Graphs containing finite induced paths of unbounded length

Maurice Pouzet^{1,2}

Imed Zaguia^{3,4}*

- ¹ Université Claude-Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, Villeurbanne, France
- ² Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada
- Department of Mathematics and Computer Science, Royal Military College of Canada, Kingston, Ontario, Canada
- ⁴ Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada

received 18th Nov. 2020, revised 5th Aug. 2021, accepted 15th Nov. 2021.

The age $\mathcal{A}(G)$ of a graph G (undirected and without loops) is the collection of finite induced subgraphs of G, considered up to isomorphy and ordered by embeddability. It is well-quasi-ordered (wqo) for this order if it contains no infinite antichain. A graph is *path-minimal* if it contains finite induced paths of unbounded length and every induced subgraph G' with this property embeds G. We construct 2^{\aleph_0} path-minimal graphs whose ages are pairwise incomparable with set inclusion and which are wqo. Our construction is based on uniformly recurrent sequences and lexicographical sums of labelled graphs.

Keywords: (partially) ordered set; incomparability graph; graphical distance; isometric subgraph, paths, well quasi order, symbolic dynamic, sturmian words, uniformly recurrent sequences

1 Introduction and presentation of the results

We consider graphs that are undirected, simple and have no loops. Among those with finite induced paths of unbounded length, we ask which ones are *unavoidable*. If a graph is an infinite path, it can be avoided in the following sense: it contains a direct sum of finite paths of unbounded length. This latter one, on the other hand, cannot be avoided. Indeed, two direct sums of finite paths of unbounded length embed in each other. Similarly, two complete sums of finite paths of unbounded length embed in each other. Hence, the direct sum, respectively the complete sum of finite paths of unbounded length are, in our sense, unavoidable. Are there other examples? This question is the motivation behind this article.

We recall that the age of a graph G is the collection Age(G) of finite induced subgraphs of G, considered up to isomorphy and ordered by embeddability (a graph H embeds in a graph H' if H is isomorphic to an induced subgraph of H'. cf. Fraïssé (2000)). It is well-quasi-ordered (wqo) for this order if it contains no infinite antichain. A path is a graph P such that there exists a one-to-one map f from the

ISSN 1365-8050

© 2021 by the author(s)

Distributed under a Creative Commons Attribution 4.0 International License

^{*}Corresponding author. Supported by Canadian Defence Academy Research Program, NSERC and LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon within the program "Investissements d'Avenir (ANR-11-IDEX-0007" operated by the French National Research Agency (ANR)

set V(P) of its vertices into an interval I of the chain \mathbb{Z} of integers in such a way that $\{u,v\}$ belongs to E(P), the set of edges of P, if and only if |f(u)-f(v)|=1 for every $u,v\in V(P)$. If $I=\{1,\ldots,n\}$, then we denote that path by P_n ; its length is n-1, so, if n=2, P_2 is made of a single edge, whereas if n=1, P_1 is a single vertex. We denote by P_∞ the path on \mathbb{N} . We say that a graph G is path-minimal if it contains induced paths of finite unbounded length and every induced subgraph G' of G with this property embeds a copy of G. Let $\bigoplus_n P_n$ respectively $\sum_n P_n$ be the direct sum, respectively the complete sum of paths P_n (these operations are a particular case of lexicographical sum defined in Subsection 4.1). These graphs are path-minimal graphs. There are others. Our main result is this.

Theorem 1 There are 2^{\aleph_0} path-minimal graphs whose ages are pairwise incomparable and wqo for embeddability.

Our construction uses uniformly recurrent sequences, and in fact Sturmian sequences (or billiard sequences) Morse and Hedlund (1940), and lexicographical sums of labelled graphs. The existence of 2^{\aleph_0} wqo ages is a nontrivial fact. It was obtained for binary relations in Pouzet (1978b) and for undirected graphs in Sobrani (1992) and in Sobrani (2002). The proofs were based on uniformly recurrent sequences. We use similar ideas in Laflamme et al. (2014) to prove that there are 2^{\aleph_0} linear orders L of order type ω on $\mathbb N$ such that L is orthogonal to the natural order on $\mathbb N$ (see Theorem 2 and Corollary 2 of Laflamme et al. (2014)).

We cannot expect to characterize path-minimal up to isomorphy. Indeed, recall that two graphs are equimorphic if they embed in each other. In general, equimorphic graphs are not isomorphic. Now, if a graph G is path-minimal then as it is easy to observe, every graph equimorphic to G is path-minimal.

We leave open the following:

Problems 1 (i) If a graph embeds finite induced paths of unbounded length, does it embed a path-minimal graph?

- (ii) If a graph is path-minimal, is its age wqo?
- (iii) If a graph G is path-minimal, how many path-minimal graphs with the same age are they?
- (iv) If a graph G is path-minimal, is G can be equipped with an equivalence relation \equiv whose blocks are paths in such a way that (G, \equiv) is path-minimal?

In some situations, $\bigoplus_n P_n$ and $\sum_n P_n$ are the only path-minimal graphs (up to equimorphy).

Theorem 2 If the incomparability graph of a poset embeds finite induced paths of unbounded length, then it embeds the direct sum or the complete sum of finite induced paths of unbounded length.

If x,y are two vertices of a graph G, we denote by $d_G(x,y)$ the length of the shortest path joining x and y if any, and $d_G(x,y) := +\infty$ otherwise. This defines a distance on the vertex set V of G whose values belong to the completion $\overline{\mathbb{N}}^+ := \mathbb{N}^+ \cup \{+\infty\}$ of non-negative integers. This distance is the *graphic distance*. If A is a subset of V, the graph G' induced by G on A is an *isometric subgraph* of G if $d_{G'}(x,y) = d_G(x,y)$ for all $x,y \in A$.

If instead of induced paths we consider isometric paths, then

Theorem 3 If a graph embeds isometric finite paths of unbounded length, then it embeds a direct sum of such paths.

We examine the primality of the graphs we obtain. Direct and complete sums of finite paths of unbounded length are not prime in the sense of Subsection 3.3, and not equimorphic to prime graphs. We construct 2^{\aleph_0} examples, none of them being equimorphic to a prime one (Theorem 25). We construct also 2^{\aleph_0} which are prime (Theorem 25). These examples are minimal in the sense of Pouzet (1979), but not in the sense of Pouzet and Zaguia (2009).

We conclude this introduction with:

1.0.1 An outline of the proof of Theorem 1.

It uses two main ingredients. One is the so called uniformly recurrent sequences (or words).

Let L be a set, a *uniformly recurrent* word over L with domain $\mathbb N$ is a sequence $u:=(u(n))_{n\in\mathbb N}$ of elements of L such that for any given integer n there is some integer m(u,n) such that every factor v of u of length at most n appears as a factor of every factor of u of length at least m(u,n). Allouche and Shallit (2003); Berthé and Rigo (2010); Lothaire (2002). To a uniformly recurrent word u we associate the labelled path P_u on $\mathbb N$ where the label of vertex n is the value u(n) of the sequence u. If the alphabet L is $\{0,1\}$, we may view P_u as a path with a loop at every vertex n for which u(n)=1 and no loop at vertices for which u(n)=0. Next comes the second ingredient.

