Open-independent, open-locating-dominating sets: structural aspects of some classes of graphs

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Let $G = (V(G), E(G))$ be a finite simple undirected graph with vertex set $V(G)$, edge set $E(G)$ and vertex subset $S \subseteq V(G)$. S is termed *open-dominating* if every vertex of G has at least one neighbor in S, and *open-independent, open-locating-dominating* (an OLD_{oind}-set for short) if no two vertices in G have the same set of neighbors in S, and each vertex in S is open-dominated exactly once by S. The problem of deciding whether or not G has an OLD_{oind}-set has important applications that have been reported elsewhere. As the problem is known to be $\mathcal{NP}$-complete, it appears to be notoriously difficult as we show that its complexity remains the same even for just planar bipartite graphs of maximum degree five and girth six, and also for planar subcubic graphs of girth nine. Also, we present characterizations of both P_4-tidy graphs and the complementary prisms of cographs that have an OLD_{oind}-set.

Keywords: open-independent, open-locating-dominating, complementary prism, planar, bipartite, P_4-tidy, complexity.

1   Introduction

Consider the situation where a graph G models a facility or a multiprocessor network with limited-range detection devices (sensing for example, movement, heat or size) that are placed at chosen vertices of G. The purpose of these devices is to detect and precisely identify the location of an intruder such as a thief, saboteur, vandal, fire or faulty processor that may suddenly be present at any vertex.

   As it is costly to install and maintain such devices it is logical to determine the locations of the minimum number of devices that can, between them, precisely determine an intruder at any location. This challenge is often called a *location-detection* or an *identification* problem and has been well-studied [21, 25, 29]. This objective is adopted throughout the present article. Sometimes such a device can determine if an intruder is in its neighborhood but cannot detect if the intruder is at its own location. In this case, it is required to find a so-called, *open-locating-dominating* vertex subset S (an OLD-set for short), which is a dominating set of G, such that every vertex in G has at least one neighbor in S, and no two vertices in G have the same set of neighbors in S. When a device may be prevented from detecting an intruder at its own location, it is necessary to install another device in its neighborhood. A natural way to analyze such situations is to make use of open neighborhood sets which may have useful additional properties, such as being open-independent, dominating, open-dominating or open-locating-dominating. A set S is *open-independent* if every member of S has at most one neighbor in S. The other terms and those in the next paragraph are made more precise later in this section.

An alternative case arises when a device can determine if the intruder is either at its own location or is in the neighborhood of its location, but which actual location cannot be detected, and furthermore, each detector cannot be located in the range of any other due to possible signal interference. Such situations can potentially be analyzed via independent, locating-dominating sets [34].

Finding an OLD-set in a given graph, if it exists, is similar to the well-studied *identifying code problem* [22]. An *identifying code* $C \subseteq V(G)$ is a dominating set where for all $u, v \in V(G)$, $u \neq v$, one has $N[u] \cap C \neq N[v] \cap C$. The minimum cardinality of an identifying code of a graph G is denoted by $IC(G)$. Although OLD-sets and identifying codes are similar notions, the parameters $OLD(G)$ and $IC(G)$ are incomparable. The concept of an open-locating-dominating set was first considered by Seo and Slater [31, 32]. The authors showed that to decide if a graph G has such a set is an $\mathcal{NP}$-complete decision problem and they provided some useful results for OLD-sets in trees and grid graphs. Foucaud et al. [11] presented a linear time algorithm that can be used to construct an open-locating dominating set of minimum size for a given cograph, based on parsing its cotree structure. Kincaid et al. [24] established the optimal density of an OLD-set for infinite triangular grids. Savic et al. [30] presented results on OLD-sets for some convex polytopes. The reader is referred to [26] for an on-line bibliography on this topic and related notions. In this paper we consider the following more restrictive problem:

OLD-OIND (existence of an open-independent, open locating dominating set)
Instance: A graph G.
Question: Does G have an open-independent, open locating dominating set?

To the best of our knowledge, Seo and Slater [33] were the first to study open-independent, open-locating-dominating sets (OLD_{oind}-sets for short). They presented some results on OLD_{oind}-sets in paths, trees and infinite grid graphs, and characterized OLD_{oind}-sets in graphs with girth at least five. The authors also demonstrated that OLD-OIND is $\mathcal{NP}$-complete. This complexity result was extended for complementary prisms by Cappelle et al. [5] who presented various properties and bounds on the sizes of minimal OLD_{oind}-sets in complementary prism graphs and showed that, if the girth of G is at least four, the OLD_{oind}-set of its complementary prism, if it exists, can be found in polynomial time.

In this paper we analyze the existence of OLD_{oind}-sets in P_4-tidy graphs and in cographs, a subclass of the P_4-tidy class. The class of P_4-tidy graphs contains several other graph families having relatively few P_4's, such as the P_4-sparse, P_4-lite, P_4-extendable and P_4-reducible graph families. The P_4-tidy graph class generalizes all of the just-mentioned graph families. It is well known that the P_4-tidy graph class is self-complementary and hereditary [12].

Haynes et al. [17] investigated several graph theoretic properties of complementary prisms, such as independence, distance and domination. For further study on domination parameters in complementary prisms, see [14, 7, 8, 19, 20] and for certain other parameters see [1, 6, 28, 35]. Cappelle et al. [2] described a polynomial-time recognition algorithm for complementary prisms. Although complementary prisms are a class of apparently well-behaved graphs, many $\mathcal{NP}$-complete problems for general graphs remain $\mathcal{NP}$-complete for this class, for example, finding an independent or a dominating set, or establishing P_3-convexity [9].

Our contributions. It appears that OLD-OIND is notoriously difficult as we show that it remains $\mathcal{NP}$-complete even for just planar bipartite graphs of maximum degree five and girth six, and also for planar subcubic graphs of girth nine. However, we study some graph classes for which the problem can be solved in polynomial time and also present characterizations of both P_4-tidy graphs and complementary prisms of cographs that have an OLD_{oind}-set.

Notation and terminology. Throughout this paper $G = (V(G), E(G))$ is assumed to be a nontrivial finite

simple undirected connected graph with vertex set $V(G)$ and edge set $E(G)$. A subgraph of G with n vertices that is a path (a cycle) is termed an n-path (an n-cycle), and is denoted by P_n (C_n). A *subcubic* graph is a graph in which each vertex has degree at most three, *i.e.* no vertex is incident with more than three edges. The *open neighborhood* of a vertex $v \in V(G)$ is denoted by $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and its *closed neighborhood* is denoted by $N_G[v] = N_G(v) \cup \{v\}$. Let $S \subseteq V(G)$, $S \neq \emptyset$. The subgraph of G induced by S is denoted by $G[S]$. S is termed *dominating* if every vertex not in S is adjacent to at least one member of S, *i.e.*, $N_G[S] = \cup_{v \in S} N[v] = V(G)$. S is *locating-dominating* if it is dominating and no two distinct vertices of $V(G) \setminus S$ have the same set of neighbors in S, *i.e.*, $\forall u, v \in V(G) \setminus S$, $u \neq v$, we have that $N(u) \cap S \neq N(v) \cap S$. Clearly, $V(G)$ is a locating-dominating set for any graph G.

S is *open-dominating* (or *total-dominating*) if every vertex $v \in V(G)$ has a neighbor in S, *i.e.*, $\cup_{u \in S} N(u) = V(G)$. In this case v is said to be *open-dominated* by S. S is *independent* if no two vertices in S are adjacent and is *open-independent* (an $OIND$-set for short) if every member of S is open-dominated by S at most once, *i.e.*, $\forall v \in S$, $|N(v) \cap S| \leq 1$. Obviously, an open-dominating set cannot be independent. S is *open-locating-dominating* (an OLD-set for short) if it is open-dominating and no two distinct members of $V(G)$ have the same open neighborhood in S, *i.e.*, for all $u, v \in V(G)$, $u \neq v$, one has $N(u) \cap S \neq N(v) \cap S$. In this case, u and v are said to be *distinguished* by S. G has an OLD-set if no two of its vertices have the same open neighborhood. The minimum cardinality of an OLD-set is denoted by $OLD(G)$, the open-locating-dominating number of G. If G has an OLD-set S, then S is called an $OLD(G)$-set if $|S| = OLD(G)$. We note that if S exists, then the fact that every leaf and its support of G must be in S is helpful in the proofs of some of the theorems stated later in the present paper.

Let G_1 and G_2 be two graphs such that $V(G_1) \cap V(G_2) = \emptyset$. We denote the disjoint union (resp. join) of G_1 and G_2 by $G_1 \oplus G_2$ (resp. $G_1 \bowtie G_2$), and the complement graph of a graph G by $\overline{G}$. An *anticomponent* of a graph G is the subgraph of G induced by the vertex set of a connected component of $\overline{G}$. More precisely, an induced subgraph H of G is an anticomponent if $\overline{H}$ is a connected component of G. Notice that if $G_1, G_2, \ldots, G_k$ are the anticomponents of G, then $G = G_1 \bowtie G_2 \bowtie \cdots \bowtie G_k$. A graph G is termed *co-connected* if $\overline{G}$ is connected. If $X \subseteq V(G)$, the subgraph of G obtained by removing both the vertices in X and the edges incident with them in G is denoted by $G \setminus X$. For a graph H and a natural number p, the graph obtained by the disjoint union of p copies of H is denoted by pH. For integer $k \geq 1$, the set $\{1, \ldots, k\}$ is denoted by $[k]$.

