

The treewidth of 2-section of hypergraphs

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Let $H = (V, F)$ be a simple hypergraph without loops. H is called linear if $|f \cap g| \leq 1$ for any $f, g \in F$ with $f \neq g$. The 2-section of H , denoted by $[H]_2$, is a graph with $V([H]_2) = V$ and for any $u, v \in V([H]_2)$, $uv \in E([H]_2)$ if and only if there is $f \in F$ such that $u, v \in f$. The treewidth of a graph is an important invariant in structural and algorithmic graph theory. In this paper, we consider the treewidth of the 2-section of a linear hypergraph. We will use the minimum degree, maximum degree, anti-rank and average rank of a linear hypergraph to determine the upper and lower bounds of the treewidth of its 2-section. Since for any graph G , there is a linear hypergraph H such that $[H]_2 \cong G$, we provide a method to estimate the bound of treewidth of graph by the parameters of the hypergraph.

Keywords: Linear hypergraph, treewidth, 2-section, supertree width

1 Introduction

The treewidth of a graph is an important invariant in structural and algorithmic graph theory. The concept of treewidth was originally introduced by Bertelé and Brioschi [2] under the name of dimension. It was later rediscovered by Halin [7] in 1976 and by Robertson and Seymour [13] in 1984. Now it has been studied by many other authors (see for example [5]-[12]). Treewidth is commonly used as a parameter in the parameterized complexity analysis of graph algorithms, since many NP-complete problems can be solved in polynomial time on graphs of bounded treewidth [3]. The relation between the treewidth and other graph parameters has been explored in a number of papers (see [10] for a recent survey). In [11], Harvey and Wood studied the treewidth of line graphs. They proved sharp lower bounds of the treewidth of the line graph of a graph G in terms of both the minimum degree and the average degree of G . Motivated by their work, in this paper, we study the treewidth of 2-section of linear hypergraphs.

A *hypergraph* is a pair $H = (V, F)$, where V is a finite set of vertices and F is a family of subsets of V such that for any $f \in F$, $f \neq \emptyset$ and $V = \cup_{f \in F} f$. The size of H is the cardinality of F . A *simple hypergraph* is a hypergraph H such that if $f \subseteq g$, then $f = g$, where $f, g \in F$. If $|f| = 1$, we call f a loop. In this paper, we just consider simple and no loop hypergraphs. The *rank* and *anti-rank* of H is defined as $r(H) = \max_{f \in F} |f|$ and $s(H) = \min_{f \in F} |f|$, respectively. If $r(H) = s(H) = 2$, then H is a graph.

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For any $v \in V$, denote $F(v) = \{f \in F | v \in f\}$. Then the degree of v , denoted by $\deg(v)$, is $|F(v)|$. The maximum and minimum degree of H will be denoted by $\Delta = \max_{v \in V} \deg(v)$ and $\delta = \min_{v \in V} \deg(v)$, respectively. If $\delta = \Delta = k$, then we call the hypergraph k -regular. The average rank of H is defined as $l(H) = \sum_{f \in F} |f|/|F|$. A hypergraph H is called *linear* if $|f \cap g| \leq 1$ for any $f, g \in F$ with $f \neq g$.

Let $H = (V, F)$ be a hypergraph (or graph), where $V = \{v_1, \dots, v_n\}$ and $F = \{f_1, f_2, \dots, f_m\}$. The *dual* of H , denoted by $H^* = (V^*, F^*)$, is a hypergraph whose vertices u_1, u_2, \dots, u_m correspond to the edges of H and with edges $g_i = \{u_j | v_i \in f_j\}$, $1 \leq i \leq n$. The *line graph* of a (hyper)graph H , denoted by $L(H)$ is a graph whose vertices w_1, w_2, \dots, w_m correspond to the edges of H and with edges $w_i w_j$ if $f_i \cap f_j \neq \emptyset$. Terminology and notation concerning hypergraph not defined here can be found in [1]. Now we give the definitions of treewidth. Let T be a tree. We will use T to denote the vertex set of T for short.

Definition 1.1 A tree decomposition of a graph $G = (V, E)$ is a pair $(T, (B_t)_{t \in T})$, where T is a tree and $(B_t)_{t \in T}$ (B_t is called a bag) is a family of subsets of V such that:

- (T1) for every $v \in V$, the set $B^{-1}(v) = \{t \in T | v \in B_t\}$ is nonempty and connected in T ;
- (T2) for every edge $uw \in E(G)$, there is $t \in T$ such that $u, w \in B_t$.

The width of the decomposition $(T, (B_t)_{t \in T})$ is the number

$$\max\{|B_t| | t \in T\} - 1.$$

The *treewidth* $tw(G)$ of G is the minimum of the widths of the tree decompositions of G .

In order to cite the definition of a generalized hypertree decomposition of a hypergraph which was given in [4], we introduce the definition of the 2-section of a hypergraph.

Definition 1.2 The 2-section of a hypergraph $H = (V, F)$, denoted by $[H]_2$, is a graph with $V([H]_2) = V$ and for any $u, v \in V([H]_2)$, $uv \in E([H]_2)$ if and only if there is $f \in F$ such that $u, v \in f$.

Definition 1.3 [4] A generalized hypertree decomposition of a hypergraph $H = (V, F)$ is a 3-tuple $(T, (B_t)_{t \in T}, (\lambda_t)_{t \in T})$, where T is a tree, $(B_t)_{t \in T}$ is a family of subsets of V and $(\lambda_t)_{t \in T}$ is a family of subsets of F such that:

- (TI) $(T, (B_t)_{t \in T})$ is a tree decomposition of $[H]_2$;
- (TII) for every $t \in T$, $B_t \subseteq \cup_{f \in \lambda_t} f$.

Now we give a new definition of decomposition of a hypergraph which is called a supertree decomposition of a hypergraph.

Definition 1.4 A supertree decomposition of a hypergraph $H = (V, F)$ is a 3-tuple $(T, (B_t)_{t \in T}, (\lambda_t)_{t \in T})$, where $(T, (B_t)_{t \in T}, (\lambda_t)_{t \in T})$ is a generalized hypertree decomposition of H such that:

- (TIII) for every $f \in F$, the set $\lambda^{-1}(f) = \{t \in T | f \in \lambda_t\}$ is nonempty and connected in T ;
- (TIV) for every $f_1, f_2 \in F$ with $f_1 \cap f_2 \neq \emptyset$, there is a $t \in T$ such that $f_1, f_2 \in \lambda_t$.

The width of the decomposition $(T, (B_t)_{t \in T}, (\lambda_t)_{t \in T})$ of H is the number

$$\max\{|\lambda_t| | t \in T\}.$$

The *supertree width* $stw(H)$ of H is the minimum of the widths of the supertree decompositions of H .

By Definition 1.4, if $(T, (B_t)_{t \in T}, (\lambda_t)_{t \in T})$ is a supertree decomposition of H , then $(T, (\lambda_t)_{t \in T})$ is a tree decomposition of $L(H)$. When $(T, (\lambda_t)_{t \in T})$ is a tree decomposition of $L(H)$, we let $B_t = \cup_{f \in \lambda_t} f$. Then $(T, (B_t)_{t \in T}, (\lambda_t)_{t \in T})$ is a supertree decomposition of H . So we have $stw(H) = tw(L(H)) + 1$.

From the Definition 1.2, we can get the following lemma.

Lemma 1.1 *Let H be a 2-regular linear hypergraph. Then $[H]_2 \cong L(H^*)$.*

Proof: Let $H = (V, F)$ be a 2-regular linear hypergraph. Then H^* is a simple graph. By the definition of dual hypergraph, there is a bijection σ between the edge set $F(H)$ (resp. the vertex set $V(H)$) and the vertex set $V(H^*)$ (resp. the edge set of $E(H^*)$) such that for any $f, g \in F(H)$ (resp. for any $u, v \in V(H)$), $\sigma(f)\sigma(g) \in E(H^*)$ if and only if $f \cap g \neq \emptyset$ (resp. $\sigma(u) \cap \sigma(v) \neq \emptyset$ if and only if there is $f \in F(H)$ with $u, v \in f$). By the definition of line graph, there is a bijection $\phi : E(H^*) \rightarrow V(L(H^*))$ such that for any $e_1, e_2 \in E(H^*)$, $\phi(e_1)\phi(e_2) \in E(L(H^*))$ if and only if $e_1 \cap e_2 \neq \emptyset$.

We will show that $[H]_2 \cong L(H^*)$. Let $\phi\sigma : V([H]_2) \rightarrow V(L(H^*))$. Then $\phi\sigma$ is a bijection. For any $u, v \in V([H]_2)$, $uv \in E([H]_2)$ if and only if there is $f \in F(H)$ such that $u, v \in f$ if and only if $\sigma(u) \cap \sigma(v) \neq \emptyset$ if only if $\phi(\sigma(u))\phi(\sigma(v)) \in E(L(H^*))$. Thus $[H]_2 \cong L(H^*)$. \square

By the definitions of line graph and the dual, we can easily get the following lemma.