Fix a map $\star: L \times L \to \{0,1\}$ and denote by $i \star j$ the image of (i,j). Let $G_{(u,\star)}$ be the lexicographical sum over the chain ω of non-negative integers of copies of the labelled path P_u . This is a labelled graph whose vertex set is $\mathbb{N} \times \mathbb{N}$ with labelling ℓ . Sets of the form $\{i\} \times \mathbb{N}$ are the *components* of $G_{(u,\star)}$. On each component $\{i\} \times \mathbb{N}$, the graph is a copy of P_u , the label $\ell(i,n)$ being u(n), two vertices (i,n) and (j,m) of $G_{(u,\star)}$, such that i < j, are linked by an edge if $\ell(i,n) \star \ell(j,m) = 1$.

Since u is uniformly recurrent, the set Fac(u) of finite factors of u is woo w.r.t the factor ordering hence, by a theorem of Higman on words Higman (1952), the ages of P_u and of $G_{(u,\star)}$ are wqo. Deleting the loops, we get a graph that we denote $\widehat{G}_{(u,\star)}$ and whose age is also wqo. Let A be a subset of V:= $V(\widehat{G}_{(u,\star)})$ such that for each non-negative integer i, the restriction of $\widehat{G}_{(u,\star)}$ to $A \cap (\{i\} \times \mathbb{N})$ is a finite path, and furthermore, the length of these various paths is unbounded. Then this restriction has the same age as $G_{(u,\star)}$ furthermore, two restrictions obtained by this process are equimorphic (i.e. embeddable in each other), thus we do not refer to the family of path and denote by $\widehat{Q}_{(u,\star)}$ any of these restrictions. We prove that $\widehat{Q}_{(u,\star)}$ is path-minimal (Proposition 17). If the operation \star is constant and equal to 0, respectively equal to 1, $Q_{(u,\star)}$ is a direct sum, respectively a complete sum of paths. To conclude the proof of the theorem, we need to prove that there is some operation \star and 2^{\aleph_0} words u such that the ages of $\widehat{G}_{(u,\star)}$ are incomparable. This is the substantial part of the proof. For that, we prove in Lemma 20 that if $L = \{0, 1\}$ and \star is the Boolean sum or a projection and u is uniformly recurrent then every long enough path in $G_{(u,\star)}$ is contained in some component. This is a rather technical fact. We think that it holds for any operation. We deduce that if Fac(u) and Fac(u') are not equal up to reversal or to addition (mod 2) of the constant word 1 the ages of $\widehat{G}_{(u,\star)}$ and $\widehat{G}_{(u',\star)}$ are incomparable w.r.t. set inclusion (Proposition 21). To complete the proof of Theorem 1, we then use the fact that there are 2^{\aleph_0} uniformly recurrent words u_{α} on the two letters alphabet $\{0,1\}$ such that for $\alpha \neq \beta$ the collections $Fac(u_{\alpha})$ and $Fac(u_{\beta})$ of their finite factors are distinct, and in fact incomparable with respect to set inclusion (this is a well known fact of symbolic dynamic, e.g. Sturmian words with different slopes will do).

2 Prerequisites

We denote by ω the order type of the chain (\mathbb{N}, \leq) of non negative integers, by ω^d the order type of its reverse, by ζ the order type of (\mathbb{Z}, \leq) .

The graphs we consider are undirected, simple and have no loops. That is, a graph is a pair G := (V, E), where E is a subset of $[V]^2$, the set of 2-element subsets of V. Elements of V are the vertices of G and elements of E its edges. We denote by $x \sim y$ the fact that the pair $\{x,y\}$ forms an edge. The graph G be given, we denote by V(G) its vertex set and by E(G) its edge set. If x, y are two vertices of G, we denote by $d_G(x,y)$ the length of the shortest path joining x and y if any, and $d_G(x,y) := +\infty$ otherwise. This defines a distance on V(G) with values in the completion $\overline{\mathbb{N}}^+ := \mathbb{N}^+ \cup \{+\infty\}$ of the non-negative integers. This distance is the graphic distance. On each connected component, this is an ordinary distance with integer values. If x is any vertex of G and $r \in \mathbb{N}$, the ball of center x, radius r, is the set $B_G(x,r) := \{ y \in V(G) : d_G(x,y) \le r \}$. More generally, if X is a subset of V, we set $B_G(X,r) := \bigcup_{x \in X} B_G(x,r)$. The diameter of G, denoted by δ_G , is the supremum of $d_G(x,y)$ for $x,y \in V$. If A is a subset of V, the graph G' induced by G on A is an isometric subgraph of G if $d_{G'}(x,y) = d_{G}(x,y)$ for all $x,y \in A$. The supremum of the length of induced finite paths of G, denoted by D_G , is sometimes called the *detour* of G Buckley and Harary (1988). We denote by $D_G(x,y)$ the supremum of the lengths of the induced paths joining x to y. Evidently, if x and y are connected by a path, $d_G(x,y) \leq D_G(x,y)$. A graph G is locally finite if the neighbourhood of every vertex x of G is finite. Equivalently, the balls $B_G(x,r)$ are finite for every vertex x and integer $r \in \mathbb{N}$. It is easy to see that if a graph G is finite, the detour of G is finite; if the detour is finite and the graph is connected then the diameter is bounded; and if the diameter is bounded and G is locally finite then G is finite. We formally state below the well-known result of König Kőnig (1927) and refer to Polat (1998) and Watkins (1986) for significant results on infinite paths.

Proposition 4 If a connected graph is infinite and locally finite then it contains an infinite isometric path

We apply to graphs concepts of the theory of relations as developed in Fraïssé (2000). We feel free to use these concepts for labelled graphs as well without giving more details. A graph G embeds in a graph G' and we set $G \le G'$ if G is isomorphic to an induced subgraph of G. Two graphs G, G' which embed in each other are said to be equimorphic. A class C of graphs is hereditary if $G' \in C$ and $G \le G'$ imply $G \in C$. We recall that the age of a graph G is the collection Age(G) of finite graphs G which embed in G. A collection G of finite graphs is the age of a graph if and only if it is hereditary and up-directed (that is any pair G is inexhaustible if any two of its members embed disjointedly into a third. Equivalently, if G is a graph with age G then for every finite subset G of G the graph induced on G has the same age as G. Most of the time we consider the members of an age up to isomorphy.

2.1 Posets, Comparability and incomparability graphs

Throughout, $P \coloneqq (V, \leq)$ denotes an ordered set (poset), that is a set V equipped with a binary relation \leq on V which is reflexive, antisymmetric and transitive. We say that two elements $x, y \in V$ are *comparable* if $x \leq y$ or $y \leq x$, otherwise, we say they are *incomparable*. The *dual* of P denoted P^d is the order defined on V as follows: if $x, y \in V$, then $x \leq y$ in P^d if and only if $y \leq x$ in P. A set of pairwise comparable elements is called a *chain*. On the other hand, a set of pairwise incomparable elements is called an *antichain*.