If an open-independent, open-locating-dominating set (an OLD_{oind}-set for short) exists in a given graph G, it is often of interest to establish a set of minimum size among such sets in G, which is denoted by $OLD_{oind}(G)$. If S is an OLD_{oind}-set for G, each component of $G[S]$ is isomorphic to K_2 (the complete graph on two vertices). See, for example, the graphs in Figures 1(a) and 1(b), where an OLD_{oind}-set of each graph is represented by the black vertices.

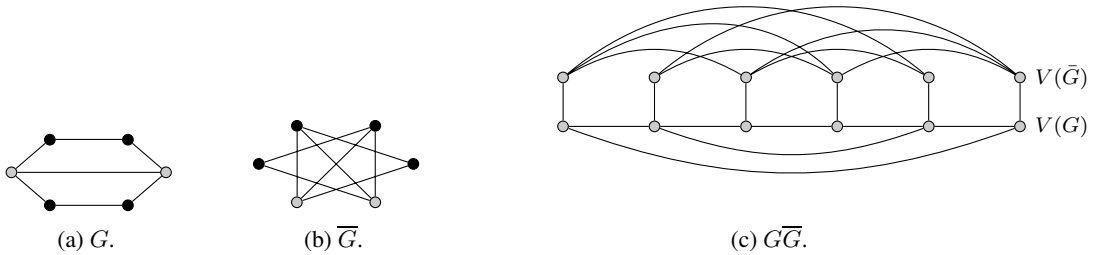(a) G. (b) $\overline{G}$. (c) $G\overline{G}$.

Fig. 1: Example of a graph, its complement and the resulting complementary prism.

Haynes et al. [17] introduced the so-called *complementary product* of two graphs as a generalization

of the well-known *Cartesian product* of the graphs. As a particular case of complementary products, the authors define the *complementary prism* of a graph G, denoted by $G\overline{G}$, as the graph formed from the disjoint union of G and its complement $\overline{G}$ by adding the edges of the perfect matching between the corresponding vertices of G and $\overline{G}$, where $V(G\overline{G}) = V(G) \cup V(\overline{G})$. For the purposes of illustration, a graph G, its complement $\overline{G}$ and the complementary prism $G\overline{G}$ are depicted respectively, in Figures 1(a), 1(b) and 1(c). To simplify matters, G and $\overline{G}$ are used to denote the subgraph copies of G and $\overline{G}$, respectively, in $G\overline{G}$. For a set $X \subseteq V(G)$, let $\overline{X}$ denote the corresponding vertices of X in $V(\overline{G})$.

2   Complexity results

Open-independent, open-locating-dominating sets were first studied by Seo and Slater [33] who stated necessary and sufficient conditions for the existence of an OLD_{oind}-set in a graph G that has girth $g(G) \geq 5$. For general graphs (with arbitrary girth), the conditions stated in Theorem 2.1 are necessary but not sufficient, as is stated in Theorem 2.2. We frequently use Theorem 2.2 in our proofs below, sometimes without mentioning the fact.

Theorem 2.1 ([33]). *If a graph G has girth satisfying $g(G) \geq 5$ and $S \subseteq V(G)$, then S is an OLD_{oind}-set iff (i) each $v \in S$ is open-dominated by S exactly once and (ii) each $v \notin S$ is open-dominated by S at least twice.*

Theorem 2.2 ([4]). *If $S \subseteq V(G)$ is an OLD_{oind}-set of a graph G, then (i) each $v \in S$ is open-dominated exactly by S once, and (ii) each $v \in V(G) \setminus S$ is open-dominated by S at least twice.*

Seo and Slater [33] presented some results about OLD_{oind}-sets in trees. The authors showed that every leaf and its neighbor are contained in any OLD_{oind}-set of any tree T, if T has such a set. Furthermore, they recursively defined the collection of trees that have unique OLD_{oind}-sets. On the other hand, they showed that OLD-OIND is an $\mathcal{NP}$-complete problem for general graphs. They present a reduction from the 3-SAT problem. Indeed, by reducing from the $\mathcal{NP}$-complete problem Planar 3-SAT, it is possible to prove $\mathcal{NP}$-completeness for planar graphs by using their construction. Recently, it was proved that OLD-OIND is $\mathcal{NP}$-complete for the complementary prisms of a given graph G [5]. We use similar ideas to those in [27] for efficient edge domination, showing that OLD-OIND is $\mathcal{NP}$-complete even for the special cases when G is either a planar bipartite graph of maximum degree five and girth six or a planar subcubic graph of girth nine, by polynomial reduction from the following decision problem.

RESTRICTED X3C (exact cover by 3-sets)
Instance: A finite set X with $|X| = 3n$ and a collection $\mathcal{S}$ of 3-element subsets of X such that each element of X is in at most 3 subsets, with $|\mathcal{S}| = m$, where n and m are positive integers.
Question: Does $\mathcal{S}$ contain an exact cover of X, *i.e.* a pairwise disjoint subcollection $\mathcal{S}' \subseteq \mathcal{S}$ such that every element of X occurs in exactly one member of $\mathcal{S}'$?

It is well known that RESTRICTED X3C is $\mathcal{NP}$-complete [16]. Each instance of RESTRICTED X3C, say $X = \{x_1, x_2, \ldots, x_{3n}\}$ and $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, can be associated with a bipartite graph $G_A = (V_A, E_A)$, where $V_A = X \cup S$ and $E_A = \{(x_i, S_j) : 1 \leq i \leq 3n, 1 \leq j \leq m, \text{ and } x_i \in S_j\}$. If the associated bipartite graph G_A is planar, the problem is termed the *planar restricted exact cover by 3-sets problem* (PLANAR RESTRICTED X3C), which is also $\mathcal{NP}$-complete [10].

Theorem 2.3. *Deciding, for a given planar bipartite graph G of maximum degree five and girth six, if G has an OLD_{oind}-set is an $\mathcal{NP}$-complete problem.*
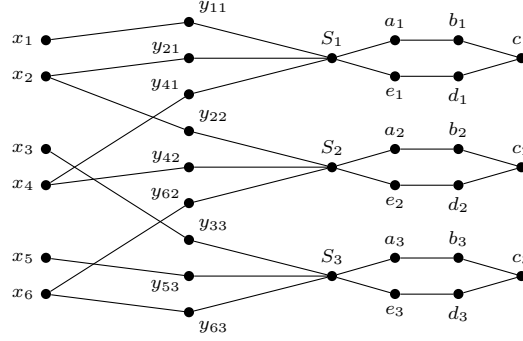
Fig. 2: Graph G that is an instance of OLD-OIND for planar bipartite graphs of maximum degree five and girth six as in the proof of Theorem 2.3). $\mathcal{S} = \{S_1, S_2, S_3\} = \{\{x_1, x_2, x_4\}, \{x_2, x_4, x_6\}, \{x_3, x_5, x_6\}\}$.

Proof: It is possible to verify in polynomial time if a given set $D \subseteq V(G)$ is an OLD_{oind}-set of G. So, OLD-OIND is in $\mathcal{NP}$. We now show that PLANAR RESTRICTED X3C is reducible in polynomial time to the problem OLD-OIND for planar bipartite graphs of maximum degree five and girth six. Let $X = \{x_1, x_2, \ldots, x_{3n}\}$ and $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ be an instance of PLANAR RESTRICTED X3C. We now construct a graph $G = (V(G), E(G))$, as illustrated in Figure 2, where $V(G) = \{x_i : 1 \leq i \leq 3n\} \cup \{y_{ij} : 1 \leq i \leq 3n, 1 \leq j \leq m \text{ and } x_i \in S_j\} \cup \{S_j, a_j, b_j, c_j, d_j, e_j : 1 \leq j \leq m\}$, and $V(E) = \{x_i y_{ij}, y_{ij} S_j : 1 \leq i \leq 3n, 1 \leq j \leq m, x_i \in S_j\} \cup \{S_j a_j, a_j b_j, b_j c_j, c_j d_j, d_j e_j, e_j S_j : 1 \leq j \leq m\}$. The resulting graph G has $3n + 9m$ vertices, and can be constructed in polynomial time.

Clearly, G is planar because (X, S) is an instance of PLANAR RESTRICTED X3C. Furthermore, G is bipartite as $V(G)$ can be partitioned into the subsets: $\{x_i : 1 \leq i \leq 3n\} \cup \{S_j, b_j, d_j : 1 \leq j \leq m\}$ and $\{y_{ij} : 1 \leq i \leq 3n, 1 \leq j \leq m \text{ and } x_i \in S_j\} \cup \{a_j, c_j, e_j : 1 \leq j \leq m\}$. It may also be noted that G has maximum degree five, since each element of X is in at most 3 subsets, and also it is easy to verify that G has girth six. We are now going to prove that (i) G has an OLD_{oind}-set if and only if (ii) there is a subset $\mathcal{S}' \subseteq \mathcal{S}$ that is an exact cover of X.

To prove that (i) is necessary for (ii), assume that $\mathcal{S}$ has an exact cover $\mathcal{S}'$. Let $D = \{x_i, y_{ij}, a_j, b_j, d_j, e_j : S_j \in \mathcal{S}' \text{ and } x_i \in S_j\} \cup \{S_j, a_j, c_j, d_j : S_j \in \mathcal{S} \setminus \mathcal{S}'\}$. Then D contains $6n + 4m$ vertices, which induce $3n + 2m$ independent edges. Note that every vertex y_{ij} that is not in D has its two neighbors in D, and the 6-cycle has four vertices in D. So it can be concluded that D is an OLD_{oind}-set of G.