Lemma 1.2 *Let H be a 2-regular linear hypergraph. Then $H^* \cong L(H)$.*

Let H be a hypergraph. There are two elementary lower bounds on treewidth of $[H]_2$. First,

$$tw([H]_2) \geq r(H) - 1 \quad (1)$$

since the vertices in a hyperedge form a clique in $[H]_2$. Second, given a minimum width tree decomposition of $[H]_2$, replace each vertex with all the hyperedges containing the vertex to obtain a supertree decomposition of H . It follows that

$$tw([H]_2) \geq \frac{1}{\Delta} stw(H) - 1. \quad (2)$$

We prove the following lower bound on $tw([H]_2)$ in terms of the minimum degree δ , maximum degree Δ and average rank $l(H)$ of a linear hypergraph H .

Theorem 1.1 *Let H be a linear hypergraph with minimum degree δ , maximum degree Δ and average rank $l(H)$. Let $\Delta \geq \delta \geq 2$. Suppose $\Delta \leq 2\delta^2 - 2\delta$. Then*

$$tw([H]_2) > \begin{cases} \frac{(2\delta^2 - 2\delta - \Delta)l(H)^2 + (2\Delta + 4\delta^2 - 4\delta)l(H)}{4\Delta\delta(\delta - 1)} - 1 & \text{if } \delta^2 - \delta \leq \Delta - 2\Delta/l(H), \\ \frac{(2\delta^2 - 2\delta - \Delta)l(H)^2 + 6\Delta l(H) - 8\Delta}{4\Delta\delta(\delta - 1)} - 1 & \text{otherwise.} \end{cases}$$

In [11], Harvey and Wood showed that for every graph G , $tw(L(G)) > \frac{1}{8}d(G)^2 + \frac{3}{4}d(G) - 2$, where $d(G)$ is the average degree of G . Let H be a 2-regular linear hypergraph of order n and size m . By Lemma 1.1, $[H]_2 \cong L(H^*)$. Note that $d(H^*) = \frac{2n}{m} = l(H)$. By Theorem 1.1, $tw(L(H^*)) = tw([H]_2) > \frac{1}{8}d(H^*)^2 + \frac{3}{4}d(H^*) - 2$, just as the result in [11].

We also prove two lower bounds on $tw([H]_2)$ in terms of the anti-rank $s(H)$ based on different condition of minimum degree of the given hypergraph H .

Theorem 1.2 *For every linear hypergraph H with anti-rank $s(H)$ and minimum degree $\delta \geq 3$, we have*

$$tw([H]_2) \geq \begin{cases} \frac{3}{8}s(H)^2 + \frac{3}{4}s(H) - 1 & \text{when } s(H) \text{ is even,} \\ \frac{3}{8}s(H)^2 + \frac{1}{2}s(H) - \frac{7}{8} & \text{when } s(H) \text{ is odd.} \end{cases}$$

Theorem 1.3 *For every linear hypergraph H with anti-rank $s(H)$ and minimum degree $\delta = 2$, we have*

$$tw([H]_2) \geq \begin{cases} \frac{1}{4}s(H)^2 + s(H) - 1 & \text{when } s(H) \text{ is even,} \\ \frac{1}{4}s(H)^2 + s(H) - \frac{5}{4} & \text{when } s(H) \text{ is odd.} \end{cases}$$

In [11], Harvey and Wood showed that for every graph G with minimum degree $\delta(G)$,

$$tw(L(G)) \geq \begin{cases} \frac{1}{4}\delta(G)^2 + \delta(G) - 1 & \text{when } \delta(G) \text{ is even,} \\ \frac{1}{4}\delta(G)^2 + \delta(G) - \frac{5}{4} & \text{when } \delta(G) \text{ is odd.} \end{cases}$$

Let H be a 2-regular linear hypergraph. By Lemma 1.1, $[H]_2 \cong L(H^*)$. Then $tw(L(H^*)) = tw([H]_2)$. Note that $\delta(H^*) = s(H)$. Thus we can obtain the same result as that in [11] by Theorem 1.3.

Now we consider upper bounds on $tw([H]_2)$. It is easy to show that

$$tw([H]_2) \leq r(H)stw(H) - 1. \quad (3)$$

To see this, we consider a minimum width supertree decomposition of H , and replace each bag λ_t by the vertices that are incident to an hyperedge of λ_t . This creates a tree decomposition of $[H]_2$, where each bag contains at most $r(H)stw(H)$ vertices. In Section 5, we improve this bound as follows.

Theorem 1.4 *For every linear hypergraph H , we have*

$$tw([H]_2) \leq \frac{2}{3}stw(H)r(H) + \frac{1}{3}(stw(H) - 1)^2 + \frac{1}{3}r(H) - 1.$$

Theorem 1.4 is of primary interest when $r(H) \gg stw(H)$, in which case the upper bound is $(\frac{2}{3}stw(H) + \frac{1}{3})r(H)$. When $r(H) < stw(H) - 1$, the bound in (3) is better than that in Theorem 1.4.

In [11], Harvey and Wood showed that for every graph G , $tw(L(G)) \leq \frac{2}{3}(tw(G) + 1)\Delta(G) + \frac{1}{3}tw(G)^2 + \frac{1}{3}\Delta(G) - 1$. Let H be a 2-regular linear hypergraph. Recall that $stw(H) = tw(L(H)) + 1$. By Lemmas 1.1 and 1.2, $[H]_2 \cong L(H^*)$ and $H^* \cong L(H)$. Then $tw(H^*) = stw(H) - 1$. Note that $\Delta(H^*) = r(H)$. By Theorem 1.4, $tw(L(H^*)) = tw([H]_2) \leq \frac{2}{3}(tw(H^*) + 1)\Delta(H^*) + \frac{1}{3}tw(H^*)^2 + \frac{1}{3}\Delta(H^*) - 1$, just the same as that in [11].

The rest of this paper is organized as follows. In Section 2, some properties of tree decompositions of 2-section of hypergraphs are given. The proof of Theorem 1.1 is given in Section 3. In Section 4, we will prove Theorems 1.2 and 1.3. In Section 5, we prove Theorem 1.4. Section 6 concludes this paper.

2 Tree decomposition of 2-section of hypergraphs

In this section, we first give some properties of tree decompositions of 2-section of hypergraphs which will be used in next sections.

Let $H = (V, F)$ be a hypergraph. For any $v \in V$, recall that $F(v) = \{f \in F \mid v \in f\}$. Let $(T, (B_t)_{t \in T})$ be a tree decomposition of $[H]_2$. For $u, v \in T$, we use $Path(u, v)$ to denote the path in T connecting u and v .

Lemma 2.1 *For every hypergraph $H = (V, F)$, there exists a minimum width tree decomposition $(T, (B_t)_{t \in T})$ of $[H]_2$ together with an assignment $b : F \rightarrow T$ such that for each vertex $v \in V$, $B^{-1}(v) = V(ST_v)$, where ST_v is the subtree of T induced by $\cup_{f_i, f_j \in F(v)} Path(b(f_i), b(f_j))$.*

Proof: Let $(T, (B_t)_{t \in T})$ be a minimum width tree decomposition of $[H]_2$ such that $\sum_{v \in V} |B^{-1}(v)|$ is minimized. For each $f \in F$, the vertices in f form a clique in $[H]_2$. Thus there exists a bag B_t containing all the vertices in f , where $t \in T$. Hence for each $f \in F$ choose one such node and declare it $b(f)$.

Let $v \in V$. For any $f \in F(v)$, we have $v \in B_{b(f)}$. It follows that $V(ST_v) \subseteq B^{-1}(v)$. If $|V(ST_v)| < |B^{-1}(v)|$, then we remove v from all bags B_t for $t \in B^{-1}(v) \setminus V(ST_v)$. Since each vertex incident to v appears in $\cup_{f \in F(v)} B_{b(f)}$, the removal of the aforementioned vertices yields another tree decomposition of $[H]_2$. However, the existence of such a tree decomposition would contradict our choice of $(T, (B_t)_{t \in T})$. Hence $V(ST_v) = B^{-1}(v)$, as required. \square

We call $b(f)$ the *base node* of f . From the proof of Lemma 2.1, we can construct a tree decomposition of $[H]_2$ so that we can assign a base node for each $f \in F$ and all the vertices in f are placed in the corresponding bag $B_{b(f)}$. In fact, we can obtain a slightly stronger result that will be used to prove our theorems.

Given $(T, (B_t)_{t \in T})$ and b guaranteed by Lemma 2.1, we can also ensure that each base node is a leaf and that b is a bijection between edges of H and leaves of T . If $b(f)$ is not a leaf, then we add a leaf adjacent to $b(f)$ and let $b(f)$ be this leaf instead. If some leaf t is the base node for several edges of H , then we add a leaf adjacent to t for each edge assigned to t . Finally, if t is a leaf that is not a base node, then delete t ; this maintains the desired properties since a leaf is never an internal node of a subtree.