The comparability graph, respectively the incomparability graph, of a poset $P \coloneqq (V, \leq)$ is the undirected graph, denoted by Comp(P), respectively Inc(P), with vertex set V and edges the pairs $\{u,v\}$ of comparable distinct vertices (that is, either u < v or v < u) respectively incomparable vertices. We set $x \sim y$ if x and y are comparable, and $x \parallel y$ otherwise. A graph $G \coloneqq (V,E)$ is a comparability graph if the edge set is the set of comparabilities of some order on V. From the Compactness Theorem of First Order Logic, it follows that a graph is a comparability graph if and only if every finite induced subgraph is a comparability graph. Hence, the class of comparability graphs is determined by a set of finite obstructions. The complete list of minimal obstructions was determined by Gallai Gallai (2001).

2.2 Words, age of words, uniformly recurrent words

We recall basic facts on words Allouche and Shallit (2003); Lothaire (2002); Fogg (2002).

Let A be a finite set, k be the number of its elements. A *word* u is a sequence of elements of A, called *letters*, whose domain is an interval I of \mathbb{Z} . If this sequence is finite, the *length* of u is the length |u| of the sequence. In this case, we may suppose that the domain is $[0, \ldots n[$. If u is a word, and I is an interval of its domain, the *restriction* of u to I is denoted by $u_{\uparrow I}$. We denote by \square the empty word. If I is infinite, we may suppose that the domain is \mathbb{N} , $\mathbb{N}^* := \{0, -1, \ldots, -n \ldots\}$ or \mathbb{Z} . If u is a finite word and v is a word, finite or infinite with domain \mathbb{N} , the *concatenation* of u and v is the word uv obtained by writing v after u. If v has domain \mathbb{N}^* , the word v is similarly defined. A word v is a factor of u if $u = u_1vu_2$. This defines an order on the collection of finite words, the factor ordering. The age of a word u is the set Fac(u) of all its finite factors endowed with the factor order. We note that a set A of finite words is the age of a word u if A is an ideal for the factor ordering, that is a non-empty set of finite words which is an initial segment (that is if $v \in A$, then every factor of v is also in A) and is up-directed (that is if $v, v' \in A$, then there exists $v'' \in A$ such that v and v' are factors of v''). Note that the domain of u is not necessarily \mathbb{N} , it can be \mathbb{N}^* or \mathbb{Z} . The age of a word u is inexhaustible if for every $v, v' \in Fac(u)$ there is some w such that $v'wv' \in Fac(u)$. If an age is inexhaustible this is the age of a word on \mathbb{N} . A word u is inexhaustible.

A word u on \mathbb{N} is *uniformly recurrent* if for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that each factor u(p), ..., u(p+n) of length n occurs as a factor of every factor of length m.

There is a relation between uniformly recurrent words and well-quasi-ordering via the notion of Jónsson poset. An ordered set (poset) P is a Jónsson poset if it is infinite and every proper initial segment has a strictly smaller cardinality than P. Jónsson posets were introduced by Oman and Kearnes Kearnes and Oman (2013). Countable Jónsson posets were studied and described in Pouzet (1978b); Pouzet and Sauer (2006); Assous and Pouzet (2018). In particular, a countable poset P is Jónsson if and only if it is well-quasi-ordered, has height ω , and for each $n < \omega$, there is $m < \omega$ such that each element of height at most n is below every element of height at least m.

Their appearance in the domain of symbolic dynamics is due to the following result.

Theorem 5 Let u be a word with domain \mathbb{N} over a finite alphabet. The following properties are equivalent:

- (i) u is uniformly recurrent;
- (ii) Fac(u) is inexhaustible and wgo;
- (iii) Fac(u) equipped with the factor ordering is a countable Jónsson poset.

The only nontrivial implication is $(ii) \Rightarrow (iii)$ (Lemma II-2.5 of Ages belordonnés in Pouzet (1978b) p. 47). For reader's convenience, we prove it below (the proof was never published).

Lemma 6 Let A be an inexhaustible age of words. If A is not Jónsson then it is not wgo.

Proof: Suppose that \mathcal{A} contains a proper initial segment \mathcal{B} which is infinite. Let $\beta_0, \ldots \beta_n \ldots$ be a sequence of members of \mathcal{B} with $|\beta_n| < |\beta_{n+1}|$ for every $n \in \mathbb{N}$. Let u be a word on \mathbb{N} with $Fac(u) = \mathcal{A}$. Let I_n be an interval of \mathbb{N} such that $u_{|I_n|} = \beta_n$.

Claim: For each $n \in \mathbb{N}$, the set \mathcal{J}_n of intervals J of \mathbb{N} such that $I_n \subseteq J$ and $u_{\uparrow J} \in \mathcal{B}$ is finite.

Proof of Claim. If this is not the case, let $[I_n \to) := \{m \in \mathbb{N} : \min(I_n) \le m\}$ and $(\leftarrow I_n[:= \{m \in \mathbb{N} : m < \min I_n\}]$. Then $Fac(u_{\upharpoonright [I_n \to)}) \subseteq \mathcal{B}$. Since \mathcal{A} is inexhaustible $Fac(u_{\upharpoonright (\leftarrow I_n[)}) = Fac(u) = \mathcal{A}$. This contradicts $\mathcal{B} \neq \mathcal{A}$. This completes the proof of the claim.

For each $n \in \mathbb{N}$ let J_n be a maximal element of $\bigcup \mathcal{J}_n$. The set I of $n \in \mathbb{N}$ such that $0 \in J_n$ is finite, otherwise the union of the J_n 's would be \mathbb{N} and \mathcal{B} would be equal to \mathcal{A} . For $n \in \mathbb{N} \setminus I$ set $x_n \coloneqq \min J_n - 1$, $y_n \coloneqq \max J_n + 1$, $\overline{J}_n \coloneqq J_n \cup \{x_n, y_n\}$ and $\gamma_n \coloneqq u_{\uparrow} \overline{J}_n$.

Since the length of the $\beta'_n s$ is increasing, there is an infinite subset K of $\mathbb{N} \setminus I$ such that $|\gamma_n| < |\gamma_m|$ for n < m in K. We claim that the set of γ_n for $n \in K$ is an infinite antichain.

Indeed, if $\gamma_n \leq \gamma_m$ then either $|\gamma_n| = |\gamma_m|$ in which case $\gamma_n = \gamma_m$, or $|\gamma_n| < |\gamma_m|$. In this latter case, either $\gamma_{n \upharpoonright J_n \cup \{x_n\}} \leq \gamma_{m \upharpoonright J_n}$ or $\gamma_{n \upharpoonright J_n \cup \{y_n\}} \leq \gamma_{m \upharpoonright J_m}$. Since $\gamma_{m \upharpoonright J_n} \in \mathcal{B}$ and \mathcal{B} is an initial segment of \mathcal{A} , either $\gamma_{n \upharpoonright J_n \cup \{x_n\}}$ or $\gamma_{n \upharpoonright J_n \cup \{y_n\}}$ belongs to \mathcal{B} , contradicting the maximality of J_n .