To prove that (ii) is necessary for (i), assume that G has an OLD_{oind}-set D. Observe that D does not contain both S_j and y_{ij}, else $a_j, e_j \notin D$, which is impossible since by the open-independence of D, at most two of the vertices in the set $\{b_j, c_j, d_j\}$ are in D and then there are two vertices of the 6-cycle $\langle S_j, a_j, b_j, c_j, d_j, e_j \rangle$ which are dominated exactly once by D. So D is not an OLD_{oind}-set of G. The fact that $\{S_j, y_{ij}\}$ is not a subset of D for all $j \in [m]$; implies that every $x_i \in D$ and it has a neighbor y_{ij} for some $j \in [m]$ such that $y_{ij} \in D$. Note that altogether, these results imply that for every S_j, the three vertices y_{ij} are either (a) all in D, or (b) none are in D. If (a) then $S_j \notin D$ and $\{a_j, b_j, d_j, e_j\} \subseteq D$. If (b) then $S_j \in D$ and exactly one of the two symmetric subsets $\{a_j, c_j, d_j\}$ and $\{b_j, c_j, e_j\}$ are contained in D. Let $\mathcal{S}' = \{S_j : 1 \leq j \leq m \text{ and } S_j \notin D\}$. Clearly, $\mathcal{S}'$ is an exact cover of $\mathcal{S}$. □

On the other hand, if odd cycles are allowed, *i.e.*, the graph can be non-bipartite, we can modify the construction in the proof of Theorem 2.3 to add the restriction that the planar graph is subcubic. In this case, the instances have girth nine.

Theorem 2.4. *Deciding, for a given planar subcubic graph G of girth nine, if G has an OLD_{oind}-set is an $\mathcal{NP}$-complete problem.*

Proof: It is possible to verify in polynomial time if a given set $D \subseteq V(G)$ is an OLD_{oind}-set of G. So, OLD-OIND is in $\mathcal{NP}$. We now show that, for planar subcubic graphs of girth nine, PLANAR RESTRICTED X3C is reducible in polynomial time to the problem OLD-OIND. Let $X = \{x_1, x_2, \ldots, x_{3n}\}$ and $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ be an instance of PLANAR RESTRICTED X3C. We now construct a graph $G = (V(G), E(G))$, as illustrated in Figure 3. Let $X = \{x_1, \ldots, x_{3n}\}$, $Y = \{y_{ij} : 1 \le i \le 3n, 1 \le j \le m$ and $x_i \in S_j\}$, and let $C_j = \{a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j, k_j\}$, for every $j \in [m]$. Let $V(G)$ be the set containing $X \cup Y$ and the set C_j, for every $j \in [m]$. Add the edges that form a 9-cycle among the vertices in C_j, for every $j \in [m]$; add the edges of a perfect matching between the three vertices y_{ij} and the vertices in the set $D_j^* = \{c_j, k_j, f_j\}$, for every $j \in [m]$; and finally, add the edges in $\{x_i y_{ij} : 1 \le i \le 3n, 1 \le j \le m$ and $x_i \in S_j\}$. The resulting graph G has $3n + 12m$ vertices.

Clearly, G is planar because (X, S) is an instance of PLANAR RESTRICTED X3C. Furthermore, G has maximum vertex degree at most 3, since each vertex x_i has at most three neighbors in Y, and it is easy to verify that G has girth nine. We are now going to prove that (i) G has an OLD_{oind}-set if and only if (ii) there is a subset $\mathcal{S}' \subseteq \mathcal{S}$ that is an exact cover of X.

For every $j \in [m]$, consider the sets $D_j^1 = \{a_j, b_j, d_j, e_j, g_j, h_j\}$, $D_j^2 = \{b_j, e_j, h_j\}$ and $D_j^3 = \{a_j, d_j, g_j\}$. To prove that (i) is necessary for (ii), assume that $\mathcal{S}$ has an exact cover $\mathcal{S}'$. Let D be the set containing, the vertices in $D_j^1 \cup \{x_i, y_{ij} : x_i \in S_j\}$, for every j such that $S_j \in \mathcal{S}'$; and, additionally, the vertices in $D_j^* \cup D_j^2$, for every j such that $S_j \notin \mathcal{S}'$. Then D contains $6n + 6m$ vertices, which induce $3n + 3m$ independent edges. Note that every vertex in every cycle induced by C_j has two neighbors in $C_j \cap D$ and every vertex y_{ij} such that $S_j \notin \mathcal{S}'$ is not in D has its two neighbors in D. So, it can be concluded that D is an OLD_{oind}-set of G.

To prove that (ii) is necessary for (i), assume that G has an OLD_{oind}-set D. First, we prove some claims.

Claim 1. *In every cycle induced by C_j, either a) vertices k_j, c_j and f_j are all in D, or b) none are in D.*

Proof of Claim 1: By inspection, we can verify that exactly six vertices of C_j are in any OLD_{oind}-set of G. So the result follows by symmetry. $\square$

Claim 2. *The set D does not contain both y_{ij} and a vertex in D_j^*, for all $j \in [m]$.*

Proof of Claim 2: By contradiction, suppose that D contains, for some $j \in [m]$, a vertex y_{ij} and its neighbor in D_j^*. This implies, by Claim 1, that D_j^* is a subset of D. Since D is open-independent, $D \cap C_j$ has exactly three vertices, and there are six vertices of C_j which are dominated by D exactly once. So D is not an OLD_{oind}-set of G. $\square$

By Claim 2, D does not contain both y_{ij} and a vertex in D_j^*. This implies that $X \subseteq D$ and every x_i has a unique neighbor y_{ij} for some $j \in [m]$ such that $y_{ij} \in D$. By Claim 1, for every $j \in [m]$, the three vertices k_j, c_j and f_j are either (a) all in D, or (b) none are in D. If (a) then exactly one of the two symmetric subsets D_j^2 and D_j^3 are contained in D. If (b) then $D_j^1 \subseteq D$. Let $\mathcal{S}' = \{S_j : 1 \le j \le m$ and $D_j^* \cap D = \emptyset\}$. Clearly, $\mathcal{S}'$ is an exact cover of $\mathcal{S}$. $\blacksquare$

3 P_4-tidy graphs

In this section, we study OLD_{oind}-sets in P_4-tidy graphs. A graph $G = (V, E)$ is termed P_4-*tidy* if, for every vertex set A inducing a P_4 in G, there is at most one vertex $v \in V \setminus A$ such that $G[A \cup \{v\}]$ contains at least two induced P_4's. This class includes spider and quasi-spider graphs, which we now define.
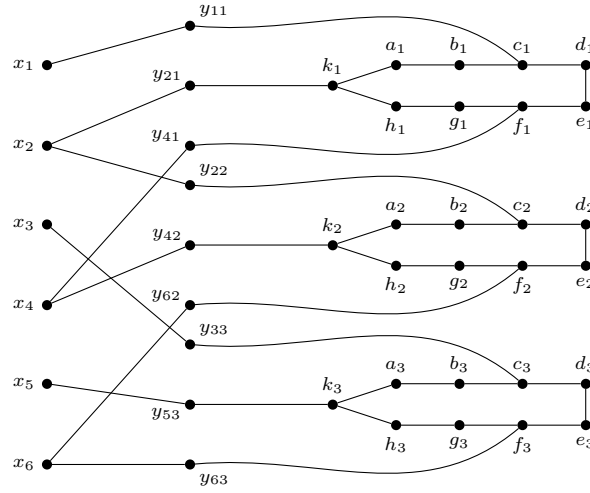
Fig. 3: Graph G that is an instance of OLD-OIND for planar subcubic graphs of girth nine as in the proof of Theorem 2.4. $\mathcal{S} = \{S_1, S_2, S_3\} = \{\{x_1, x_2, x_4\}, \{x_2, x_4, x_6\}, \{x_3, x_5, x_6\}\}$.

A *spider* is a graph whose vertex set has a partition (C, X, H), where $C = \{c_1, \ldots, c_k\}$ and $X = \{x_1, \ldots, x_k\}$ for a given integer $k \geq 2$ are respectively, a clique and an independent set; x_i is adjacent to c_j iff $i = j$ (a thin spider), or x_i is adjacent to c_j if and only if $i \neq j$ (a thick spider); and every vertex of H is adjacent to each vertex of C and is not adjacent to any vertex of X. The size k of both C and X is called the *weight* of the spider and the set H in the partition is called its *head*. Notice that if $k = 2$, then the thick and thin spider graphs are isomorphic. A *quasi-spider* is a graph obtained from a spider that has vertex partition (C, X, H) by replacing at most one vertex of $C \cup X$ by a K_2 or a $\overline{K_2}$ (where each vertex of the K_2 or the $\overline{K_2}$ has the same adjacency structure as the vertex it replaced).

(a) A thin spider of weight three with a nonempty head.

(b) A thick quasi-spider of weight three.

(c) A thin quasi-spider of weight three.

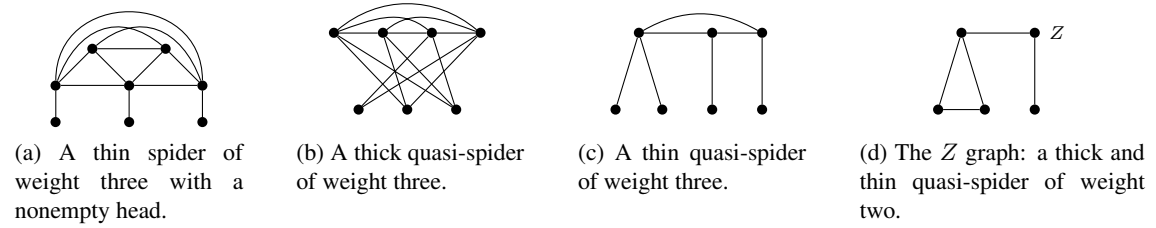(d) The Z graph: a thick and thin quasi-spider of weight two.