We can improve this further. Given a tree T , we can root it at a node and orient all edges away from the root. In such a tree, a leaf is a node with outdegree 0. Say a rooted tree is binary if every non-leaf node has outdegree 2. Given a tree decomposition, by the same way described in [11], we can root it and then modify the underlying tree so that each non-leaf node has outdegree 2. The above results give the following lemma.

Lemma 2.2 *For every hypergraph $H = (V, F)$ there exists a minimum width tree decomposition $(T, (B_t)_{t \in T})$ of $[H]_2$ together with an assignment $b : F \rightarrow T$ such that T is a binary tree, b is a bijection onto the leaves of T and for each $v \in V$, $B^{-1}(v) = V(ST_v)$.*

By Lemma 2.2, we have the following lower bound on $tw([H]_2)$ that is slightly stronger than (2).

Lemma 2.3 *Let $H = (V, F)$ be a hypergraph with $\Delta \geq 2$. Then we have $tw([H]_2) \geq \frac{1}{\Delta-1}(stw(H) - 1) - 1$.*

Proof: Let $k = tw([H]_2) + 1$ and $(T, (B_t)_{t \in T})$ be a tree decomposition of $[H]_2$ of width $k - 1$, together with an assignment b as ensured by Lemma 2.2. For each $v \in V$ with $\deg(v) \geq 2$, we can assume that $|B^{-1}(v)| \geq 2$; otherwise, we can simply add a leaf t' adjacent to t and let $B_{t'} = B_t$, where $t \in B^{-1}(v)$. We now partially construct a supertree decomposition $(T', (B'_t)_{t \in T'}, (\lambda_t)_{t \in T'})$ of H as follows: first let $(T', (B'_t)_{t \in T'}) = (T, (B_t)_{t \in T})$. Let $v \in V$. If $\deg(v) = 1$, then we just put the hyperedge adjacent to v in λ_t , where $t \in B^{-1}(v)$. Assume that $\deg(v) \geq 2$. We arbitrarily choose two hyperedges in $F(v)$, say f_v^1 and f_v^2 . We put $F(v) \setminus \{f_v^2\}$ in λ_t for all $t \in B^{-1}(v) \setminus \{b(f_v^2)\}$ and put $F(v) \setminus \{f_v^1\}$ in $\lambda_{b(f_v^2)}$. Then for all t , $|\lambda_t| \leq k(\Delta - 1)$ since each vertex contributes at most $\Delta - 1$ hyperedges to a given bag. An edge tt' in T is called the *edge corresponding to a 3-tuple* (v, f_v^1, f_v^2) if $f_v^1 \in \lambda_t \setminus \lambda_{t'}$ and $f_v^2 \in \lambda_{t'} \setminus \lambda_t$. If tt' is an edge corresponding to (v, f_v^1, f_v^2) and $(v', f_{v'}^1, f_{v'}^2)$ simultaneously, say $f_v^1, f_{v'}^1 \in \lambda_t$, then we subdivide tt' by adding a new node t'' and let $B'_{t''} = B'_t \cap B'_{t'}$, $\lambda_{t''} = (\lambda_t \setminus \{f_v^1\}) \cup \{f_{v'}^1\}$. Repeat this process so that every edge in T' corresponds to at most one 3-tuple. We also use T' to denote the tree after all of these subdivisions. Note that $(T', (B'_t)_{t \in T'})$ is still a tree decomposition of $[H]_2$. Now if there is an edge tt' corresponding to (v, f_v^1, f_v^2) , then we add f_v^1 into $\lambda_{t'}$, which increases the size of $\lambda_{t'}$ at most 1.

We are going to show that $(T', (B'_t)_{t \in T'}, (\lambda_t)_{t \in T'})$ is a supertree decomposition of H . By the construction of $(T', (B'_t)_{t \in T'}, (\lambda_t)_{t \in T'})$, (TI) and (TIV) in Definitions 2.3 and 2.4 hold. So we just need to show (TII) and (TIII).

First, we prove that for all $t \in T'$, $B'_t \subseteq \cup_{f \in \lambda_t} f$. If $t \in T$, then the conclusion holds because for each $v \in B_t$, we put at least one edge incident to v in λ_t . If t is the subdivided node, then the result holds by the construction.

Now we show that for any $f \in F$, $\lambda^{-1}(f)$ is connected in T' . Suppose there is $f \in F$ such that $\lambda^{-1}(f)$ is nonconnected in T' . Then there must be $t, t' \in T'$ such that for all $t'' \in \text{Path}(t, t') \setminus \{t, t'\}$, $f \in (\lambda_t \cap \lambda_{t'}) \setminus \lambda_{t''}$. If there is a vertex $v \in f$ such that $\text{Path}(t, t') \subseteq B^{-1}(v)$, then we should have $f \in \lambda_{t''}$ for all $t'' \in \text{Path}(t, t')$ even if t'' is a subdivided node by the construction, a contradiction. Suppose there exist two vertices $v, v' \in f$ such that $t \in B^{-1}(v) \setminus B^{-1}(v')$, $t' \in B^{-1}(v') \setminus B^{-1}(v)$, then there is $t'' \in \text{Path}(t, t') \setminus \{t, t'\}$ such that $t'' \in B^{-1}(v) \cap B^{-1}(v')$ by $vv' \in E([H]_2)$ which implies that $f \in \lambda_{t''}$, a contradiction.

Thus we have $stw(H) \leq (\Delta - 1)k + 1 = (\Delta - 1)(tw([H]_2) + 1) + 1$, as required. \square

3 Lower bound in terms of average rank

In this section, we prove Theorem 1.1. Let $H = (V, F)$ be a hypergraph with size m . Let $V_i = \{v \in V(H) \mid \deg(v) = i\}$ and $n_i = |V_i|$, where $i = \delta, \dots, \Delta$. By the definition of average rank, $l(H) = (\delta n_\delta + \dots + \Delta n_\Delta) / m$. Given two sets $X, Y \subseteq F$. Let $\sigma_i^j(X) = \#\{v \in V_i \mid \deg_X(v) = j\}$ and let $\sigma_i^{j,l}(X, Y) = \#\{v \in V_i \mid \deg_X(v) = j \text{ and } \deg_Y(v) = l\}$, where $\deg_X(v) = \#\{f \in F(v) \cap X\}$. We say a hypergraph H is *minimal* if $l(H_S) < l(H)$ for every nonempty proper subset S of F , where

$$l(H_S) = \left[\delta \left(n_\delta - \sum_{j=1}^{\delta} \sigma_\delta^j(S) \right) + \dots + \Delta \left(n_\Delta - \sum_{j=1}^{\Delta} \sigma_\Delta^j(S) \right) \right] / (m - |S|).$$

Firstly, we need some lemmas before we start bounding the tree width of $[H]_2$.

Lemma 3.1 *If H is a minimal hypergraph and S is a nonempty proper subset of $F(H)$, then*

$$\frac{1}{\Delta}l(H) < \frac{1}{|S|} \left(\sum_{f \in S} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sigma_i^j(S) \left(j - \frac{i}{\Delta} \right) \right).$$

Proof: Since H is minimal, we have $l(H) > l(H_S)$ which implies

$$\frac{\sum_{i=\delta}^{\Delta} in_i}{m} > \frac{\sum_{i=\delta}^{\Delta} in_i - \sum_{i=\delta}^{\Delta} i \sum_{j=1}^i \sigma_i^j(S)}{m - |S|}.$$

Hence $l(H) = \left(\sum_{i=\delta}^{\Delta} in_i \right) / m < \left(\sum_{i=\delta}^{\Delta} \sum_{j=1}^i i \sigma_i^j(S) \right) / |S|$. We easily get $\sum_{f \in S} |f| = \sum_{i=\delta}^{\Delta} \sum_{j=1}^i j \sigma_i^j(S)$. Thus

$$\frac{1}{\Delta}l(H) < \frac{1}{|S|} \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \frac{i}{\Delta} \sigma_i^j(S) = \frac{1}{|S|} \left(\sum_{f \in S} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sigma_i^j(S) \left(j - \frac{i}{\Delta} \right) \right),$$

and we have the conclusion. \square

Let $(T, (B_t)_{t \in T})$ be a tree decomposition of $[H]_2$ as guaranteed by Lemma 2.2. For each node t of T , let T_t denote the subtree of T rooted at t containing exactly t and the descendants of t . And let $z(T_t)$ be the set of edges of H with the base nodes in T_t . (Recall all base nodes are leaves.)

Lemma 3.2 *Let $H = (V, F)$ be a minimal hypergraph and $(T, (B_t)_{t \in T})$ be a tree decomposition of $[H]_2$ as guaranteed by Lemma 2.2. If $t \in T$ is a non-leaf, non-root node and a, b are the children of t , then*

$$|B_t| > \frac{1}{\Delta} (|z(T_a)| + |z(T_b)|) l(H) - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_a)) - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_b)).$$

Proof: Denote

$$g(z(T_a), z(T_b)) = \sum_{f \in z(T_a)} |f| + \sum_{f \in z(T_b)} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u, w \geq 0} \sigma_i^{u, w}(z(T_a), z(T_b)) \left(j - \frac{i}{\Delta} \right).$$

We first show that $g(z(T_a), z(T_b)) > \frac{1}{\Delta} (|z(T_a)| + |z(T_b)|) l(H)$.