In the sequel, we suppose that $A := \{0, ... k - 1\}$. To the word u with domain \mathbb{N} we may associate the labelled path P_u on \mathbb{N} where vertex n is labelled by u(n). We may consider P_u as a binary structure made of the path and k unary relations $U_0, ..., U_{k-1}$ where $U_i(n) = 1$ if u(n) = i. If the alphabet is $\{0, 1\}$, this can be expressed in a much simpler form: P_u is the path where vertex n has a loop if and only if u(n) = 1.

Due to Theorem 5 and a theorem of Higman, there is a simple correspondence between properties of words and properties of labelled paths. First we recall Higman's theorem on words.

Let $P \coloneqq (V, \leq)$ be an ordered set and let P^* be the set of finite sequences of elements of V that we see as words on the alphabet V. The *subword ordering* on P^* is defined as follows. If $u \coloneqq u_0, ..., u_{n-1}$ and $v \coloneqq v_0, ..., v_{m-1}$ are two words then we let $u \leq v$ if there is a 1-1 order preserving map φ from the interval [0, n[into the interval [0, m[such that $u_i \leq v_{\varphi(i)}$ for each i < n.

Theorem 7 Higman (1952)If P is wgo, then P^* is wgo.

Next, we state the correspondence.

Theorem 8 Let u be a word with domain \mathbb{N} on a finite alphabet. The following properties are equivalent:

- (i) u is uniformly recurrent;
- (ii) $Aqe(P_n)$ is inexhaustible and wgo.

Proof: Suppose that u is uniformly recurrent. According to $(i) \Rightarrow (ii)$ of Theorem 5, Fac(u) is inexhaustible and wqo. Members of $Age(P_u)$ are finite direct sums of labelled paths. They can be viewed as words on the alphabet Fac(u). From Higman's theorem this set of words is wqo. Hence $Age(P_u)$ is wqo. The fact that $Age(P_u)$ is inexhaustible is obvious. Conversely, suppose that $Age(P_u)$ is inexhaustible

and wqo. Then trivially, Fac(u) is inexhaustible and wqo. Hence, from $(ii) \Rightarrow (i)$ of Theorem 5, u is uniformly recurrent.

As it is well known there are 2^{\aleph_0} uniformly recurrent words with distinct ages (e.g. Sturmian words with different slopes, e.g., see Chapter 6 of Fogg (2002)). Despite the fact that the age of a labelled path P_u does not determine the age of u (it does it up to reversal), this suffices to prove that:

Lemma 9 There are 2^{\aleph_0} labelled paths with distinct ages.

We will need this fact and more, namely that the graphs $\widehat{G}_{(u,\star)}$, that we define in Subsection 4.1, have distinct ages (Proposition 21). For this, we start with the following notation and observations. If $v \coloneqq v_0 \dots v_{n-1}$ is a finite word, let $v^d \coloneqq v_{n-1} \cdots v_0$ its reverse; if X is a set of words, set $X^d \coloneqq \{v^d : v \in X\}$. If Fac(u) is the age of a word u on \mathbb{N} , then $Fac(u)^d$ is the set of factors of a word on $\mathbb{N}^* = \{0, -1, \dots, -n \dots\}$ but in general, it is not necessarily the age of a word on \mathbb{N} . Except if the word u is recurrent and especially if the word is uniformly recurrent. Next, let 1 be the constant word equal to 1; if $v \coloneqq v_0 \dots v_{n-1}$ is a word on $\{0, 1\}$, denote by $v \dotplus 1$ the word $\{v_0 \dotplus 1 \dots v_{n-1} \dotplus 1\}$ where the sum \dotplus is modulo 2. If X is a set of words, set $X \dotplus 1 \coloneqq \{v \dotplus 1 : v \in X\}$. Note that if X is the age of a uniformly recurrent word, then $X \dotplus 1$ is too.

Definition 1 Two ages \mathcal{A} and \mathcal{A}' of uniformly recurrent words are equivalent if $\mathcal{A} \in \{\mathcal{A}', \mathcal{A}'^d, \mathcal{A}' \dotplus \mathbf{1}, \mathcal{A}'^d \dotplus \mathbf{1}\}$.

This is an equivalence relation and each equivalence class has at most four elements. From 2^{\aleph_0} uniformly recurrent words with distinct ages we can extract 2^{\aleph_0} ages of words which are not pairwise equivalent. In particular these words yield 2^{\aleph_0} labelled paths with distinct ages. This proves Lemma 9 (and more).

3 A general setting for minimality

Definition 2 Let C be a class of finite graphs. A graph G is C-minimal if Age(G), the age of G, contains C and every induced subgraph G' of G such that $C \subseteq Age(G')$ embeds G. Equivalently, G is minimal among the graphs G'' such that $C \subseteq Age(G'')$, the collection of these G'' being quasi-ordered by embeddability.

Example 1 If \mathfrak{P} is the collection of all finite paths, we say that a \mathfrak{P} -minimal graph is path-minimal.

A natural question is:

Question 1 Given a class C of finite graphs, does every graph such that $C \subseteq Age(G)$ embeds some C-minimal graph?

We have no answer, except in some special cases.

For example, we have:

Fact 1 If G is a path-minimal graph and $Age(G) = \downarrow \mathfrak{P}$, the set of graphs which embed in some finite path, then G is a direct sum of finite paths of unbounded length.

Proof: Since Age(G) is equal to $\downarrow \mathfrak{P}$, the graph G is the direct sum of paths, some finite and some infinite. If no connected component of G is infinite, then the length of these connected components is unbounded and G is equimorphic to a direct sum of finite paths of arbitrarily large length. Else if some connected component of G is infinite it embeds a direct sum of finite paths of unbounded length. \Box

For an other example, see Theorem 2.

Question 2 Let G be a graph such that $C \subseteq Age(G)$. Does G embed a graph G' such that Age(G') is minimal in the sense that for no induced subgraph G'' of G' such that $C \subseteq Age(G'')$, Age(G'') strictly included in Age(G')?

The following are obvious observations.

- (1) A positive answer to Question 1 yields a positive answer to Question 2.
- (2) If $\{G': G' \leq G \text{ and } C \subseteq Age(G')\}$ is well founded, then the answer to Question 1 is positive.
- (3) If the collection of ages of induced subgraphs G' of G containing C is well founded when ordered by set inclusion, then the answer to Question 2 is positive.

3.1 Links to other notions of minimality

Definition 3 A graph G is minimal for its age (or simply minimal) if it embeds in every induced subgraph G' of G with the same age as G Pouzet (1979).

Fact 2 *If* G *is* C-minimal, then it is minimal.

Proof: Let G' be an induced subgraph of G such that Age(G') = Age(G). Since G is C-minimal, $C \subseteq Age(G)$. Since Age(G') = Age(G) we have $C \subseteq Age(G')$. From the C-minimality of G we deduce that G embeds into G' proving that G is minimal.