Fig. 4: Examples of spider and quasi-spider graphs.

The following is a structural theorem for P_4-tidy graphs in terms of spider and quasi-spider graphs. Spiders and quasi-spiders are co-connected graphs.

Theorem 3.1 ([12]). *G is a P_4-tidy graph iff if exactly one of the following statements holds:*

1. *G is the union or the join of two P_4-tidy graphs;*
2. *G is a spider or a quasi-spider graph with partition (C, X, H) such that either H induces a P_4-tidy graph or is empty;*

3. *G is isomorphic to C_5, P_5, $\overline{P_5}$, or K_1.*

The following two theorems settle which spiders and quasi-spiders have an OLD_{oind}-set. Let the graph in Figure 4(d) be denoted by Z.

Theorem 3.2. *No spider has an OLD_{oind}-set.*

Proof: Let $G = (V(G), E(G))$ be a spider with vertex set $V(G)$ partitioned into (C, X, H), where $|C| = |X| = k \geq 2$. For a proof by contradiction, suppose that G has an OLD_{oind}-set $S \subseteq V(G)$. We consider the subgraph $G[C \cup X]$ and show that either S is not an open-independent set or S is not an open-dominating set. As the cases $H \neq \emptyset$ and $H = \emptyset$ are analogous, they are not considered further in the following proof. If G is thin it contains at least two pendant vertices in X say, x_i and x_j. If both $x_i, x_j \in S$, then both $c_i, c_j \in S$. As C is a clique, c_i and c_j are adjacent in C and thus, $c_i(c_j)$ is open-dominated by both x_i and $c_j(x_j$ and $c_i)$. By Theorem 2.2(i), S cannot be an OLD_{oind}-set. If at least one of $x_i, x_j \notin S$, then that vertex cannot be open-dominated by S at least twice. By Theorem 2.2(ii), S cannot be an OLD_{oind}-set. Suppose instead that G is thick. If $k = 2$, then G is isomorphic to a thin spider and the result follows. So, assume that $k \geq 3$. As C is a clique, $|C \cap S| \leq 2$, and the following three subcases demonstrate that S is not open-dominating.
   1. If $C \cap S = \emptyset$, each $x_i \in X$ must belong to S, but no x_i is open-dominated by any vertex in S.
   2. If $|C \cap S| = 1$, say $c_i \in S$ for a unique $i \in [k]$ then, as G is thick, x_i is not open-dominated by c_i. Whether or not $x_i \in S$, x_i is not open-dominated by any member of S.
   3. If $|C \cap S| = 2$, say $c_i, c_j \in S$, where $i, j \in [k]$, then x_i (resp. x_j) is not a member of S and is open-dominated by only c_j (resp. c_i). Hence, both x_i and x_j are open-dominated exactly once each by S, and thus S is not an OLD_{oind}-set.

□

Theorem 3.3. *Any quasi-spider that has an OLD_{oind}-set is isomorphic to the thick and thin quasi-spider Z depicted in Fig. 4(d).*

Proof: Let $G = (V(G), E(G))$ be a quasi-spider obtained from a given spider G_s that has a vertex partition (C, X, H), where once again, $|C| = |X| = k \geq 2$. Suppose G is produced by applying the vertex replacement operation to a vertex $v \in C \cup X$ in G_s and that G has an OLD_{oind}-set $S \subseteq V(G)$. Again, we consider the subgraph $G[C \cup X]$ and show that unless $k = 2$, either S is not an open-locating-dominating set or is not an open-independent set of G and thus cannot be an OLD_{oind}-set.

First we consider $k = 2$. If $H = \emptyset$, since headless thick and thin spiders of size two are isomorphic, then G_s is a 4-path, and there are four possibilities for G. In the first possibility, where either vertex in X is replaced by a K_2, G is isomorphic to Z and thus G has an OLD_{oind}-set. In the other three possibilities, where a vertex in X is replaced by a $\overline{K_2}$, or a vertex in C is replaced by a K_2 or a $\overline{K_2}$; it is easy to establish by inspection that G does not have an OLD_{oind}-set. If $H \neq \emptyset$, then G can possibly have an OLD_{oind}-set only if H has a dominating subset. If this is so there are eight cases - where G_s is either thick or thin and a vertex in either C or X is replaced by either a K_2 or a $\overline{K_2}$. It is straightforward to establish by inspection that none of the cases has an an OLD_{oind}-set.

Now, we consider $k \geq 3$. Whether G_s is thick or thin, there are four cases, depending on which of the following vertex replacement operations is applied: either $v \in C$ or $v \in X$ is replaced by either a K_2 or a $\overline{K_2}$. Let c_i^ℓ (resp. x_i^ℓ), $\ell = 1, 2$, be the two vertices that replace vertex v in C (resp. X). When no index ℓ is indicated, c_i (resp. x_i) can be either of the two vertices c_i^1 and c_i^2 (resp. x_i^1 and x_i^2). Replacing a vertex in C induces in G either a clique or a clique with one edge missing, denoted in either case by $\overline{C}$. As S must be open-independent to be an OLD_{oind}-set, it follows that $|\overline{C} \cap S| \leq 2$. Instead, replacing a vertex in X

induces in G either an independent set or an independent set with one additional edge, denoted in either case by $\overline{X}$.

If G_s is thin, after any of the four types of replacement, the pair of vertices in G that replace v cannot be distinguished and thus, in this case, G does not have an OLD_{oind}-set. If G_s is thick, similar arguments to those used in the proof of Theorem 3.2 for thick spiders can be applied to establish that, once again, G does not have an OLD_{oind}-set, as follows:

1. If $\overline{C} \cap S = \emptyset$ then, unless a vertex $x_i \in X$ is replaced by a K_2, no vertex in X is open-dominated by any vertex in S. If indeed, x_i is replaced by a K_2, there are still at least two vertices in X that are not open-dominated by any vertex in S.
2. If $|\overline{C} \cap S| = 1$, say $c_i \in S$ for a unique $i \in [k]$ then, as G is thick, x_i is not open-dominated by c_i. Whether or not $x_i \in S$, x_i is not open-dominated by any member of S.
3. If $|\overline{C} \cap S| = 2$, say $c_i, c_j \in S$, where $i, j \in [k]$ (with $i = j$ when $c_i^1, c_i^2 \in S$), then x_i (resp. x_j) is not a member of S and is open-dominated by only c_j (resp. c_i). Hence, both x_i and x_j are open-dominated exactly once each by S, and thus S is not an OLD_{oind}-set.

$\square$

We now consider the existence of an OLD_{oind}-set in a P_4-tidy graph. If G is a disconnected P_4-tidy graph having an OLD_{oind}-set, then each component of G has an OLD_{oind}-set. Thus we may consider only connected P_4-tidy graphs.

Theorem 3.4. *Let G be a co-connected P_4-tidy graph. G has an OLD_{oind}-set iff G is isomorphic to either P_5 or to the graph Z shown in Figure 4(d).*

Proof: It can be verified by inspection that P_5 and Z are co-connected P_4-tidy graphs having OLD_{oind}-sets. Let G be a co-connected P_4-tidy graph. By Theorem 3.1, G is either a spider, a quasi-spider or one of the graphs C_5, P_5 and $\overline{P_5}$. Suppose that G has an OLD_{oind}-set. Clearly, G is not isomorphic to either C_5 or $\overline{P_5}$. By Theorems 3.2 and 3.3 G is isomorphic to the graph Z. $\square$

Theorem 3.5. *Let G be a connected P_4-tidy graph of order $n \geq 2$. G has an OLD_{oind}-set iff either G is isomorphic to one of the graphs K_2, K_3, P_5, $P_5 \bowtie K_1$, Z, and $Z \bowtie K_1$ or it can be obtained from them recursively by applying the following operation. Let $t \geq 2$ and $G_1, \ldots, G_t$ be connected P_4-tidy graphs each having an OLD_{oind}-set. Set $G := (G_1 \oplus \cdots \oplus G_t) \bowtie K_1$.*

Proof: Clearly, K_2, K_3, P_5, $P_5 \bowtie K_1$, Z, and $Z \bowtie K_1$ are connected P_4-tidy graphs, each having an OLD_{oind}-set. Suppose instead that $G \notin \{K_2, K_3, P_5, P_5 \bowtie K_1, Z, Z \bowtie K_1\}$. Let $G_1, \ldots, G_t$ be graphs such that G_i has an OLD_{oind}-set S_i, for each $i \in [t]$ and let G be a graph obtained by the described operation. Let $S = \bigcup_{i \in [t]} S_i$ and $V(K_1) = \{v\}$. It is easy to see that every vertex in $V(G) \setminus \{v\}$ is both distinguished and open-dominated by S, and that v is the only vertex of G that is adjacent to all of the vertices of S. Hence S is also an OLD_{oind}-set of G.