Since $t \in T$ is a non-leaf, non-root node, $z(T_a), z(T_b) \neq \emptyset$, $z(T_a) \cap z(T_b) = \emptyset$ and $z(T_t) = z(T_a) \cup z(T_b) \subsetneq F$. By Lemma 3.1, we have

$$\frac{1}{\Delta}l(H) < \frac{1}{|z(T_a) \cup z(T_b)|} \left(\sum_{f \in z(T_a) \cup z(T_b)} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sigma_i^j(z(T_a) \cup z(T_b)) \left(j - \frac{i}{\Delta} \right) \right).$$

So $g(z(T_a), z(T_b)) > \frac{1}{\Delta}(|z(T_a)| + |z(T_b)|)l(H)$. We consider B_t , the bag of the target node t , which consists of vertices covered by edges in $z(T_a)$ (resp. $z(T_b)$) and $F \setminus z(T_a)$ (resp. $F \setminus z(T_b)$) at the same time. Thus,

$$\begin{aligned}
|B_t| &= \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sigma_i^j(z(T_a)) + \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sigma_i^j(z(T_b)) - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u, w > 0} \sigma_i^{u, w}(z(T_a), z(T_b)) \\
&\quad - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_a)) - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_b)) \\
&= \sum_{f \in z(T_a)} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sigma_i^j(z(T_a))(j-1) + \sum_{f \in z(T_b)} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sigma_i^j(z(T_b))(j-1) \\
&\quad - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u, w > 0} \sigma_i^{u, w}(z(T_a), z(T_b)) - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_a)) - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_b)).
\end{aligned}$$

Notice that $\sigma_i^j(z(T_a)) = \sum_{u=0}^{i-j} \sigma_i^{j, u}(z(T_a), z(T_b))$ and $\sigma_i^j(z(T_b)) = \sum_{u=0}^{i-j} \sigma_i^{j, u}(z(T_b), z(T_a))$. We have $\sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sigma_i^j(z(T_x)) = \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sum_{u=0}^{i-j} \sigma_i^{j, u}(z(T_x), z(T_y)) = \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u > 0, w \geq 0} \sigma_i^{u, w}(z(T_x), z(T_y))$ holds for $x = a, y = b$ or $x = b, y = a$.

So the above formula can be written as

$$\begin{aligned}
|B_t| &= \sum_{f \in z(T_a) \cup z(T_b)} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u, w > 0} \sigma_i^{u, w}(z(T_a), z(T_b))(1+u-1+w-1) \\
&\quad - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_a)) - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_b)) - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u \cdot w = 0} \sigma_i^{u, w}(z(T_a), z(T_b))(j-1) \\
&= \sum_{f \in z(T_a) \cup z(T_b)} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u, w \geq 0} \sigma_i^{u, w}(z(T_a), z(T_b))(u-1+w-1+1) \\
&\quad - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_a)) - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_b)) \\
&= \sum_{f \in z(T_a) \cup z(T_b)} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u, w \geq 0} \sigma_i^{u, w}(z(T_a), z(T_b))(j-1) - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_a)) \\
&\quad - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_b))
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
|B_t| &\geq \sum_{f \in z(T_a) \cup z(T_b)} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u, w \geq 0} \sigma_i^{u, w}(z(T_a), z(T_b)) \left(j - \frac{i}{\Delta} \right) - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_a)) \\
&\quad - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_b)) \\
&= g(z(T_a), z(T_b)) - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_a)) - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_b)) \\
&> \frac{1}{\Delta} (|z(T_a)| + |z(T_b)|) l(H) - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_a)) - \sum_{i=\delta}^{\Delta} \sigma_i^i(z(T_b)),
\end{aligned}$$

thus we have the conclusion. \square

Theorem 1.1 follows from the following lemma since every hypergraph H with $\delta \geq 2$ contains a minimal subgraph H' with $l(H') \geq l(H)$, in which case $[H']_2 \subseteq [H]_2$ and $tw([H]_2) \geq tw([H']_2)$.

Lemma 3.3 *Let H be a minimal linear hypergraph with minimum degree δ , maximum degree Δ and average rank $l(H)$. Let $\Delta \geq \delta \geq 2$. Suppose $\Delta \leq 2\delta^2 - 2\delta$. Then*

$$tw([H]_2) > \begin{cases} \frac{(2\delta^2 - 2\delta - \Delta)l(H)^2 + (2\Delta + 4\delta^2 - 4\delta)l(H)}{4\Delta\delta(\delta-1)} - 1 & \text{if } \delta^2 - \delta \leq \Delta - 2\Delta/l(H), \\ \frac{(2\delta^2 - 2\delta - \Delta)l(H)^2 + 6\Delta l(H) - 8\Delta}{4\Delta\delta(\delta-1)} - 1 & \text{otherwise.} \end{cases}$$

Proof: If $0 \leq l(H) < 2$, we have $\delta^2 - \delta > 0 \geq \Delta - 2\Delta/l(H)$. Then $tw([H]_2) \geq 0 \geq \frac{2}{\Delta} - 1 = \frac{2^2(2\delta^2 - 2\delta - \Delta) + 6 \cdot 2 \cdot \Delta - 8\Delta}{4\Delta\delta(\delta-1)} - 1 > \frac{(2\delta^2 - 2\delta - \Delta)l(H)^2 + 6\Delta l(H) - 8\Delta}{4\Delta\delta(\delta-1)} - 1$, as required. So we assume that $l(H) \geq 2$. Since $\delta \geq 2$ and H is a linear hypergraph, we have $l(H) < |F|$.

Let $(T, (B_t)_{t \in T})$ be a tree decomposition of $[H]_2$ as guaranteed by Lemma 2.2. Call a node t of T *significant* if $|z(T_t)| > \frac{l(H)}{2}$ but $|z(T_{t'})| \leq \frac{l(H)}{2}$ for each child t' of t .

Claim 1. There exists a non-root, non-leaf significant node t .

Proof of Claim 1: Starting at the root of T , begin traversing down the tree by the following rule: if some child t of the current node has $|z(T_t)| > \frac{l(H)}{2}$, then traverse to t ; otherwise stop. Clearly this algorithm halts.

Since $|z(T_t)| = 1 \leq \frac{l(H)}{2}$ for any leaf t , the algorithm will stop at a non-leaf.

Let t be the node where the above algorithm stops. Suppose that t is the root. Then $|z(T_t)| = |F|$. Assume t_1 and t_2 are the children of t . Then $|z(T_{t_1})|, |z(T_{t_2})| \leq \frac{l(H)}{2}$. Thus $|z(T_t)| = |z(T_{t_1})| + |z(T_{t_2})| \leq l(H) < |F|$, a contradiction. Hence the algorithm does not stop at the root. \square

Let t be a non-root, non-leaf significant node and a, b be the children of t . Set $A = z(T_a)$ and $B =$

$z(T_b)$. Then $|A|, |B| \leq \frac{l(H)}{2}$ but $|z(T_t)| = |A \cup B| > \frac{l(H)}{2} \geq 1$. By Lemma 3.2, we can get

$$|B_t| > \frac{1}{\Delta}(|A| + |B|)l(H) - \sum_{i=\delta}^{\Delta} \sigma_i^i(A) - \sum_{i=\delta}^{\Delta} \sigma_i^i(B).$$

We consider the following linear programming.

$$\begin{aligned} \mathbf{max} \quad & \sum_{i=\delta}^{\Delta} \sigma_i^i(A) \\ \mathbf{s.t} \quad & \sum_{i=\delta}^{\Delta} \frac{i(i-1)}{2} \sigma_i^i(A) \leq \frac{|A|(|A|-1)}{2}, \\ & \sigma_i^i(A) \geq 0, \quad \delta \leq i \leq \Delta, \end{aligned}$$

where the constraint condition is based on H being a linear hypergraph, that is, each pair (f_i, f_j) can only be calculated at most one in $\sigma_i^i(A)$. From the above linear programming, we can easily know that $\sum_{i=\delta}^{\Delta} \sigma_i^i(A) \leq \frac{|A|(|A|-1)}{\delta(\delta-1)}$. Thus

$$|B_t| > \frac{1}{\Delta}(|A| + |B|)l(H) - \frac{|A|(|A|-1)}{\delta(\delta-1)} - \frac{|B|(|B|-1)}{\delta(\delta-1)}.$$

Define α, β, s such that $|A| = \alpha l(H)$, $|B| = \beta l(H)$ and $s = \frac{1}{l(H)}$. Recall $|A|, |B| \leq \frac{1}{2}l(H)$ and $|A| + |B| > \frac{1}{2}l(H)$. Hence $|A|, |B| \geq 1$. Thus $s \leq \alpha, \beta \leq \frac{1}{2}$ and $\alpha + \beta > \frac{1}{2}$. Now we have

$$\begin{aligned} |B_t| &> \frac{1}{\Delta}(\alpha l(H) + \beta l(H))l(H) - \frac{\alpha l(H)(\alpha l(H) - 1)}{\delta(\delta - 1)} - \frac{\beta l(H)(\beta l(H) - 1)}{\delta(\delta - 1)} \\ &= l(H)^2 \left(\frac{\alpha}{\Delta} + \frac{\beta}{\Delta} + \frac{1}{\delta(\delta - 1)}(-\alpha^2 + s\alpha - \beta^2 + s\beta) \right). \end{aligned}$$

In **Appendix A**, we prove that $f(\alpha, \beta) = \frac{\alpha}{\Delta} + \frac{\beta}{\Delta} + \frac{1}{\delta(\delta-1)}(-\alpha^2 + s\alpha - \beta^2 + s\beta) \geq \min \left\{ f\left(\frac{1}{2}, s\right), f\left(\frac{1}{2} - s, s\right) \right\}$. Since $f\left(\frac{1}{2}, s\right) > f\left(\frac{1}{2} - s, s\right)$ if and only if $\delta^2 - \delta > \Delta - 2\Delta s$ and $tw([H]_2) + 1 \geq |B_t|$, the result holds immediately. \square

By Theorem 1.1, we can get the following corollary.