Question 3 Does there exist a graph such that no induced subgraph with the same age is minimal? (Pouzet (1979)).

In Pouzet (1979), examples of multirelations (not graphs) yield a positive answer to that question.

Fact 3 If G is a graph as in Question 3, then for C = Age(G), G does not embed a C-minimal graph.

Proof: Suppose that some induced subgraph of G' of G, with $C \subseteq Age(G')$, is C-minimal. Then it follows from Fact 2 that G' is minimal. A contradiction.

3.2 Indivisibility and minimality

Definition 4 An age A of finite graphs is indivisible if for every graph G such that Age(G) = A and every partition of the vertex set of G into two parts A and B, one the induced graphs of G on A or B has age A. Equivalently, an age A of finite graphs is indivisible if for every $S \in A$ there is $S \in A$ such that for every partition A, B of the vertex set of S, one of the induced graphs S_{A} , S_{B} embeds S (cf. Pouzet (1979) p.331 and also Fraïssé (2000)).

Let $G_n = (V_n, E_n)$ for $n \in \mathbb{N}$ be a family of graphs having pairwise disjoint vertex sets. We define the direct sum of $(G_n)_{n \in \mathbb{N}}$, denoted $\bigoplus_n G_n$, is the graph whose vertex set is $\bigcup_{n \in \mathbb{N}} V_n$ and edge set $\bigcup_{n \in \mathbb{N}} E_n$. The complete sum of $(G_n)_{n \in \mathbb{N}}$, denoted $\sum_n G_n$, is the graph whose vertex set is $\bigcup_{n \in \mathbb{N}} V_n$ and edge set $\bigcup_{i \neq j} \{\{v, v'\} : v \in V_i \land v' \in V_j\} \cup \bigcup_{n \in \mathbb{N}} E_n$.

For an indivisible age A, graphs that are A-minimal are easy to describe.

Theorem 10 If A is an indivisible age of finite graphs, then every graph G with $A \subseteq Age(G)$ embeds some graph G' which is A-minimal. In fact, either G embeds a direct sum $\bigoplus_n G_n$ or a complete sum $\sum_n G_n$ of finite graphs $G_0, G_1, \dots, G_n, \dots$ such that $A = \downarrow \{G_n : n < \omega\}$, the set of graphs which embed in some G_n .

This result is essentially Lemma 4.2 of Milner and Pouzet (1992). It is based on Propositions IV.2.3.3, IV.2.3.1 and IV.2.3.2, p.338, 339 of Pouzet (1979). We do not give a proof. We prefer to give the following illustration (Lemma 4.1 of Milner and Pouzet (1992)).

Proposition 11 If a graph embeds finite cliques of unbounded cardinality, then either it embeds an infinite clique or a direct sum of finite cliques of unbounded cardinality.

This proposition is readily obtained from Ramsey's Theorem on quadruples. For the reader's convenience we provide a proof. For each integer m>0, choose a complete graph G_m on m vertices and set $V(G_m)\coloneqq\{v_{0,m},\cdots,v_{i,m},\cdots,v_{m-1,m}\}$. Let $[\mathbb{N}]^2$ be the set of pairs $\{n,m\}$ with n< m and $f:[\mathbb{N}]^2\to G$ be the map defined by $f(n,m)=v_{n,m}$. Partition the set $[\mathbb{N}]^4$ of 4-tuples $\{i,j,k,l\}$ into three classes according to whether f(i,j) is equal, adjacent, or not adjacent to f(k,l). From Ramsey's Theorem there exists an infinite subset X of \mathbb{N} such that $[X]^4$ is in one class. The conclusion follows.

3.3 Minimality and primality

Let G := (V, E) be a graph. According to Gallai Gallai (2001), a subset A of V is *autonomous* (also called *module* in the literature) in G if for every $v \notin A$, either v is adjacent to all vertices of A or v is not adjacent to any vertex of A. Clearly, the empty set, the singletons in V and the whole set V are autonomous in G; they are called *trivial*. An undirected graph is called *indecomposable* if all its autonomous sets are trivial. With this definition, graphs on a set of size at most two are indecomposable. Also, there are no indecomposable graphs on a three-element set. An indecomposable graph with more than three elements will be called *prime*. The graph P_4 , the path on four vertices, is prime. In fact, as it is well known, every prime graph contains an induced P_4 (Sumner Sumner (1973) for finite graphs and Kelly Kelly (1985) for infinite graphs). Furthermore, every infinite prime graph contains an induced countable prime graph Ille (2005).

Definition 5 Pouzet and Zaguia (2009) A an infinite graph G is minimal prime if G is prime and every prime induced subgraph with the same cardinality as G embeds G.

In Pouzet and Zaguia (2009), the authors provided among other things, the list of minimal prime graphs with no infinite cliques or no infinite independent sets.

Definition 6 An age A of graphs is minimal prime if A contains prime graphs of unbounded cardinality but every proper hereditary subclass contains only finitely many prime graphs.

Problem 1 Oudrar (2015)(Problem 11 page 99 subsection 5.3.1) If a graph G (more generally a binary relational structure) is minimal prime, is its age minimal prime?

4 Path-minimality

This section is devoted to the proof of Theorem 1. In Subsection 4.1 we define the lexicographical sum of labelled graphs, the set of labels being endowed with an operation * with 0-1 values. This allows us to

construct a path-minimal graph $\widehat{Q}_{(u,\star)}$ attached to each uniformly recurrent word u (Proposition 17). In Subsection 4.2 we consider path-minimal graphs attached to some periodic sequences and prove that their ages are pairwise incomparable (Proposition 18). In Subsection 4.3 we show first that for some operations \star on $L := \{0,1\}$ each long enough path in $\widehat{Q}_{(u,\star)}$ is contained in some component (Lemma 20). Then we prove that there are 2^{\aleph_0} such $\widehat{Q}_{(u,\star)}$'s whose ages are wqo and incomparable w.r.t. inclusion. This completes the proof of Theorem 1.

4.1 Lexicographical sum of labelled graphs

Definition 7 A labelled graph is a pair (G, ℓ) where G := (V, E) is a graph and ℓ is a map from V into a set L of labels. If H denotes a labelled graph, then \widehat{H} denotes the underlying graph. In our context, the set L has no order structure, so when we say that (G, ℓ) embeds into (G', ℓ') and write $(G, \ell) \le (G', \ell')$, we mean that there is an embedding h of G into G' such that $\ell'(h(v)) = \ell(v)$ for all $v \in V$.

If $L = \{0, 1\}$, we may replace a labelling ℓ by a loop at every vertex v such that $\ell(v) = 1$ and suppress the label 0.

In the sequel L is endowed with a map $\star : L \times L \to \{0,1\}$ and we set $i \star j := \star (i,j)$. We denote by \star^d the dual of \star , defined by $i \star^d j := j \star i$.