Conversely, suppose that G is a connected P_4-tidy graph distinct from K_2 and K_3 such that G has an OLD_{oind}-set S. Hence $n \geq 4$. If $\overline{G}$ is connected, G is co-connected and by Theorem 3.4, G is isomorphic to one of P_5 and Z. If $\overline{G}$ is disconnected, G can be obtained by the join of two graphs, say H_1 and H_2. Firstly, we claim that one of H_1 and H_2 has exactly one vertex v say, and $v \notin S$. To the contrary, suppose $S_1 = S \cap V(H_1) \neq \emptyset$ and $S_2 = S \cap V(H_2) \neq \emptyset$. In order to be open-independent $|S_1| = |S_2| = 1$, and $|S| = 2$, which implies that G has at most three vertices. Thus, one of H_1 and H_2 has an empty intersection with S. Without loss of generality, we may assume that $V(H_2) \cap S = \emptyset$. If H_2 has at least two vertices u and v say, then $N_G(u) \cap S = N_G(v) \cap S$ which contradicts the fact that S is an OLD-set of G. So H_2 is isomorphic to K_1 and, $S = S_1$ is an OLD_{oind}-set of H_1.

We now analyze the possibilities for H_1. First suppose that H_1 is connected. If $|S| = 2$, G has at most three vertices. Since $n \geq 4$, $|S| \geq 4$. Observe that, if H_1 is the join of two graphs, the vertices of S must all belong to the same subgraph and any vertex of any other subgraph is adjacent to every vertex of S. Altogether, these facts imply that there exists a vertex in H_1, y say, such that $S \subseteq N_{H_1}(y)$. In this case, v and y are not distinguished by S in G. Thus we may assume that H_1 is co-connected. By Theorem 3.4, H_1 is isomorphic either to P_5 ($G = P_5 \bowtie K_1$) or Z ($G = Z \bowtie K_1$). Now instead, assume that H_1 is disconnected. In this case $H_1 = G_1, \ldots, G_t$ is a collection of $t \geq 2$ connected P_4-tidy graphs. As every vertex in $V(G_i)$ has exactly one additional neighbor in G (*i.e.* vertex $v \notin S$), if there exists $i \in [t]$ such that G_i does not have an OLD_{oind}-set, it is easy to see that S is not an OLD_{oind}-set of G. Thus, we can conclude that every G_i with $i \in [t]$ has an OLD_{oind}-set and that G is the graph $H_1 \bowtie K_1$.   □

A *cograph* is a graph that can be constructed from a given solitary vertex using the repeated application of the disjoint union and join operations. Another standard characterization of cographs is that they are those graphs that do not contain a four-vertex path as an induced subgraph. All complete graphs, complete bipartite graphs, cluster graphs and threshold graphs are cographs. Since any cograph G has the property that either G or $\overline{G}$ is disconnected, we can conclude that any cograph having an OLD_{oind}-set can be obtained from either K_2 or K_3 by the operations described in Theorem 3.5. Hence, Corollary 3.6 follows.

Corollary 3.6. *Let G be a connected cograph of order $n \geq 2$. G has an OLD_{oind}-set iff G is either (i) isomorphic to either K_2 or K_3 or (ii) it can be obtained from them recursively by applying the following operation. Let $t \geq 2$ and $G_1, \ldots, G_t$ be connected cographs each having an OLD_{oind}-set. Set $G := (G_1 \oplus \cdots \oplus G_t) \bowtie K_1$.*

4   Complementary prisms of cographs

Corollary 3.6 provides a characterization of the class of cographs that have an OLD_{oind}-set. We now consider OLD_{oind}-sets in the complementary prisms of cographs. If G is a cograph then $\overline{G}$ and $G \oplus \overline{G}$ are also cographs. However, if G is a nontrivial cograph, $G\overline{G}$ is is a P_7-free and is not a cograph.

If G is a connected cograph, then $\overline{G}$ is disconnected. Henceforth it is assumed that G is a connected cograph and we denote the connected components of $\overline{G}$ by $\overline{G_1}, \overline{G_2}, \ldots, \overline{G_t}$; and the anticomponents of G by $G_1, G_2, \ldots, G_t$. The number of vertices of the subgraph G_i (resp. $\overline{G_i}$) is denoted by $|G_i|$ (resp. $|\overline{G_i}|$). There are infinite families of cographs that have complementary prisms with OLD_{oind}-sets, for example, the family described in Theorem 4.1 below.

Theorem 4.1. *[4] If G is a nontrivial graph with a unique universal vertex, then $G\overline{G}$ has an OLD_{oind}-set iff $G = K_1 \bowtie \overline{mK_2}$, where $m \geq 1$.*

Cappelle et al. [4] reported some properties of an OLD_{oind}-set in a complementary prism $G\overline{G}$. For instance, they proved that in any OLD_{oind}-set in this class of graphs there is at most one edge that directly connects a vertex in $V(G)$ with a vertex in $V(\overline{G})$. The authors also proved that, given a general graph G, deciding whether or not $G\overline{G}$ has an OLD_{oind}-set is an $\mathcal{NP}$-complete problem [5]. However, the special case where G has girth of at least four can be decided in polynomial time. Here, we show that the connected cographs having an OLD_{oind}-set S such that $|S \cap V(G)| = 1$ are exactly those described in Theorem 4.1.

By Theorem 4.1, if G is a nontrivial cograph with a unique universal vertex, then $G\overline{G}$ has an OLD_{oind}-set iff either (i) $|\overline{G_i}| = 1$, for a unique i, where $1 \leq i \leq t$, or (ii) $|\overline{G_i}| = 2$. If a cograph G does not have a universal vertex, then $|\overline{G_i}| \geq 2$, $i \in [t]$. In this case, we show that if $G\overline{G}$ has an OLD_{oind}-set S, then at least one, and at most three, components of $\overline{G}$ have at least three vertices. Let $S = S_0 \cup \overline{S}_1$ with $S_0 \subseteq V(G)$ and $\overline{S}_1 \subseteq V(\overline{G})$. We consider the case $|S_0| = 2$ in Theorem 4.5 and the general case in Theorem 4.6. For each

component $\overline{G_i}$ of $\overline{G}$, let $\overline{D_i} = \overline{G_i} \setminus \overline{S_1}$ and $\overline{D} = \bigcup_{i \in [t]} \overline{D_i}$. Let D_i (resp. D) be the corresponding vertices of $\overline{D_i}$ (resp. $\overline{D}$) in G_i (resp. G), with $S_0 \subseteq D$. Let $n_i = |V(G_i)|$.

Theorem 4.2. *Let G be a nontrivial connected cograph such that $G\overline{G}$ has an OLD_{oind}-set $S = S_0 \cup \overline{S_1}$ with $S_0 \subseteq V(G)$ and $\overline{S_1} \subseteq V(\overline{G})$. Then $|S_0| = 1$ iff $G = K_1 \bowtie \overline{mK_2}$, where $m \geq 1$.*

Proof: If $G = K_1 \bowtie \overline{mK_2}$, where $m \geq 1$, it is easy to see that $G\overline{G}$ has an OLD_{oind}-set $S_0 \cup \overline{S_1}$ say, where $\overline{S_1} = V(\overline{G})$ and S_0 contains only a universal vertex of G and thus $|S_0| = 1$. For the converse, suppose G is such that $G\overline{G}$ has an OLD_{oind}-set $S_0 \cup \overline{S_1}$ with $|S_0| = 1$, where $S_0 = \{v\}$, say. Since G is nontrivial it has at least two vertices and every vertex in $V(G) \setminus \{v\}$ has at most one neighbor in $\overline{S_1}$. In order to be dominated at least twice, the neighbor must be in $N_G(v)$ and thus, v is a universal vertex of G. By Theorem 4.1, $\overline{G} = K_1 \oplus sK_2$, where $s \geq 1$. $\qquad\square$

For each component $\overline{G_i}$ of $\overline{G}$, let $\overline{D_i} = \overline{G_i} \setminus \overline{S_1}$ and $\overline{D} = \bigcup_{i \in [t]} \overline{D_i}$. Let D_i (resp. D) be the corresponding vertices of $\overline{D_i}$ (resp. $\overline{D}$) in G_i (resp. G), with $S_0 \subseteq D$. Let $n_i = |V(G_i)|$.

Lemma 4.3. *If G is a connected cograph such that $|\overline{G_i}| \geq 2$, $i \in [t]$, where $t \geq 2$, and that $G\overline{G}$ has an OLD_{oind}-set $S = S_0 \cup \overline{S_1}$ with $S_0 \subseteq V(G)$ and $\overline{S_1} \subseteq V(\overline{G})$, then the following statements hold:*
  (i) *$S_0 \neq \emptyset$ and $\overline{S_1} \neq \emptyset$.*
  (ii) *For each $i \in [t]$, $V(\overline{G_i}) \cap \overline{S_1} \neq \emptyset$.*
  (iii) *If $i \in [t]$, $n_i \geq 3$ and $|S_0 \cap V(G_i)| \leq 1$, then $\overline{D_i}$ is an independent set of $\overline{G}$ with $1 \leq |\overline{D_i}| \leq 2$.*
  (iv) *If $|S_0| = 2$, then G has two anticomponents G_i and G_j say, having nonempty intersection with S_0, and $n_i, n_j \geq 3$. Moreover, $2 \leq |\overline{D}| \leq 3$.*
  (v) *There exists at most one $i \in [t]$ such that $n_i \geq 3$ and $S_0 \cap V(G_i) = \emptyset$.*
  (vi) *$\overline{G}$ has at least one and at most three components of size at least three.*

Proof: We prove each item separately.

  (i) For a proof by contradiction, suppose that $S_0 = \emptyset$. Let $v \in V(G)$. By Theorem 2.2, vertex v has to be open-dominated at least twice. Since v has at most one neighbor in $\overline{S_1}$, we can conclude that S is not an OLD_{oind}-set in $G\overline{G}$. The proof for $\overline{S_1}$ follows analogously.

