a separated family of gapped k -codes $L_k \subset [k]^d$. Let $0 < \varepsilon \leq S^{-2}$. We put

$$\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(\mathcal{L}) = \bigcup_{2 \leq k \leq S} \mathcal{P}_{L_k}, \quad (33)$$

where \mathcal{P}_{L_k} is as in (22).

In Lemma 15 below, we compile the properties that we need of \mathcal{U}_ε . For the relevant notation, recall (4), (5) and Definition 3.

Lemma 15. Suppose $\mathcal{L} = (L_k)_{2 \leq k \leq S}$ is a separated family of non-empty gapped k -codes $L_k \subset [k]^d$. Suppose $0 < \varepsilon \leq S^{-2}$. Let $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(\mathcal{L})$ be the family of all the hypercubes $Q(w) = Q^{(k)}(w) \subset [0, 1]^d$ with $w \in L_k$ and $2 \leq k \leq S$. Then the following assertions hold: (i) the hypercubes in \mathcal{U}_ε are pairwise disjoint and form an ε -packing; (ii) for every $2 \leq k \leq S$, we have $\nu_k(\mathcal{U}_\varepsilon) = |L_k|$; (iii) $|K(\mathcal{U}_\varepsilon)| = S - 1$.

Proof: Let us first check that the hypercubes $Q(w)$ in \mathcal{U}_ε are pairwise disjoint. We remark that, when introducing the notions of gapped and separated codes, we already discussed the reason why the $Q(w)$ in \mathcal{U}_ε are indeed pairwise disjoint. However, we give a formal proof here for completeness. Let $w = (w_i)_{1 \leq i \leq d} \in L_k$ and $w' = (w'_i)_{1 \leq i \leq d} \in L_{k'}$ with $2 \leq k \leq k' \leq S$ be given. Consider $Q(w) = Q^{(k)}(w)$ and $Q(w') = Q^{(k')}(w')$. We have to show that

$$Q(w) \cap Q(w') = \emptyset. \quad (34)$$

Suppose first that $k = k'$. In that case, both w and w' are in $L_k = L_{k'}$ and we may suppose that $w \neq w'$. Thus, there is some $1 \leq i \leq d$ such that $w_i \neq w'_i$. Furthermore, since L_k is gapped, either $k - 1$ or k is missed by L_k at i . In particular, the pair $\{w_i, w'_i\}$ cannot be the pair $\{k - 1, k\}$ and therefore

$$I^{(k)}(w_i) \cap I^{(k)}(w'_i) = \emptyset \quad (35)$$

(recall Fact 7(ii)). Expression (19) applied to $Q(w)$ and $Q(w')$, together with (35), confirms (34) when $k = k'$.

Suppose now that $k < k'$. Since L_k and $L_{k'}$ are separated, there is some $1 \leq i_0 \leq d$ such that $w_{i_0} < k < k' = w'_{i_0}$. Fact 7(i) tells us that

$$I^{(k)}(w_{i_0}) \cap I^{(k')}(w'_{i_0}) = \emptyset. \quad (36)$$

Expression (19) applied to $Q(w)$ and $Q(w')$, together with (36), confirms (34) in this case also. We therefore conclude that \mathcal{U}_ε is indeed a packing.

The hypercubes in \mathcal{U}_ε are copies of the hypercubes Q_k^+ for $2 \leq k \leq S$, and therefore \mathcal{U}_ε is an ε -packing. This concludes the proof of Lemma 15(i). Assertions (ii) and (iii) are clear. \square

We are now ready to prove Lemma 4.

Proof of Lemma 4: Let d_0 be as in Lemma 13. We may and shall suppose that $d_0 \geq e^2$ and that d_0 is large enough so that, for every $d \geq d_0$, the last inequality in (37) below holds. We prove that Lemma 4 holds with this choice of d_0 . Let $d \geq d_0$ and $0 < \varepsilon \leq d^{-2}$ be given. Let $S = \lceil 2d/9 \log d \rceil$. Note that $\varepsilon \leq d^{-2} \leq S^{-2}$. Let $\mathcal{L} = (L_k)_{2 \leq k \leq S}$ be a separated family of gapped k -codes as given by Lemma 13. Lemma 15 tells us that $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(\mathcal{L})$ is an ε -packing with

$$\begin{aligned} w(\mathcal{U}_\varepsilon) &= \sum_{k \in K(\mathcal{U}_\varepsilon)} \frac{\nu_k(\mathcal{U}_\varepsilon)}{(k-1)^d} = \sum_{k \in K(\mathcal{U}_\varepsilon)} \frac{|L_k|}{(k-1)^d} \\ &\geq \frac{10}{11}(S-1) = \frac{10}{11} \left(\left\lceil \frac{2d}{9 \log d} \right\rceil - 1 \right) \geq \frac{d}{5 \log d}. \end{aligned} \quad (37)$$

Thus, to prove Lemma 4, it suffices to take $\mathcal{U} = \mathcal{U}_\varepsilon$. \square

5 Concluding remarks

We have not optimized the numerical constants in our calculations above. In particular, the constant 10 in Theorem 5 can be made arbitrarily close to 4, although d_0 would grow as we do so. We note that, since the problem posed by Epstein and van Stee (2005) is of an asymptotic nature ($d \rightarrow \infty$), the specific value of d_0 is not particularly relevant.

Our approach for finding a certain good packing in this paper is based on establishing the existence of certain specific families of compatible codes by the probabilistic method. We hope similar ideas will be useful in other related contexts.

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