Let $(G_i, \ell_i)_{i \in I}$ be a family of labelled graphs, indexed by a chain $C := (I, \leq)$, and suppose that the sets $V(G_i)$, for $i \in I$, are pairwise disjoint. The *lexicographical sum* $\sum_{\star i \in C} (G_i, \ell_i)$ of the family $(G_i, \ell_i)_{i \in I}$ is the labelled graph (G, ℓ) whose vertex set is $\bigcup_{i \in I} V(G_i)$ and edge set

$$E := (\cup_{i \in I} E_i) \cup \{\{x_i, x_i\} : i < j, x_i \in V(G_i), x_i \in V(G_i) \text{ and } \ell(x_i) \star \ell(x_i) = 1\}.$$

If \star takes only the value 0, then $\sum_{\star i \in I} (G_i, \ell_i)$ is the direct sum of $(G_i, \ell_i)_{i \in I}$. On the other hand, if \star takes only the value 1, then $\sum_{\star i \in I} (G_i, \ell_i)$ is the complete sum of $(G_i, \ell_i)_{i \in I}$.

If the sets $V(G_i)$'s are not pairwise disjoint, we replace each (G_i,ℓ_i) by a copy on $\{i\} \times V(G_i)$. If all (G_i,ℓ_i) 's are equal to the same labelled graph (G,ℓ) then the lexicographical sum of copies of (G,ℓ) indexed by the chain C is the labelled graph $(G,\ell) \cdot_{\star} C$ on $I \times V(G)$ such $\ell(i,x) \coloneqq \ell_i(x)$ for each (i,x) and for i < j in C, (i,x) and (j,y) form an edge if and only if $\ell(i,x) \star \ell(j,y) = 1$ (our notation agrees with the notation of lexicographical sum of chains). In the sequel we will mostly consider lexicographical sum of copies of a labelled graph (G,ℓ) indexed by the chain ω of the non-negative integers. If G is a labelled graph over a set L of labels and \star denotes the map from $L \times L$ to $\{0,1\}$, we denote by $G \cdot_{\star} \omega$ the lexicographical sum of copies of G indexed by the chain ω of the non-negative integers; we denote by $G \cdot_{\star} \omega$ the graph obtained by removing the labels of $G \cdot_{\star} \omega$. In the sequel, our graphs G will be paths labelled by uniformly recurrent sequences.

Remark 1 The fact that our family of labelled graphs is indexed by a chain is enough to serve our purpose. But one could consider a more general notion of lexicographical sum in a way that the fact that two vertices x_i and x_j are adjacent depends only upon the values of the labels $\ell(x_i)$ and $\ell(x_j)$. This amounts to impose that the sets V_i , for $i \in I$, are *Fraïssé intervals* (a generalization of autonomous sets; see Fraïssé (1984) and for a general development, see Ehrenfeucht et al. (1999)) of (G, ℓ) , that is:

for every
$$x_i, x_i' \in V_i, y \notin V_i$$
,

if
$$\ell(x_i) = \ell(x_i')$$
, then $\{x_i, y\}$ is an edge if and only if $\{x_i', y\}$ is an edge. (1)

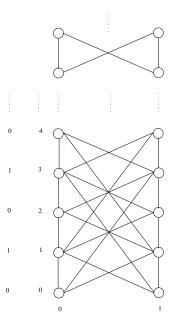


Fig. 1: Sum of two labeled paths. For $n \in \mathbb{N}$ set $\ell(2n) = 0$ and $\ell(2n+1) = 1$. The operation \star is the Boolean sum.

A crucial property of lexicographical sums is the following

Lemma 12 Let $(G_i, \ell_i)_{i \in I}$ and $(G'_{i'}, \ell'_{i'})_{i' \in I'}$ be two families of labelled graphs, indexed by the chains $C := (I, \leq)$ and $C' := (I', \leq)$ respectively. If there is an embedding φ from C in C' and for each $i \in I$ an embedding f_i of (G_i, ℓ_i) in $(G'_{\varphi(i)}, \ell'_{\varphi(i)})$ then $\bigcup_{i \in I} f_i$ is an embedding of $(G, \ell) \cdot_{\star} C$ in $(G', \ell') \cdot_{\star} C'$.

The proof is immediate: the fact two vertices (i, x) and (j, y) with i < j form an edge depends only upon the values of $\ell_i(x) \star \ell_j(y)$.

We have also:

Lemma 13 If C^d is the dual of a chain C then $(G, \ell) \cdot_{\star} C := (G, \ell) \cdot_{\star^d} C^d$.

Let \mathcal{C} be a class of L-labelled graphs. Let $\widetilde{\mathcal{C}}_{\star}$ be the collection of L-labelled graphs which are lexicographical sums of L-labelled graphs belonging to \mathcal{C} indexed by some chain and let $\widetilde{\mathcal{C}}_{\star < \omega}$ be those that are finite sums.

Theorem 14 Let C be a class of L-labelled graphs.

- (1) If C is a hereditary class of L-labelled finite graphs, then $\widetilde{C}_{\star < \omega}$ is hereditary;
- (2) If C is an inexhaustible age of L-labelled graphs, then $\widetilde{C}_{\star < \omega}$ is an inexhaustible age too;
- (3) If C is wgo then $\widetilde{C}_{\star < \omega}$ is wgo.
- (4) If C consists of direct sums of labelled paths of bounded length then the length of labelled paths in $\widetilde{C}_{\star < \omega}$ is bounded.

Proof: Only the last two statements are nontrivial. They are consequence of Higman's Theorem Higman (1952).

- (3) Let P be the class $\mathcal C$ quasi ordered by embeddability. To a finite sequence $(G_0,...,G_{n-1}) \in P^*$ associate the sum $\sum_{i \in \{1,...,n-1\}} G_i$ and observe that the operation Σ is order preserving from P^* into $\widetilde{\mathcal C}_{\star < \omega}$: if $(G_0,...,G_{n-1}) \leq (G'_0,...,G'_{n'-1})$ then $\sum_{i \in \{1,...,n-1\}} G_i \leq \sum_{i \in \{1,...,n'-1\}} G'_i$. Since P is wqo, P^* is wqo by Higman's Theorem. Its image is $\widetilde{\mathcal C}_{\star < \omega}$ which is wqo too.
- (4) The proof relies on a strengthening of the notion of wqo due to the first author (Pouzet (1972), Fraı̈ssé (2000) 13.2.1 page 355). A collection \mathcal{C} of labelled graphs is 1-well if the collection $\mathcal{C}_{[1]}$ made of pairs (G,U) where $G\in\mathcal{C}$ and U is a unary relation on V(G) is wqo. A collection of paths (labelled or not) of unbounded length is not 1-well. Indeed, to each path G associate (G,U) where G take value 1 on the end-vertices of G and G on the other vertices. Then paths with different length are incomparable. Now, suppose that G consists of direct sums of labelled paths of bounded length then, with Higman's Theorem, G is wqo, and again by Higman's Theorem $(C_{*<\omega})_{[1]}$ is wqo. Hence the length of paths in $C_{*<\omega}$ is bounded.