Corollary 3.4 *Let $H = (V, F)$ be a h -regular linear hypergraph with average rank $l(H)$. Suppose $h \geq 2$. Then*

$$tw([H]_2) > \frac{(2h-3)l(H)^2 + 6l(H) - 8}{4h(h-1)} - 1.$$

Let $k \geq 2$ be a positive integer. We construct a hypergraph $H = (V, F)$ where $F = \{f_1, f_2, \dots, f_n\}$. The vertex set of H is determined as the following rule: for any positive integers i, j with $i \neq j$, if

$|i - j| \leq k$, then we put the vertex $v_{i,j}$ into both f_i and f_j , where we let $v_{i,j} = v_{j,i}$. Then $V = \cup_{i=1}^n f_i$. We can easily know that H is a 2-regular linear hypergraph with $l(H) = 2k - \gamma$, where $\gamma \rightarrow 0$ when $n \rightarrow \infty$. By Corollary 3.4, $tw([H]_2) > \frac{1}{2}k^2 + \frac{3}{2}k - 2 - \gamma(\frac{1}{2}k + \frac{3}{4} - \frac{1}{8}\gamma)$. Since $\frac{1}{2}k^2 + \frac{3}{2}k - 2$ is an integer, $tw([H]_2) \geq \frac{1}{2}k^2 + \frac{3}{2}k - 2$. Note that $[H]_2 \cong L(P_n^k)$, where P_n^k is the k^{th} -power of an n -vertex path with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{(v_i, v_j) \mid |i - j| \leq k, 1 \leq i < j \leq n\}$. In [11], Harvey and Wood showed that $tw(L(P_n^k)) \leq \frac{1}{2}k^2 + \frac{3}{2}k - 1$. Thus when $h = 2$, Corollary 3.4 is almost precisely sharp for treewidth.

4 Lower bounds in terms of anti-rank

We use similar techniques to those in Section 3 to prove a lower bound on $tw([H]_2)$ in terms of $s(H)$ instead of $l(H)$.

Proof of Theorems 1.2 and 1.3 : If $s(H) < 2$, then Theorems 1.2 and 1.3 are trivial, since $tw([H]_2) \geq 0$ whenever $[H]_2$ contains at least one vertex. Now we assume that $s(H) \geq 2$. Since H is a linear hypergraph and $\delta \geq 2$, $s(H) < |F|$.

Let $(T, (B_t)_{t \in T})$ be a tree decomposition for $[H]_2$ as guaranteed by Lemma 2.2. For each node t of T , let T_t denote the subtree of T rooted at t containing exactly t and the descendants of t . Let $z(T_t)$ be the set of edges of H with the base nodes in T_t . (Recall all base nodes are leaves.)

Call a node t of T significant if $|z(T_t)| > \frac{s(H)}{2}$ but $|z(T_{t'})| \leq \frac{s(H)}{2}$ for each child t' of t . By an argument similar to Claim 1, there exists a non-root, non-leaf significant node t . Let a, b be the children of t , and let $A = z(T_a)$ and $B = z(T_b)$. Then $|A|, |B| \geq 1$, $|A|, |B| \leq \frac{s(H)}{2}$ but $|A \cup B| > \frac{s(H)}{2}$. Since $|A|, |B|$ are integers, if $s(H)$ is odd (resp. even) then $|A| + |B| \geq \frac{s(H)}{2} + \frac{1}{2}$ (resp. $|A| + |B| \geq \frac{s(H)}{2} + 1$). Define α, β, s such that $|A| = \alpha s(H)$, $|B| = \beta s(H)$ and $s = \frac{1}{s(H)}$. Then $s \leq \alpha, \beta \leq \frac{1}{2}$ and

$$\alpha + \beta \geq \begin{cases} \frac{1}{2} + s & \text{when } s(H) \text{ is even,} \\ \frac{1}{2} + \frac{s}{2} & \text{when } s(H) \text{ is odd.} \end{cases}$$

We first give the **proof of Theorem 1.2** where $\delta \geq 3$. As in Lemma 3.2,

$$\begin{aligned} |B_t| &= \sum_{f \in A \cup B} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u, w \geq 0} \sigma_i^{u,w}(A, B)(j-1) - \sum_{i=\delta}^{\Delta} \sigma_i^i(A) - \sum_{i=\delta}^{\Delta} \sigma_i^i(B) \\ &= \sum_{f \in A \cup B} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u, w \geq 0} \sigma_i^{u,w}(A, B)(j-1) - \sum_{i=\delta}^{\Delta} \sum_{u+w=i, u, w=0} \sigma_i^{u,w}(A, B) \end{aligned}$$

Furthermore, we have

$$\begin{aligned}
|B_t| &\geq \sum_{f \in A \cup B} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u, w \geq 0} \sigma_i^{u,w}(A, B)(j-1) - \sum_{i=\delta}^{\Delta} \sum_{u+w=i, u, w \geq 0} \sigma_i^{u,w}(A, B) \\
&= \sum_{f \in A \cup B} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^{i-1} \sum_{u+w=j, u, w \geq 0} \sigma_i^{u,w}(A, B)(j-1) - \sum_{i=\delta}^{\Delta} \sum_{u+w=i, u, w \geq 0} i \sigma_i^{u,w}(A, B) \\
&= \sum_{f \in A \cup B} |f| - \sum_{i=\delta}^{\Delta} \sum_{j=1}^{i-1} \sigma_i^j(A \cup B)(j-1) - \sum_{i=\delta}^{\Delta} i \sigma_i^i(A \cup B) \\
&= \sum_{f \in A \cup B} |f| - \sum_{j=2}^{\Delta} \sum_{i=j+1}^{\Delta} \sigma_i^j(A \cup B)(j-1) - \sum_{i=\delta}^{\Delta} i \sigma_i^i(A \cup B),
\end{aligned}$$

where $\sigma_i^j(A \cup B) = 0$ if $i < \delta$ or $i > \Delta$. In **Appendix B**, we show that

$$-\sum_{j=2}^{\Delta} \sum_{i=j+1}^{\Delta} \sigma_i^j(A \cup B)(j-1) - \sum_{i=\delta}^{\Delta} i \sigma_i^i(A \cup B) \geq -\frac{|A \cup B|(|A \cup B| - 1)}{2}.$$

Thus we have

$$\begin{aligned}
|B_t| &\geq \sum_{f \in A \cup B} |f| - \frac{|A \cup B|(|A \cup B| - 1)}{2} \\
&\geq (|A| + |B|)s(H) - \frac{(|A| + |B|)(|A| + |B| - 1)}{2} \\
&= \left(\left(1 + \frac{1}{2}s\right) \alpha - \frac{1}{2}\alpha^2 + \left(1 + \frac{1}{2}s\right) \beta - \frac{1}{2}\beta^2 - \alpha\beta \right) s(H)^2.
\end{aligned}$$

Now we calculate the minimum value of $f(\alpha, \beta) = (1 + \frac{1}{2}s)\alpha - \frac{1}{2}\alpha^2 + (1 + \frac{1}{2}s)\beta - \frac{1}{2}\beta^2 - \alpha\beta$ under the condition $0 \leq s \leq \alpha, \beta \leq \frac{1}{2}$ and

$$\alpha + \beta \geq \begin{cases} \frac{1}{2} + s & \text{when } s(H) \text{ is even,} \\ \frac{1}{2} + \frac{s}{2} & \text{when } s(H) \text{ is odd.} \end{cases}$$

Consider the partial derivative of $f(\alpha, \beta)$ with respect to α and β , we have

$$\frac{\partial f(\alpha, \beta)}{\partial \alpha} = \frac{s}{2} - \alpha - \beta + 1 > 0, \quad \frac{\partial f(\alpha, \beta)}{\partial \beta} = \frac{s}{2} - \alpha - \beta + 1 > 0.$$