  (ii) For a proof by contradiction, suppose that there is an $i \in [t]$ such that $V(\overline{G_i}) \cap \overline{S_1} = \emptyset$. Let $\overline{v} \in V(\overline{G_i})$. By Theorem 2.2, vertex $\overline{v}$ has to be open-dominated at least twice. Since $\overline{v}$ has at most one neighbor in S_0, we can conclude that S is not an OLD_{oind}-set in $G\overline{G}$.

  (iii) If $\overline{D_i}$ is empty, as G is connected, $\overline{G_i}$ is isomorphic to K_2 and thus $n_i = 2$. Suppose $|\overline{D_i}| \geq 3$. Then the vertices in D_i have the same neighborhood as those in in $G \setminus V(G_i)$. Since $|V(G_i) \cap S_0| \leq 1$, D_i has at least two vertices that are not distinguished by S. Now, suppose that $\overline{D_i}$ has two vertices and they are adjacent in $\overline{G}$. In this case the vertices in D_i are not adjacent in G_i. If one of them is a member of S_0, the other vertex is dominated once, since S_0 is open independent, it can possess at most one vertex of another anticomponent G_j say, with $i \neq j$. If neither of the vertices is a member of S_0, they have the same neighborhood in S_0 and they are not distinguished by S.

  (iv) Suppose $|S_0| = 2$. We first prove that G has two anticomponents G_i and G_j say, that have nonempty intersection with S_0, and that $n_i, n_j \geq 3$. For a proof by contradiction, suppose that there is an unique anticomponent G_i of G with $|G_i \cap S_0| = 2$. Let $S_0 = \{u, v\}$. Since G_i is disconnected, there is a vertex $z \in V(G_i)$ that is adjacent to neither u nor v. Since z has at most one neighbor in $\overline{S_1}$, it follows that S is not an OLD_{oind}-set. Now, suppose that $|V(G_i) \cap S_0| = 1$ and that $n_i = 2$. Let $V(G_i) = \{u, v\}$. Without loss of generality, assume $S_0 = \{u\}$. If $\overline{u} \in \overline{S_1}$, then $\overline{v} \notin \overline{S_1}$ and $\overline{v}$ is

dominated once by S, contradicting the fact that S is an OLD_{oind}-set of $G\overline{G}$. If $\overline{u} \notin \overline{S}_1$, since it has to be dominated twice, $\overline{v} \in \overline{S}_1$ and, since S_0 is open independent, $v \in S_0$. Thus $u, v \in S_0$, which contradicts the fact that there are two anticomponents in G with nonempty intersection with S_0.

Next, we prove that $2 \leq |\overline{D}| \leq 3$. By (i) above, there are two anticomponents of G of size at least three, G_i and G_j say, that have nonempty intersection with S_0. Since $|\overline{D}_i|, |\overline{D}_j| \geq 1$, it follows that $|\overline{D}| \geq 2$. For the upper bound, since $|S_0 \cap V(G_k)| \leq 1$, for every $k \in [t]$, by Lemma 4.3 (iii) the vertices of $\overline{D}_k$ are independent in $\overline{G}$. Also, as the vertices of the distinct components of $\overline{G}$ are not adjacent, $\overline{D}$ is an independent set of $\overline{G}$ and D induces a complete graph in G. If $|D| \geq 4$, since at most two of these vertices are in S_0, and they have no neighbors in $\overline{S}_1$, the remainder (at least two vertices) are not distinguished by S.

(v) Suppose that there are two anticomponents of G of order at least 3, say G_i and G_j, that do not have vertices in common with S_0. By (iii), $|D_i \cup D_j| \geq 2$. However, the vertices in $D_i \cup D_j$ are not distinguished by S_0, since they have the same neighborhood in $G \setminus (V(G_i) \cup V(G_j))$.

(vi) Firstly, suppose that each component of $\overline{G}$ is a K_2. By (i), $S_0 \neq \emptyset$. Suppose that $v \in V(G_i)$ with $v \in S_0$. If $\overline{v} \in \overline{S}_1$, its unique neighbor in $\overline{G}$, $\overline{u}$ say, does not belong to $\overline{S}_1$ and then, in order to dominate $\overline{u}$ twice, $u \in S_0$. This implies that u has another neighbor in S_0 that is a member of an anticomponent different from G_i. This contradicts the assumption that S is open-independent. If $\overline{v} \notin \overline{S}_1$, the vertex $\overline{u}$ is in $\overline{S}_1$ and we have an analogous situation. Thus, it can be concluded that $S_0 = \emptyset$, which contradicts the premise of (i). Therefore, $\overline{G}$ has at least one component of size at least three. Secondly, for a proof by contradiction, suppose that at least four components of $\overline{G}$ have size at least three. Since the elements of S_0 are open-independent, S_0 contains vertices that are members of at most two distinct anticomponents of G. Thus, there are at least two anticomponents of G with order at least 3, say G_i and G_j, that do not have vertices in common with S_0. By (v), S is not an OLD_{oind}-set.

$\square$

For $\ell, m \geq 1$, the graph $\overline{K_\ell} \bowtie mK_2$ that contains $\ell + 2m$ vertices is denoted by $R_{\ell,m}$. Also, $R^*_{\ell,m}$ denotes the graph obtained from $R_{\ell,m}$ with one edge missing between one vertex of $\overline{K_\ell}$ and exactly one copy of K_2. Note that $R_{\ell,m}$ is a cograph; $R^*_{\ell,1}$ for $\ell = 1, 2$, is also a cograph; $R_{1,1}$ is isomorphic to K_3; and $R^*_{1,1}$ is isomorphic to P_3.

Lemma 4.4. *Let G be a connected cograph such that $|\overline{G}_i| \geq 2$, $i \in [t]$, where $t \geq 2$, and that $G\overline{G}$ has an OLD_{oind}-set $S = S_0 \cup \overline{S}_1$ with $S_0 \subseteq V(G)$ and $\overline{S}_1 \subseteq V(\overline{G})$ and let $i \in [t]$ such that $n_i \geq 3$.*
   (i) If $V(G_i) \cap S_0 = \emptyset$, then $\overline{G}_i \cong R_{1,m}$.
   (ii) If $|V(G_i) \cap S_0| = 1$, then $\overline{G}_i$ is isomorphic to one of $R^*_{\ell,1}$ and $R_{\ell,m}$, for $\ell = 1, 2$.

Proof:

To prove (i), suppose that $n_i \geq 3$ and $V(G_i \cap S_0 = \emptyset)$. By Lemma 4.3 (ii), $V(\overline{G}_i) \cap \overline{S}_1 \neq \emptyset$. Let $\overline{A}_i = V(\overline{G}_i) \cap \overline{S}_1$. By Lemma 4.3 (iii), $\overline{D}_i$ induces an independent set with $1 \leq |\overline{D}_i| \leq 2$. The set $\overline{A}_i$ induces $m = |\overline{A}_i|/2$ independent edges. Since $\overline{G}_i$ is connected, at least one vertex of every edge induced by $\overline{A}_i$ has a neighbor in $\overline{D}_i$. If $|\overline{D}_i| = 2$, the two vertices of D_i are not distinguished by S, since they have the same neighborhood in $G \setminus V(G_i)$. Thus, we may assume $|\overline{D}_i| = 1$. Let $\overline{D}_i = \{\overline{u}\}$. If $m = 1$, since $\overline{u}$ has at least two neighbors in $\overline{S}_1$, it follows that $\overline{G} \cong R_{1,1}$. If $m \geq 2$, we claim that $\overline{u}$ is adjacent to every vertex in $\overline{A}_i$. Suppose that this is not true and consider an edge ab induced by vertices in $\overline{A}_i$ such that $\overline{u}a \notin E(\overline{G}_i)$. In this case, $ab\overline{u}c$ is an induced 4-path in $\overline{G}_i$, where c is an vertex of some other edge induced by $\overline{A}_i$. So, $\overline{G}_i$ is not a cograph. Therefore, $\overline{G}_i \cong R_{1,m}$.

Now, we prove (ii). By Lemma 4.3 (ii), $V(\overline{G_i}) \cap \overline{S}_1 \neq \emptyset$. Let $\overline{A}_i = V(\overline{G_i}) \cap \overline{S}_1$. By Lemma 4.3 (iii), $\overline{D}_i$ induces an independent set with $1 \leq |\overline{D}_i| \leq 2$. The set $\overline{A}_i$ induces $m = |\overline{A}_i|/2$ independent edges. Since $\overline{G_i}$ is connected, at least one vertex incident with every edge induced by $\overline{A}_i$ has a neighbor in $\overline{D}_i$. If $m = 1$, we have four possible cographs, $R_{\ell,1}^*$, $R_{\ell,1}$, for $\ell = 1, 2$. If $m \geq 2$, we claim that every vertex in $\overline{D}_i$ is adjacent to every vertex in $\overline{A}_i$. For a proof by contradiction, suppose that this is not true and consider two edges ab and cd, induced by vertices in $\overline{A}_i$. Let $\overline{u} \in \overline{D}_i$. Without loss of generality, suppose that $\overline{u}a \notin E(\overline{G_i})$, for an edge ab induced by vertices in $\overline{A}_i$. In this case, $ab\overline{u}c$ is an induced 4-path in $\overline{G_i}$, where c is an vertex of some other edge induced by $\overline{A}_i$. So, $\overline{G_i}$ is not a cograph. Therefore, $\overline{G_i}$ is isomorphic to $R_{\ell,m}$, for $\ell = 1, 2$. $\square$

Next, we present in Theorems 4.5 and 4.6 a general recursive characterization of the class C of connected cographs such that if G is a member of C then $G\overline{G}$ has an OLD_{oind}-set. Theorem 4.5 (4.6) is illustrated in Figure 5(a) (5(b)).