Lemma 15 Let G be a labelled graph and let $G \cdot_{\star} C$ be the lexicographic sum of copies of G indexed by the chain $C := (I, \leq)$. Let J be an infinite subset of I, $(A_j)_{j \in J}$ be a family of subsets of V(G), $S := (G_j)_{j \in J}$ where $G_j := G_{ \upharpoonright A_j}$ and $G_{(\star, S)}$ be the graph induced by $G \cdot_{\star} C$ on $\bigcup_{i \in J} \{j\} \times A_j$. Then

- (1) $Age(G_{(\star,S)}) = Age(G_{\star}C)$ if $Age(G) = \bigcup_{i \in J} Age(G_i)$.
- (2) Furthermore, if $C_{\uparrow J}$ has order type ω or ω^d and $(A_j)_{j \in J}$, $(A'_j)_{j \in J}$ are two families of finite subsets of V(G) such that $Age(G) = \bigcup_{i \in J} Age(G_i) = \bigcup_{i \in J} Age(G'_i)$ then $G_{(\star,S)}$ and $G_{(\star,S')}$ are equimorphic.

Proof: (1) Let $H \in Age(G \cdot_{\star} C)$. Let $F \subseteq I \times V(G)$ such that $(G \cdot_{\star} C)_{\upharpoonright F}$ is isomorphic to H. Let K be the projection of F on I. For $k \in K$, set $F_k := \{x \in V(G) : (k,x) \in F\}$. Since $Age(G) = \bigcup_{i \in J} Age(G_i)$, $G_{\upharpoonright F_k}$ embeds in some G_j . Write k_0, \ldots, k_{m-1} the elements of K in an increasing order, so $k_i < k_{i+1}$ for $i \in \{0, ..., m-2\}$. For each $i, 0 \le i < m$, we may select n_{k_i} such that $G_{\upharpoonright F_{k_i}}$ embeds in $G_{n_{k_i}}$, this $G_{n_{k_i}}$ being isomorphic to $G_{\upharpoonright \{n_{k_i}\} \times A_{n_{k_i}}}$ and the indices n_{k_i} strictly increasing with i. Then, with Lemma 12, one can see that H embeds into $(G \cdot_{\star} C)_{\upharpoonright \bigcup_{0 \le i < m} \{n_{k_i}\} \times A_{n_{k_i}}}$, hence in $G_{(\star, \mathcal{S})}$, thus $H \in Age(G_{(\star, \mathcal{S})})$. This proves our assertion.

(2)We prove that $G_{(\star,S)}$ embeds in $G_{(\star,S')}$. We may suppose that $C_{\uparrow J}$ has order type ω and identify it with $\mathbb N$. We built a strictly increasing self map φ on $\mathbb N$ such that $(G \cdot_{\star} C_{\uparrow J})_{\uparrow \{n\} \times A_n}$ embeds in $(G \cdot_{\star} \omega)_{\uparrow \{\varphi(n)\} \times A'_{\varphi(n)}}$. The conclusion follows with Lemma 12.

Corollary 16 $G \cdot_{\star} \zeta$, $G \cdot_{\star} \omega^d$ and $G \cdot_{\star} \omega$ have the same age.

Definition 8 To a uniformly recurrent word u on \mathbb{N} over the alphabet L we associate four graphs $G_{(u,\star)}$, $Q_{(u,\star)}$, $\widehat{G}_{(u,\star)}$ and $\widehat{Q}_{(u,\star)}$. The graph $G_{(u,\star)} \coloneqq P_u \cdot_\star \omega$ is the lexicographical sum of ω copies of the labelled graph P_u . To emphasize, the vertex set $V(G_{(u,\star)})$ is $\mathbb{N} \times \mathbb{N}$; on each component $\{i\} \times \mathbb{N}$, the graph induced by $G_{(u,\star)}$ is a copy of the path P_u labelled by u; the label of vertex (i,x) is u(x), two vertices (i,x) and (j,y) of $G_{(u,\star)}$ with i < j are linked by an edge if and only if $u(x) \star u(y) = 1$. Let $(I_n)_{n \in \mathbb{N}}$ be a family of intervals of \mathbb{N} of unbounded finite length then $Q_{(u,\star)}$ is the induced subgraph of

 $G_{(u,\star)}$ on $I := \bigcup_{n \in \mathbb{N}} \{n\} \times I_n$. Let $\widehat{G}_{(u,\star)}$ and $\widehat{Q}_{(u,\star)}$ be the graphs obtained by removing the labels of $G_{(u,\star)}$ and $Q_{(u,\star)}$ respectively. Hence, $\widehat{Q}_{(u,\star)}$ is the restriction to I of $\widehat{G}_{(u,\star)}$.

Proposition 17 The graphs $\widehat{G}_{(u,\star)}$ and $\widehat{Q}_{(u,\star)}$ have the same age and this age is wqo. Furthermore $\widehat{Q}_{(u,\star)}$ is path-minimal.

Proof: First, we prove that $\widehat{G}_{(u,\star)}$ and $\widehat{Q}_{(u,\star)}$ have the same age. This follows from the fact that $G_{(u,\star)}$ and $Q_{(u,\star)}$ have the same age. This fact is an immediate consequence of Lemma 15.

Next we prove that $Age(\widehat{G}_{(u,\star)})$, the age of $\widehat{G}_{(u,\star)}$, is wqo. Again, this fact follows from the fact that $Age(G_{(u,\star)})$ is wqo. This last fact follows from Item (3) of Theorem 14. Indeed, according to Theorem 8, $Age(P_u)$ is wqo. If $\mathcal{C} := Age(P_u)$ then $\widetilde{\mathcal{C}}_{\star < \omega}$ is wqo by Item (3) of Theorem 14. Since $Age(G_{(u,\star)}) \subseteq \widehat{\mathcal{C}}_{\star < \omega}$, $Age(G_{(u,\star)})$ is wqo.

We prove that any other family of intervals of finite unbounded length yields an equimorphic graph to $\widehat{Q}_{(u,\star)}$ and furthermore that this graph as well as $\widehat{Q}_{(u,\star)}$ is path-minimal. For that, we apply the following claim.

Claim: Let $G := G_{(u,\star)}$, $Q := Q_{(u,\star)}$ and \widehat{G} , \widehat{Q} be the graphs obtained by deleting the labels.

- (1) \widehat{Q} contains finite induced paths of unbounded length.
- (2) Let $A \subseteq V(G)$. If $\widehat{G}_{\uparrow A}$ contains finite paths of unbounded length, then \widehat{Q} embeds into $\widehat{G}_{\uparrow A}$.

Proof of Claim: For $n \in \mathbb{N}$ let $A_n \coloneqq A \cap (\{n\} \times P_u)$. We claim that for each integer m there exists some n_m such that $\widehat{G}_{\upharpoonright A_{n_m}}$ contains a path of length at least m. Suppose that this is not the case. This means that there is some integer k such that the length of paths in each A_n is bounded by k. In this case, $\widehat{G}_{\upharpoonright A_n}$ decomposes into a direct sum of labelled paths of length at most k. According to (4) of Theorem 14, the length of paths in $G_{\upharpoonright A}$ is bounded, contradicting our hypothesis. This proves our claim. For each m, let I_{n_m} be an interval of \mathbb{N} of length m such that $\{n_m\} \times I_{n_m}$ is included into some A_{n_m} , the A_{n_m} being distinct and in an increasing order. The set $S \coloneqq \{P_u \upharpoonright I_{m_n} : m \in \mathbb{N}\}$ is an infinite subset of $Age(P_u)$. Since u is uniformly recurrent, it follows from Theorem 8 that S is cofinal in $Age(P_u)$, that is every element of $Age(P_u)$ embeds in some element of S. Thus Lemma 15 applies, S0 embeds into S1. This completes the proof of the claim.