Since $f(\alpha, \beta) = f(\beta, \alpha)$, we have $f(\alpha, \beta) \geq \min \{f(\frac{1}{2}, \frac{1}{2}), f(\frac{1}{2}, s)\}$ when $s(H)$ is even and $f(\alpha, \beta) \geq \min \{f(\frac{1}{2}, \frac{1}{2}), f(\frac{1}{2}, s), f(s, \frac{1}{2} - \frac{1}{2}s)\}$ when $s(H)$ is odd. Since $f(\frac{1}{2}, \frac{1}{2}) = \frac{s}{2} + \frac{1}{2}$, $f(\frac{1}{2}, s) = \frac{3s}{4} + \frac{3}{8}$ and $f(\frac{1}{2} - \frac{1}{2}s, s) = \frac{s^2}{8} + \frac{s}{2} + \frac{3}{8}$, we have

$$f(\alpha, \beta) \geq \begin{cases} \frac{3s}{4} + \frac{3}{8} & \text{when } s(H) \text{ is even,} \\ \frac{s^2}{8} + \frac{s}{2} + \frac{3}{8} & \text{when } s(H) \text{ is odd.} \end{cases}$$

Thus

$$tw([H]_2) + 1 \geq |B_t| \geq \begin{cases} \frac{3}{8}s(H)^2 + \frac{3}{4}s(H) & \text{when } s(H) \text{ is even,} \\ \frac{3}{8}s(H)^2 + \frac{1}{2}s(H) + \frac{1}{8} & \text{when } s(H) \text{ is odd.} \end{cases}$$

Now we give the **Proof of Theorem 1.3** where $\delta = 2$. We have

$$\begin{aligned} |B_t| &= \sum_{f \in A \cup B} |f| - \sum_{i=2}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u, w \geq 0} \sigma_i^{u,w}(A, B)(j-1) - \sum_{i=2}^{\Delta} \sigma_i^i(A) - \sum_{i=2}^{\Delta} \sigma_i^i(B) \\ &= \sum_{f \in A \cup B} |f| - \sum_{i=3}^{\Delta} \sum_{j=1}^i \sum_{u+w=j, u, w \geq 0} \sigma_i^{u,w}(A, B)(j-1) - \sum_{i=3}^{\Delta} \sigma_i^i(A) - \sum_{i=3}^{\Delta} \sigma_i^i(B) \\ &\quad - \sum_{j=1}^2 \sum_{u+w=j, u, w \geq 0} \sigma_i^{u,w}(A, B)(j-1) - \sigma_2^2(A) - \sigma_2^2(B) \\ &= \sum_{f \in A \cup B} |f| - \sum_{j=2}^{\Delta} \sum_{i=j+1}^{\Delta} (j-1)\sigma_i^j(A \cup B) - \sum_{i=3}^{\Delta} i\sigma_i^i(A \cup B) - \sigma_2^2(A \cup B) - \sigma_2^2(A) \\ &\quad - \sigma_2^2(B). \end{aligned}$$

In **Appendix C**, we show that

$$\begin{aligned} & - \sum_{j=2}^{\Delta} \sum_{i=j+1}^{\Delta} (j-1)\sigma_i^j(A \cup B) - \sum_{i=3}^{\Delta} i\sigma_i^i(A \cup B) - \sigma_2^2(A \cup B) - \sigma_2^2(A) - \sigma_2^2(B) \\ & \geq - \frac{|A \cup B|(|A \cup B| - 1)}{2} - \frac{|A|(|A| - 1)}{2} - \frac{|B|(|B| - 1)}{2}. \end{aligned}$$

Thus

$$\begin{aligned} |B_t| &\geq \sum_{f \in A \cup B} |f| - \frac{|A \cup B|(|A \cup B| - 1)}{2} - \frac{|A|(|A| - 1)}{2} - \frac{|B|(|B| - 1)}{2} \\ &\geq (|A| + |B|)s(H) - \frac{(|A| + |B|)(|A| + |B| - 1)}{2} - \frac{|A|(|A| - 1)}{2} - \frac{|B|(|B| - 1)}{2} \\ &= ((1+s)\alpha - \alpha^2 + (1+s)\beta - \beta^2 - \alpha\beta)s(H)^2. \end{aligned}$$

Let $f(\alpha, \beta) = (1+s)\alpha - \alpha^2 + (1+s)\beta - \beta^2 - \alpha\beta$. By the same argument as above, we have

$$f(\alpha, \beta) \geq \begin{cases} s + \frac{1}{4} & \text{when } s(H) \text{ is even,} \\ -\frac{s^2}{4} + s + \frac{1}{4} & \text{when } s(H) \text{ is odd.} \end{cases}$$

Thus

$$tw([H]_2) \geq \begin{cases} \frac{1}{4}s(H)^2 + s(H) - 1 & \text{when } s(H) \text{ is even,} \\ \frac{1}{4}s(H)^2 + s(H) - \frac{5}{4} & \text{when } s(H) \text{ is odd,} \end{cases}$$

and we have the conclusion. \square

When $s(H)$ is even, let $H = (C_n^k)^*$, where C_n^k is the k^{th} -power of an n -vertex cycle with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{(v_i, v_j) \mid \min\{|i-j|, i+n-j\} \leq k, 1 \leq i < j \leq n\}$. We can see that in this case H is a 2-regular linear hypergraph and $s(H) = \delta(C_n^k) = 2k$. By Theorem 1.3, $tw([H]_2) \geq \frac{1}{4}s(H)^2 + s(H) - 1 = k^2 + 2k - 1$. In [11], Harvey and Wood showed that $tw(L(C_n^k)) \leq k^2 + 2k - 1$. Hence Theorem 1.3 is precisely sharp when $s(H)$ is even.

When $s(H)$ is odd, choose two matchings $X_1 = \{1(n-k+1), 2(n-k+2), \dots, kn\}$ and $X_2 = \{(k+1)(k+2), (k+3)(k+4), \dots, (n-k-1)(n-k)\}$ in C_n^k . If n is odd (resp. even), let $H = (C_n^k \setminus X_1)^*$ (resp. $H = (C_n^k \setminus (X_1 \cup X_2))^*$). Then we have $s(H) = 2k - 1$. By Theorem 1.3, $tw([H]_2) \geq k^2 + k - 1$. In [11], Harvey and Wood showed that

$$tw([H]_2) = tw(L(H^*)) \leq \begin{cases} k^2 + k - 1 & \text{when } n \text{ is even,} \\ k^2 + k - 2 & \text{when } n \text{ is odd.} \end{cases}$$

Thus when $s(H)$ is odd, Theorem 1.3 is precisely sharp when n is even; and within '+1' when n is odd.

5 Upper bound

Proof of Theorem 1.4: Let $(T, (B_t)_{t \in T}, (\lambda_t)_{t \in T})$ be a supertree decomposition of a linear hypergraph H with width k such that T has maximum degree at most 3. By the discussion in Section 1, we may assume that $r(H) \geq k - 1$. (The existence of such a supertree decomposition $(T, (B_t)_{t \in T}, (\lambda_t)_{t \in T})$ is well known and follows by a similar argument to Lemma 2.2.)

Say a hyperedge $f \in F$ is small if $|f| \leq k - 1$ and large otherwise. For each $f \in F$, we use $T(f)$ to denote the subtree of T induced by $\lambda^{-1}(f)$. For each edge e in T , let $A(e), B(e)$ denote the two subtrees of $T - e$. If e is also an edge of $T(f)$ for some $f \in F(H)$, then let $A(e, f), B(e, f)$ denote two subtrees of $T(f) - e$, where $A(e, f) \subseteq A(e)$ and $B(e, f) \subseteq B(e)$. For $t \in \lambda^{-1}(f)$, let $\gamma_t(f) = \{v \in V(H) \mid v \in f \cap g, g \in \lambda_t \setminus \{f\}\}$. Since H is a linear hypergraph, we have $|\gamma_t(f)| \leq k - 1$. Denote $\alpha(e, f) = \cup_{t \in A(e, f)} \gamma_t(f)$ and $\beta(e, f) = \cup_{t \in B(e, f)} \gamma_t(f)$. We have the following claim.

Claim 2. For every large $f \in F$ there is an edge e in $T(f)$ such that $|\alpha(e, f)|, |\beta(e, f)| \leq \frac{2}{3}|f| + \frac{1}{3}(k-1)$.

Proof of Claim 2: Assume for the sake of a contradiction that no such e exists. Hence for all e in $T(f)$, either $|\alpha(e, f)|$ or $|\beta(e, f)|$ is too "large". Direct the edge e towards $A(e, f)$ or $B(e, f)$ respectively. (If both $|\alpha(e, f)|, |\beta(e, f)|$ are too large, then direct e arbitrarily.) Given this orientation of $T(f)$, there must be a sink (all the edges incident to it direct to it), which we label t_0 .