Theorem 4.5. *Let G be a connected cograph such that $|\overline{G_i}| \geq 2$, $i \in [t]$ with $t \geq 2$. $G\overline{G}$ has an OLD_{oind}-set $S = S_0 \cup \overline{S}_1$ with $S_0 \subseteq V(G)$ and $\overline{S}_1 \subseteq V(\overline{G})$ such that $|S_0| = 2$ iff there are at least two, and at most three, components $\overline{G_i}$ with $n_i \geq 3$ and each component of $\overline{G}$ is isomorphic to one of the graphs K_2, $R_{\ell,1}^*$ or $R_{\ell,m}$, for $\ell = 1, 2$ and $m \geq 1$, with the following further conditions:*

(i) at most one component of $\overline{G}$ is either $R_{2,1}^*$ or $R_{2,m}$ and if $\overline{G}$ has such a component, $\overline{G}$ has exactly two components of size at least three; and

(ii) at most two components of $\overline{G}$ are isomorphic to $R_{1,1}^*$, $R_{2,1}^*$ or $R_{2,m}$.

Proof: Suppose that G is as described and that $G\overline{G}$ has an OLD_{oind}-set $S = S_0 \cup \overline{S}_1$ with $S_0 \subseteq V(G)$, $\overline{S}_1 \subseteq V(\overline{G})$, and $|S_0| = 2$. Let t' be the number of components of $\overline{G}$ of size at least three. By Lemma 4.3 (vi), $t' \leq 3$. By Lemma 4.3 (iv), two anticomponents of G have nonempty intersection with S_0 and both have size at least three, thus $2 \leq t' \leq 3$.

Since $|V(G_i) \cap S_0| \leq 1$, for every component $\overline{G_i}$ of $\overline{G}$ with $n_i \geq 3$, by Lemma 4.4 (ii), the components of $\overline{G}$ of size at least three are $R_{\ell,1}^*$ or $R_{\ell,m}$, for $\ell = 1, 2$. To prove (i), firstly suppose that there are at least two components of $\overline{G}$ that are one of the graphs $R_{2,1}^*$ and $R_{2,m}$, say $\overline{G_i}$ and $\overline{G_j}$. By Lemma 4.3 (iii), $\overline{D}_i$ (resp. $\overline{D}_j$) is independent with size at least one and at most two. If $|\overline{D}_i| = 1$ (resp. $|\overline{D}_j| = 1$) the remaining vertices of $\overline{G_i}$ (resp. $\overline{G_j}$) cannot induce independent edges. Hence $|\overline{D}_i| = |\overline{D}_j| = 2$ which implies that $|\overline{D}| \geq 4$. By Lemma 4.3 (iv), $|S_0| \neq 2$, contradicting the premise of the theorem. Hence we can conclude that at most one component of $\overline{G}$ is either $R_{2,1}^*$ or $R_{2,m}$. Secondly, suppose that $\overline{G}$ has exactly one component, say $\overline{G_i}$, which is isomorphic to either $R_{2,1}^*$ or $R_{2,m}$. Thus $|\overline{D}_i| = 2$. Suppose that $\overline{G}$ has other two components, say $\overline{G_j}$ and $\overline{G_p}$, each one with size at least three. By Lemma 4.3 (iii), $|\overline{D}_j \cup \overline{D}_p| \geq 2$. Since $|\overline{D}_i| = 2$, $|\overline{D}| \geq 4$. Again, by Lemma 4.3 (iv), $|S_0| \neq 2$, contradicting the premise of the theorem. This completes the proof of (i). Now we prove (ii). If $t' = 3$, since $|S_0| = 2$ one of the components of $\overline{G}$ with size at least three, say $\overline{G_j}$, is such that G_j has empty intersection with S_0. Thus by Lemma 4.4 (i), $\overline{G_j} \cong R_{1,m}$, which implies that at most two components of $\overline{G}$ are isomorphic to the graphs $R_{1,1}^*$, $R_{2,1}^*$, and $R_{2,m}$. The difference $t - t'$ is the number of components of $\overline{G}$ of size two. Clearly, each of these components (if any) is isomorphic to K_2.

Conversely, suppose $G\overline{G}$ is as described, with at least two, and at most three, components $\overline{G_i}$ with $n_i \geq 3$, where the components of $\overline{G}$ are the graphs K_2, $R_{\ell,1}^*$, and $R_{\ell,m}$, for $\ell = 1, 2$. Assume that conditions (i) and (ii) hold. We show how to select the two vertices in S_0 and the vertices in $\overline{S}_1$ such that $S = S_0 \cup \overline{S}_1$ is an OLD_{oind}-set of $G\overline{G}$. For every component $\overline{G_i}$ of $\overline{G}$ there is a partition of this vertex set into sets $\overline{D}_i$ and $\overline{M}_i$ such that $\overline{D}_i$ is an independent set with $|\overline{D}_i| \in \{0, 1, 2\}$ and $\overline{M}_i$ contains the $2m$ vertices of the $m \geq 1$ independent edges of $\overline{G_i}$. Let $\overline{D}_i$ and $\overline{M}_i$ be such sets. Let $\overline{D} = \bigcup_{i \in [t]} \overline{D}_i$. Then, $\overline{S}_1 = \bigcup_{i \in [t]} \overline{M}_i = \overline{M}$. Let D_i, M_i, D, and M be the sets of corresponding vertices in G. Note that in G, every vertex in M

has a different neighbor in $\overline{M}$. Since we shall show that S_0 has vertices in two distinct anticomponents, every vertex of G has at least one neighbor in S_0. Hence, the vertices in $M \cup \overline{M}$ are distinguished and open-dominated by S and we need to consider only the vertices in $D \cup \overline{D}$.

We proceed by setting the vertices in S_0. We first consider the case $t' = 2$. Without loss of generality, assume $|G_1|, |G_2| \geq 3$. Since conditions (i) and (ii) hold, by symmetry, we can assume that $\overline{G_1} \in \{R_{1,1}^*, R_{2,1}^*, R_{1,m}, R_{2,m}\}$ and $\overline{G_2} \in \{R_{1,1}^*, R_{1,m}\}$.

If $|D_1| = 1$, let $D_1 = \{v\}$. If $|D_1| = 2$ and $\overline{G_1}$ is isomorphic to $R_{2,1}^*$, let $D_1 = \{u, v\}$, with v being the vertex of degree two in G_1. If $|D_1| = 2$ and $\overline{G_1}$ is isomorphic to $R_{2,m}$, let $D_1 = \{u, v\}$. Let $\overline{u}, \overline{v}$ be the corresponding vertices of u, v in $\overline{G_1}$. We set $v \in S_0$, which gives rise to the following cases:

1. If $\overline{G_1}$ is isomorphic to $R_{1,1}^*$, then $\overline{v}$ has one neighbor in $\overline{M_1}$ and $v \in N_G(\overline{v})$.
2. If $\overline{G_1}$ is isomorphic to $R_{2,1}^*$, then $\overline{u}$ has two neighbors in $\overline{M_1}$ and $\overline{v}$ has exactly one neighbor in $\overline{M_1}$. Since $v \in S_0$, $\overline{u}$ and $\overline{v}$ are open-dominated twice and distinguished by S.
3. If $\overline{G_1}$ is isomorphic to $R_{2,m}$, then $\overline{u}$ and $\overline{v}$ are both neighbors of every vertex in $\overline{M_1}$. Since $v \in S_0$, these two vertices are open-dominated and distinguished by S.
4. If $\overline{G_1}$ is isomorphic to $R_{1,m}$, then $\overline{v}$ is the unique vertex that is the neighbor of all vertices of M_1.

As $\overline{G_2} \in \{R_{1,1}^*, R_{1,m}\}$, $|D_2| = 1$. Let $D_2 = \{v\}$. We set $v \in S_0$. We have two possibilities for $\overline{G_2}$ that are analogous to the cases 1 and 4, described above. We can conclude that all vertices in $\overline{G}$ are distinguished and open-dominated by S, and S is open-independent.

Now we consider the vertices in D. Since at most one component of $\overline{G}$ is isomorphic to either $R_{2,1}^*$ or $R_{2,m}$, it follows that $2 \leq |D| \leq 3$. Two of the vertices in D are also in S_0. So, there is at most one vertex of G, z say, such that neither z nor $\overline{z}$ is in S. The choosing of S_0 guarantees that z is the only vertex of G that is dominated only by S_0.

Finally, we consider the case $t' = 3$. Without loss of generality, we assume that $|G_j| \geq 3$, for every $j \in [3]$. Since conditions (i) and (ii) hold, and by symmetry, we assume that $\overline{G_1}, \overline{G_2} \in \{R_{1,1}^*, R_{1,m}\}$ and $\overline{G_3}$ is isomorphic to $R_{1,m}$. Thus, $|D_1| = |D_2| = 1$ and we set $S_0 = D_1 \cup D_2$.