Let e_1, \dots, e_d be the edges in $T(f)$ incident to t_0 , where $d \in \{1, 2, 3\}$. Without loss of generality say that e_i was directed towards $B(e_i, f)$ for all e_i . Hence $|\beta(e_i, f)| > \frac{2}{3}|f| + \frac{1}{3}(k-1)$ for all i . Let $\alpha'(e_i, t_0, f) = \alpha(e_i, f) \setminus \gamma_{t_0}(f)$. Then $\alpha'(e_i, t_0, f) \cap \alpha'(e_j, t_0, f) = \emptyset$ when $i \neq j$.

If $d = 3$, then $\sum_{i=1}^3 |\beta(e_i, f)| > 2|f| + k - 1$. But $\sum_{i=1}^3 |\beta(e_i, f)| = \sum_{i=1}^3 \sum_{j \neq i} (|\gamma_{t_0}(f)| + |\alpha'(e_j, t_0, f)|) = 2(|\gamma_{t_0}(f)| + \sum_{j=1}^3 |\alpha'(e_j, t_0, f)|) + |\gamma_{t_0}(f)| \leq 2|f| + k - 1$, a contradiction. If $d = 2$, then $\sum_{i=1}^2 |\beta(e_i, f)| > \frac{4}{3}|f| + \frac{2}{3}(k-1)$. However, $\sum_{i=1}^2 |\beta(e_i, f)| = \sum_{i=1}^2 (\sum_{j \neq i} (|\alpha'(e_j, t_0, f)|) + |\gamma_{t_0}(f)|) = |\gamma_{t_0}(f)| + \sum_{j=1}^2 |\alpha'(e_j, t_0, f)| + |\gamma_{t_0}(f)| \leq |f| + k - 1$, a contradiction with $|f| > k - 1$. If

$d = 1$, then $|\beta(e_1, f)| > \frac{2}{3}|f| + \frac{1}{3}(k-1)$. However, $|\beta(e_1, f)| = |\gamma_{t_0}(f)| \leq k-1$, a contradiction with $|f| > k-1$. \square

For each small hyperedge f of H , arbitrarily select a base node in $\lambda^{-1}(f)$. For each large hyperedge f of H , select an edge e in the subtree $T(f)$ as guaranteed by Claim 2. Subdivide e and declare the new node to be $b(f)$, the base node of f . If e is selected for several different hyperedges, then subdivide it multiple times and assign a different base node for each hyperedge of H that selected e . Denote the tree after all of these subdivisions as T' . Together, this underlying tree T' and the assignment b gives a tree decomposition of $[H]_2$ in the same form as Lemma 2.1. Label the set of bags for this tree decomposition by B' . So the tree decomposition of $[H]_2$ is $(T', (B'_{t'})_{t' \in T'})$ and for each vertex $v \in V$, $B'^{-1}(v) = V(ST_v)$, where ST_v is the subtree of T' induced by $\cup_{f_i, f_j \in F(v)} \text{Path}(b(f_i), b(f_j))$. It remains to bound the width of this tree decomposition of $[H]_2$.

For each bag $B'_{t'}$ of T' , define a *corresponding bag* in T as follows. If $t' \in T'$ is also in T , then the corresponding bag of $B'_{t'}$ is simply $\lambda_{t'}$. If $t' \in T'$ is a subdivision node created by subdividing the edge $t_1 t_2$, then the corresponding bag of $B'_{t'}$ is λ_{t_1} or λ_{t_2} , chosen arbitrarily.

The following two claims give enough information to bound the width of $(T', (B'_{t'})_{t' \in T'})$.

Claim 3. If $B'_{t'}$ is a bag of T' with corresponding bag λ_t ($t \in T$) and $v \in B'_{t'}$, then there is an edge $f \in F(v)$ such that $f \in \lambda_t$.

Proof of Claim 3: Suppose that $f \notin \lambda_t$ for all $f \in F(v)$. Then $t \notin \cup_{f \in F(v)} \lambda^{-1}(f)$. If there are $f_i, f_j \in F(v)$ such that $T(f_i)$ and $T(f_j)$ are contained in different components of $T-t$, then $\lambda^{-1}(f_i) \cap \lambda^{-1}(f_j) = \emptyset$ a contradiction with $v \in f_i \cap f_j$. Thus for all $f \in F(v)$, $T(f)$ are all contained in the same component of $T-t$. Note that $b(f)$ is assigned inside of $\lambda^{-1}(f)$ (perhaps after some edges are subdivided, but this doesn't alter their positions relative to λ_t). Hence the subtree ST_v in T' doesn't include t' which implies $v \notin B'_{t'}$, a contradiction. \square

Claim 4. Suppose f is a large hyperedge and $t' \in T'$ is not $b(f)$. If λ_t ($t \in T$) is the corresponding bag of $B'_{t'}$, then we have $|\gamma_t(f)| \leq \frac{2}{3}|f| + \frac{1}{3}(k-1)$.

Proof of Claim 4: Since t' is not $b(f)$, there exists a component of $T' - b(f)$, say T'' , containing t' . Let $v \in B'_{t'} \cap f$. Then $B'_{t'}$ is a bag in the subtree ST_v in T' . Hence there is $f' \in F(v) \setminus \{f\}$ such that $b(f') \in V(T'')$ since $v \in B'_{t'}$.

Since f is a large hyperedge, $b(f)$ is a subdivision node. Let e be the edge in $T(f)$ that was subdivided to create $b(f)$. (The edge e is also guaranteed by Claim 2.) Hence $V(T'')$ has non-empty intersection with exactly one of $V(A(e))$ and $V(B(e))$, say $V(T'') \cap V(A(e)) \neq \emptyset$. Since $b(f') \in V(T'')$, there must be $v \in \alpha(e, f)$ by $v \in f \cap f'$. Then $|\alpha(e, f)| \leq \frac{2}{3}|f| + \frac{1}{3}(k-1)$ by Claim 2. Hence $B'_{t'}$ contains at most $\frac{2}{3}|f| + \frac{1}{3}(k-1)$ vertices in f , which means $\gamma_t(f) \leq \frac{2}{3}|f| + \frac{1}{3}(k-1)$. \square

We now determine an upper bound on the size of a bag $B'_{t'}, t' \in T'$. We count the vertices of $B'_{t'}$ by considering the number of vertices in a given hyperedge f of H contributes to $B'_{t'}$. By Claim 3, at most k hyperedges of the corresponding bag λ_t contribute to $B'_{t'}$.

If f is small, it contributes at most $k - 1$ vertices to $B'_{t'}$. If f is large and $t' \neq b(f)$, by Claim 4, f contributes at most $\frac{2}{3}r(H) + \frac{1}{3}(k - 1) \geq k - 1$ vertices to $B'_{t'}$. If f is large and $t' = b(f)$, then f contributes at most $r(H) \geq \frac{2}{3}r(H) + \frac{1}{3}(k - 1)$ vertices. Therefore, we conclude the highest possible contribution of a large f with $t' = b(f)$ is greater than that of a large f with $t' \neq b(f)$ which is greater than that of a small f . Note that $t' = b(f)$ for at most one $f \subseteq F$. Hence

$$|B'_{t'}| \leq (k - 1) \left(\frac{2}{3}r(H) + \frac{1}{3}(k - 1) \right) + r(H) = \frac{2}{3}kr(H) + \frac{1}{3}(k - 1)^2 + \frac{1}{3}r(H),$$

where there exists $k - 1$

If we set $(T, (B_t)_{t \in T}, (\lambda_t)_{t \in T})$ to be a minimum width hypertree decomposition of H , then $k = stw(H)$ and so

$$tw([H]_2) \leq \frac{2}{3}stw(H)r(H) + \frac{1}{3}(stw(H) - 1)^2 + \frac{1}{3}r(H) - 1.$$

□

Remark If $tw(L(H)) \leq c$ for some constant c , we can get the exact value of $stw(H)$ in polynomial time by $stw(H) = tw(L(H)) + 1$. Then Theorem 1.4 can give a useful upper bound for $tw([H]_2)$.

6 Conclusion

Treewidth is an important graph parameter in structural graph theory and in algorithmic graph theory. This paper studies the treewidth of corresponding graphs of linear hypergraphs. Let $G = (V, E)$ be a graph. There is a linear hypergraph $H = (V, F)$ such that $[H]_2 \cong G$. For example, we first find cliques C_1, C_2, \dots, C_s in G such that $\cup_{i=1}^s V(C_i) = V$ and $|V(C_i) \cap V(C_j)| \leq 1$ for $i \neq j$ and $1 \leq i, j \leq s$. Now we construct a hypergraph $H = (V, F)$ with $V(H) = V(G)$ and $F = \{C_1, \dots, C_s\} \cup \{\{x, y\} | x, y \in V, xy \in E(G) \setminus \cup_{i=1}^s E(C_i)\}$. Obviously, H is a linear hypergraph and $[H]_2 \cong G$. Thus we provide a method to estimate the bound of treewidth of graph by the parameters of the hypergraph.

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Appendix A

We calculate the minimum value of $f(\alpha, \beta) = \frac{\alpha}{\Delta} + \frac{\beta}{\Delta} + \frac{1}{\delta(\delta-1)}(-\alpha^2 + s\alpha - \beta^2 + s\beta)$, under the condition $s \leq \alpha, \beta \leq \frac{1}{2}, \alpha + \beta > \frac{1}{2}, 0 < s \leq \frac{1}{2}$ and $\Delta \geq \delta \geq 2$.