As $\overline{G_1}, \overline{G_2} \in \{R_{1,1}^*, R_{1,m}\}$, we can use similar arguments to those used in the case $t' = 2$ in order to conclude that each vertex in $\overline{D_1} \cup \overline{D_2}$ is both dominated at least twice and distinguished by S. Since $S_0 = D_1 \cup D_2$, these vertices are distinguished by S. Again, the vertex in D_3 is the only vertex of G that is dominated only by S_0. Therefore, S is an OLD_{oind}-set of $G\overline{G}$ with $|S_0| = 2$. See Fig. 5(a) for an example. ☐

Theorem 4.6. *If G is a connected cograph then $G\overline{G}$ has an OLD_{oind}-set iff either (i) G is isomorphic to one of the graphs $K_1 \bowtie \overline{mK_2}$, $m \geq 1$, and graph G is as described in Theorem 4.5, or (ii) G can be obtained from them recursively by the following operation, where H is a connected cograph such that $H\overline{H}$ has an OLD_{oind}-set. Set $G = \overline{H} \bowtie F \bowtie (\overline{rK_2})$, where $F \in \{\overline{R_{1,m}}, \overline{K_2}\}$, $r \geq 0$, $m \geq 1$.*

Proof: Suppose that G is a connected cograph such that $G\overline{G}$ has an OLD_{oind}-set $S = S_0 \cup \overline{S}_1$ where $S_0 \subseteq V(G)$ and $\overline{S}_1 \subseteq V(\overline{G})$. Assume that $\overline{G}$ has components $\overline{G_1} \ldots, \overline{G_t}$, with $t \geq 2$. By Lemma 4.3 (i), S_0 is nonempty. If $|S_0| = 1$, then by Theorem 4.2, G is the graph $K_1 \bowtie \overline{mK_2}$, and if $|S_0| = 2$, then G is the graph described in Theorem 4.5. So, we may assume that $|S_0| \geq 3$. By Lemma 4.3 (vi), at least one, and at most three, components of $\overline{G}$ have at least three vertices. Since $|S_0| \geq 3$, there exists exactly one anticomponent G_i say, of G with nonempty intersection with S_0. (If this were not so, since every vertex of G_i is adjacent to every vertex of any other anticomponent G_p with $p \neq i$, then S would not be an open-independent set.) By Lemma 4.3 (v), there exists at most one anticomponent G_i say, of G with size at least three with empty intersection with S_0. This implies that G has at most two anticomponents G_i, with size at least three. If G has an anticomponent G_i of size at least three having empty intersection with S_0, by

(a) The components of $\overline{G}$ are the graphs $R_{2,2}$, $R_{1,1}^*$, and K_2.

(b) G can be obtained recursively from $K_1 \bowtie \overline{2K_2}$. $G = \overline{\overline{K_1 \bowtie \overline{2K_2}}} \bowtie \overline{K_2} \bowtie \overline{K_2} \bowtie \overline{R_{1,1}} \bowtie \overline{R_{1,1}} \bowtie \overline{K_2} = ((((K_1 \bowtie \overline{2K_2}) \oplus K_2) \bowtie \overline{K_2}) \oplus R_{1,1}) \bowtie \overline{R_{1,1}} \bowtie \overline{K_2}$.
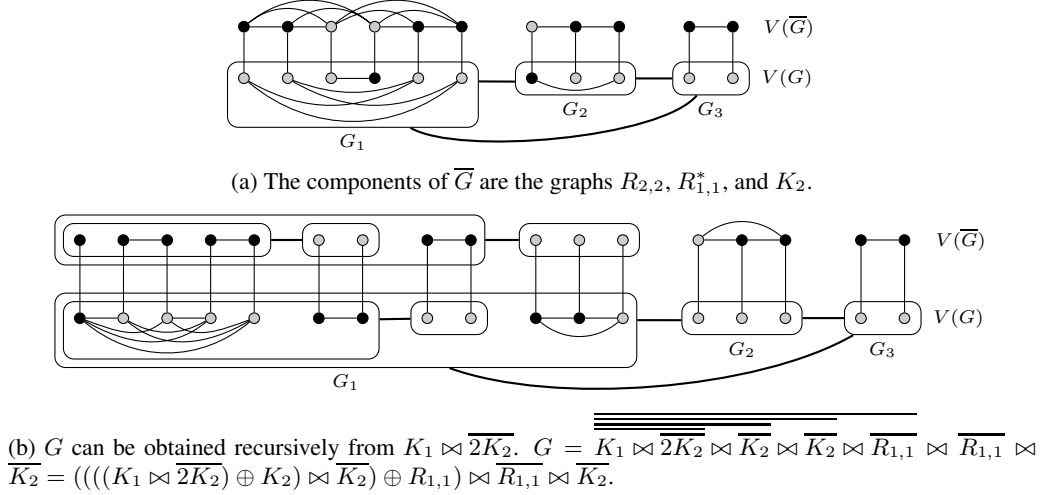
Fig. 5: The complementary prisms of cographs having OLD_{oind}-sets. 5(a) illustrates Theorem 4.5 and 5(b) illustrates Theorem 4.6. An edge between two rectangles indicates adjacency between all pairs of vertices, one in each rectangle.

Lemma 4.4 (i), this anticomponent is isomorphic to $\overline{R_{1,m}}$. Otherwise, if $|G_i| = 2$, then G_i is isomorphic to $\overline{K_2}$. Without loss of generality, we assume that G_1 is an anticomponent of G with $S_0 \subseteq V(G_1)$ and that G_2 is an anticomponent that is isomorphic to either $\overline{R_{1,m}}$ or to $\overline{K_2}$. Note that each of the $t - 2$ anticomponents of G (if any) is isomorphic to $\overline{K_2}$. So, until now, we have that $G = G_1 \bowtie G_2 \bowtie (\overline{rK_2})$, with $r \geq 0$, for some disconnected graph G_1 (since $\overline{G_1}$ is connected). Now, suppose that $G_1\overline{G_1}$ does not have an OLD_{oind}-set. Thus, $S' = S \cap V(G_1\overline{G_1})$ is not an OLD_{oind}-set of $G_1\overline{G_1}$, and there is a vertex v say, of $G_1\overline{G_1}$ that is neither distinguished nor open-dominated by S'. Since $S_0 \subseteq V(G_1)$, and there is no edge between $\overline{G_1}$ and any other component of $\overline{G}$, we have that $N_{G\overline{G}}(v) \cap S \subseteq V(G_1 \cup \overline{G_1})$. This implies that v is neither distinguished nor dominated by S and hence, S is not an OLD_{oind}-set of $G\overline{G}$. Therefore, G can be obtained by the described operation from a disconnected graph G_1 such that $G_1\overline{G_1}$ has an OLD_{oind}-set.

Conversely, suppose that G is a connected cograph obtained as described in the statement of the theorem. Since, by Theorem 4.2, $K_1 \bowtie \overline{mK_2}$ has an OLD_{oind}-set, and is also the graph described in Theorem 4.5, we assume that G is none of these graphs. Hence we assume that $G = \overline{H} \bowtie F \bowtie (\overline{rK_2})$ and H is a connected cograph such that $H\overline{H}$ has an OLD_{oind}-set $B = B_0 \cup \overline{B_1}$ with $B_0 \subseteq V(H)$ and $\overline{B_1} \subseteq V(\overline{H})$. We show how to select the vertices of a set $S = S_0 \cup \overline{S_1}$ where $S_0 \subseteq V(G)$ and $\overline{S_1} \subseteq V(\overline{G})$ such that S is an OLD_{oind}-set of $G\overline{G}$. We begin setting $S_0 = \overline{B_1}$ and $\overline{S_1} = B_0$. Hence, the vertices of $H \cup \overline{H}$ are all distinguished and open-dominated by S and we need to consider only the remainder of the vertices. We analyze the two relevant cases. In the first case, assume that F is isomorphic to $\overline{R_{1,m}}$. Here, we must add to $\overline{S_1}$ the vertices that induce the m independent edges in $\overline{F} = R_{1,m}$. Let $\overline{v}$ be the universal vertex in $\overline{F}$. Then $\overline{v}$ is dominated at least twice by the vertices incident with the $m \geq 1$ edges of $\overline{F}$ and is the unique vertex with this neighborhood in S. The vertices in F are adjacent to all the vertices in S_0. Since $\overline{H}$ is disconnected and hence, $H\overline{H}$ has an OLD_{oind}-set, by Lemma 4.3 (ii), $\overline{B_1}$ has at least one vertex in each component of $\overline{H}$. Let v be the neighbor of $\overline{v}$ in F. Each vertex in $V(F) \setminus \{v\}$ has a distinct neighbor in $\overline{S_1}$ and v is the unique vertex whose neighborhood in S is exactly S_0. Observe that $\overline{B_1}$ has size at least three and recall that S_0 was set to it. In the second case, assume that F is isomorphic to $\overline{K_2}$. In this case we add the vertices of $\overline{F} = K_2$ to the set $\overline{S_1}$ and, by using similar arguments to those in the first case, it is easy to see that the vertices of F

are open-dominated and distinguished by S. If $t \geq 3$, each of the $t - 2$ other components of $\overline{G}$ (if any) is isomorphic to K_2 and we proceed in the same way as in the former case. Therefore, S is an OLD_{oind}-set of $G\overline{G}$. See Fig. 5(b) for an example of a graph $G\overline{G}$ obtained by this operation from $K_1 \bowtie \overline{2K_2}$. $\qquad\square$

5   Summary

The problem of deciding whether or not a graph G has an OLD_{oind}-set has important applications and was shown above to be $\mathcal{NP}$-complete even for the special cases when G is either a planar bipartite graph of maximum degree five and girth six, or a planar subcubic graph of girth nine. Characterizations of both the P_4-tidy graphs and the cographs that have OLD_{oind}-sets have been presented. Also, necessary and sufficient conditions for a complementary prism of a connected cograph to have an OLD_{oind}-set are derived. For future work, it might be fruitful to study the complexity of identifying OLD_{oind}-sets in other families of graphs, such as those that are Hamiltonian, Eulerian or n-partite when $n \geq 3$.

Acknowledgements

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