Note that $f(\alpha, \beta) = f(\beta, \alpha)$. Consider the partial derivative of $f(\alpha, \beta)$ with respect to α and β , we have

$$\frac{\partial f(\alpha, \beta)}{\partial \alpha} = \frac{1}{\Delta} - \frac{2\alpha}{\delta(\delta-1)} + \frac{s}{\delta(\delta-1)}, \quad \frac{\partial f(\alpha, \beta)}{\partial \beta} = \frac{1}{\Delta} - \frac{2\beta}{\delta(\delta-1)} + \frac{s}{\delta(\delta-1)}.$$

In the boundary $\alpha + \beta = \frac{1}{2}$, we have

$$f\left(\alpha, \frac{1}{2} - \alpha\right) = \frac{-8\Delta\alpha^2 + 4\Delta\alpha + 2\delta^2 - 2\delta - \Delta + 2\Delta s}{4\Delta\delta(\delta-1)}$$

and

$$\frac{\partial f(\alpha, \frac{1}{2} - \alpha)}{\partial \alpha} = \frac{1 - 4\alpha}{\delta(\delta - 1)}.$$

Thus we have $f(\alpha, \beta) \geq \min \{f(\frac{1}{2}, \frac{1}{2}), f(\frac{1}{2}, s), f(\frac{1}{2} - s, s)\}$. Note that

$$\begin{aligned} f\left(\frac{1}{2}, \frac{1}{2}\right) &= \frac{2\Delta s + 2\delta^2 - 2\delta - \Delta}{2\Delta(\delta - 1)}, \\ f\left(\frac{1}{2}, s\right) &= \frac{2\Delta s - 2\delta - \Delta - 4\delta s + 4\delta^2 s + 2\delta^2}{4\Delta(\delta - 1)}, \\ f\left(\frac{1}{2} - s, s\right) &= \frac{-8\Delta s^2 + 6\Delta s + 2\delta^2 - 2\delta - \Delta}{4\Delta(\delta - 1)}. \end{aligned}$$

Since $\Delta \leq 2\delta^2 - 2\delta$, $f(\frac{1}{2}, \frac{1}{2}) \geq f(\frac{1}{2}, s)$. Thus $f(\alpha, \beta) \geq \min \{f(\frac{1}{2}, s), f(\frac{1}{2} - s, s)\}$.

Appendix B

Here we show that $-\sum_{j=2}^{\Delta} \sum_{i=j+1}^{\Delta} \sigma_i^j(A \cup B)(j-1) - \sum_{i=\delta}^{\Delta} i\sigma_i^i(A \cup B) \geq -\frac{|A \cup B|(|A \cup B| - 1)}{2}$.

We consider the following linear programming

$$\begin{aligned} \min \quad & -\sum_{j=2}^{\Delta} \sum_{i=j+1}^{\Delta} (j-1)\sigma_i^j(A \cup B) - \sum_{i=\delta}^{\Delta} i\sigma_i^i(A \cup B) \\ \text{s.t.} \quad & \sum_{j=2}^{\Delta} \sum_{i=j}^{\Delta} \frac{j(j-1)}{2} \sigma_i^j(A \cup B) \leq \frac{n'(n'-1)}{2}, \\ & -\sigma_i^j(A \cup B) \leq 0, \quad 2 \leq j \leq i \leq \Delta, \end{aligned}$$

where $n' = |A \cup B|$. The *KKT* condition of this linear programming is

$$\begin{aligned} 1 - j + u \frac{j(j-1)}{2} - u_{i,j} &= 0, \quad 2 \leq j < i \leq \Delta, \\ -i + u \frac{i(i-1)}{2} - u_{i,i} &= 0, \quad i = \delta, \dots, \Delta, \\ u, u_{i,j} &\geq 0, \quad 2 \leq j \leq i \leq \Delta, \\ u \left(\sum_{j=2}^{\Delta} \sum_{i=j}^{\Delta} \frac{j(j-1)}{2} \sigma_i^j(A \cup B) - \frac{n'(n'-1)}{2} \right) &= 0, \\ u_{i,j} \sigma_i^j(A \cup B) &= 0, \quad 2 \leq j \leq i \leq \Delta. \end{aligned}$$

We can easily get $u = 1$, $u_{i_0, 2} = 0$ and $\sigma_{i_0}^2(A \cup B) = \frac{n'(n'-1)}{2}$ for some $\delta \leq i_0 \leq \Delta$, and any other $u_{i,j} = \sigma_i^j(A \cup B) = 0$. Thus $-\sum_{j=2}^{\Delta} \sum_{i=j+1}^{\Delta} \sigma_i^j(A \cup B)(j-1) - \sum_{i=\delta}^{\Delta} i\sigma_i^i(A \cup B) \geq -\frac{|A \cup B|(|A \cup B| - 1)}{2}$.

Appendix C

Here we show the lower bound of

$$-\sum_{j=2}^{\Delta} \sum_{i=j+1}^{\Delta} (j-1)\sigma_i^j(A \cup B) - \sum_{i=3}^{\Delta} i\sigma_i^i(A \cup B) - \sigma_2^2(A \cup B) - \sigma_2^2(A) - \sigma_2^2(B).$$

We consider the following linear programming

$$\begin{aligned} \min \quad & -\sum_{j=2}^{\Delta} \sum_{i=j+1}^{\Delta} (j-1)\sigma_i^j(A \cup B) - \sum_{i=3}^{\Delta} i\sigma_i^i(A \cup B) - \sigma_2^2(A \cup B) - \sigma_2^2(A) - \sigma_2^2(B) \\ \text{s.t} \quad & \sum_{j=2}^{\Delta} \sum_{i=j}^{\Delta} \frac{j(j-1)}{2} \sigma_i^j(A \cup B) \leq \frac{n'(n'-1)}{2}, \\ & \sigma_2^2(A) + \sigma_2^2(B) \leq \sigma_2^2(A \cup B), \\ & 0 \leq \sigma_2^2(A) \leq \frac{n'_1(n'_1-1)}{2}, \\ & 0 \leq \sigma_2^2(B) \leq \frac{n'_2(n'_2-1)}{2}, \\ & -\sigma_i^j(A \cup B) \leq 0, \quad 2 \leq j \leq i \leq \Delta, \end{aligned}$$

where $n' = |A \cup B|$, $n'_1 = |A|$ and $n'_2 = |B|$. The *KKT* condition of this linear programming is

$$\begin{aligned} 1 - j + u_1 \frac{j(j-1)}{2} - u_{i,j} &= 0, \quad 2 \leq j < i \leq \Delta, \\ -i + u_1 \frac{i(i-1)}{2} - u_{i,i} &= 0, \quad i = 3, \dots, \Delta, \\ -1 + u_1 - u_2 - u_{2,2} &= 0, \\ -1 + u_2 + u_3 - u_5 &= 0, \\ -1 + u_2 + u_4 - u_6 &= 0, \\ u_{i,j} &\geq 0, \quad 2 \leq j \leq i \leq \Delta, \\ u_l &\geq 0, \quad l = 1, \dots, 5, \\ u_1 \left(\sum_{j=2}^{\Delta} \sum_{i=j}^{\Delta} \frac{j(j-1)}{2} \sigma_i^j(A \cup B) - \frac{n'(n'-1)}{2} \right) &= 0, \\ u_2 (\sigma_2^2(A) + \sigma_2^2(B) - \sigma_2^2(A \cup B)) &= 0, \quad u_3 \left(\sigma_2^2(A) - \frac{n'_1(n'_1-1)}{2} \right) = 0, \\ u_4 \left(\sigma_2^2(B) - \frac{n'_2(n'_2-1)}{2} \right) &= 0, \quad u_5 \sigma_2^2(A) = 0, \quad u_6 \sigma_2^2(B) = 0, \\ u_{i,j} \sigma_i^j(A \cup B) &= 0, \quad 2 \leq j \leq i \leq \Delta. \end{aligned}$$

We can get $\sigma_2^2(A \cup B) = \frac{n'(n'-1)}{2}$, $\sigma_2^2(A) = \frac{n'_1(n'_1-1)}{2}$, $\sigma_2^2(B) = \frac{n'_2(n'_2-1)}{2}$ and all other $\sigma_i^j(A \cup B) = 0$.

Thus

$$\begin{aligned}
& - \sum_{j=2}^{\Delta} \sum_{i=j+1}^{\Delta} (j-1)\sigma_i^j(A \cup B) - \sum_{i=3}^{\Delta} i\sigma_i^i(A \cup B) - \sigma_2^2(A \cup B) - \sigma_2^2(A) - \sigma_2^2(B) \\
& \geq -\frac{|A \cup B|(|A \cup B| - 1)}{2} - \frac{|A|(|A| - 1)}{2} - \frac{|B|(|B| - 1)}{2}.
\end{aligned}$$