The proof of the Proposition 17 is now complete.

4.2 Congruences and path-minimal graphs

In this subsection, we start with a simple construction. It gives an illustration of our techniques and allows us to show that there are countably many path-minimal graphs with incomparable ages (Proposition 18).

Let $k \ge 2$ be an integer and \equiv_k be the congruence modulo k on the integers. Let \widehat{G}_k be the graph with vertex set $\mathbb{N} \times \mathbb{N}$ such that two vertices (x,y) and (x',y') form an edge if either x=x' and |y-y'|=1 or $x \ne x'$ and $y \ne_k y'$. Let \widehat{Q}_k be the restriction of \widehat{G}_k to $\{(n,m): m < n+5\}$.

The graph G_k can be obtained by the process described in Subsection 4.1. Indeed, let $u \coloneqq (u(n))_{n \in \mathbb{N}}$ where u(n) is the residue of n modulo k. Hence u is a periodic word on the alphabet $\{0, ..., k-1\}$. Define \star from $\{0, ..., k-1\} \times \{0, ..., k-1\}$ into $\{0, 1\}$ by setting $i \star i = 0$ for i = 0, ..., k-1 and $i \star j = 1$ otherwise. To the word u we associate the labelled path P_u on \mathbb{N} where vertex n is labelled by u(n). Let G be the lexicographical sum of copies of P_u indexed by the chain ω , say $G \coloneqq P_u \cdot_\star \omega$, and \widehat{G} be the

graph obtained by removing the labels of G. Then $\widehat{G}_k = \widehat{G}$. Note that $\widehat{G}_2 = \widehat{G}_{(u,\star)}$ where u is the periodic sequence $010101\ldots$ and \star is the Boolean sum on $\{0,1\}$.

Proposition 18 For each integer $k \ge 2$, \widehat{Q}_k is a path-minimal graph and for different values of k the ages of \widehat{Q}_k are incomparable w.r.t. inclusion.

The fact that \widehat{Q}_k is path-minimal follows from Proposition 17. We give a self-contained proof of the full proposition above avoiding the use of wqo. It relies on claims 2, 3 and 4 stated below.

For $i \in \mathbb{N}$, let $\mathbb{N}_i := \{(i,j) : j \in \mathbb{N}\}$ and let $\widehat{G}_k(i)$ be the graph induced by \widehat{G}_k on \mathbb{N}_i . This is an infinite path that we call a *component* of \widehat{G}_k . The restriction \widehat{Q}_k of \widehat{G}_k to the set $X_i := \{(i,j) : j < i+5\}$ is an induced path of length i+5 that we call also a component of \widehat{Q}_k . We note that \widehat{Q}_k is a union of finite paths of unbounded length.

Claim 1: $\delta(\widehat{G}_2) = 3$ and $\delta(\widehat{G}_k) = 2$ for $k \ge 3$.

Proof: Let a:=(x,y) and a':=(x',y') be two distinct vertices. We claim that their distance is at most 3 if $x \neq x'$ or k > 2, and at most 3 if x = x' and k = 2. Suppose that $x \neq x'$, e.g., x < x'. If y' = y then $(x,y) \sim (x',y+1) \sim (x',y)$ and the distance is 2. Suppose that $y \neq y'$, e.g., y < y'. If $y' \not\equiv_k y$ then $(x,y) \sim (x',y')$ and the distance is 1. If $y' \equiv_k y$ then $(x,y) \sim (x',y+1) \sim (x',y')$ and the distance is 2. Suppose that x = x'. We may suppose that $y \neq y'$ otherwise the distance is 0. We may thus also suppose y < y'. Let r be the residue of $y' - y \mod k$, i.e. y' = my + r with $0 \le r < k$. If r = 0 then $(x,y) \sim (x+1,y+1) \sim (x',y')$ and thus $d_{\widehat{G}_k}(a,a') \le 2$. If k = 2 and r = 1 then either m = 1 and thus $(x,y) \sim (x,y')$, and the distance is 1, or m > 1, in which case $(x,y) \sim (x+1,y+1) \sim (x,y'-1) \sim (x,y')$ and hence $d_{\widehat{G}_k}(a,a') \le 2$. If k > 2 let r' := 2 if r = 1 and r' := 1 if r > 1. Then $(x,y) \sim (x+1,y+r') \sim (x,y')$ and hence $d_{\widehat{G}_k}(a,a') \le 2$.

Claim 2: An induced path of length at least six of \widehat{G}_k is necessarily included in a component of \widehat{G}_k .

Proof: Let $R := ((x_i, y_i))_{i \ge 0}$ be an induced path in \widehat{G}_k of length at least four. Then the following four statements below hold:

- (1) For all i and for all $j \ge i+2$ we have $x_i = x_j$ or $x_i = x_{j+1}$. Proof: Suppose $x_i \ne x_j$ and $x_i \ne x_{j+1}$. Since $(x_i, y_i) \not\leftarrow (x_j, y_j)$ and $(x_i, y_i) \not\leftarrow (x_{j+1}, y_{j+1})$, $y_j \equiv_k y_i \equiv_k y_{j+1}$. Since $(x_j, y_j) \sim (x_{j+1}, y_{j+1})$ and $y_j \equiv_k y_{j+1}$ we must have $x_j = x_{j+1}$ and $|y_j - y_{j+1}| = 1$ which is impossible since $k \ge 2$.
- (2) For all i and for all $j \ge i+2$, x_i is equal to at least three elements among $x_j, x_{j+1}, x_{j+2}, x_{j+3}$. *Proof:* Suppose for a contradiction that $j \le r < s \le j+3$ such that $x_i \notin \{x_r, x_s\}$. Since R is induced we have $y_r \equiv_k y_i \equiv_k y_s$. It follows from (1) that $s-r \ne 1$. There are three cases to consider.

Case 1: r = j and s = j + 2.

It follows from (1) that $x_{j+1} = x_i = x_{j+3}$. Since $x_{j+3} \neq x_j$ and R is induced we infer that $y_{j+3} \equiv_k y_j$. But then $y_{j+3} \equiv_k y_{j+2}$ contradicting $x_{j+3} \neq x_{j+2}$.

Case 2: r = j + 1 and s = j + 3.

This follows from Case 1 and symmetry.

Case 3: r = j and s = j + 3.

It follows from (1) that $x_{j+1} = x_i = x_{j+2}$ and hence $|y_{j+1} - y_{j+2}| = 1$. Furthermore, $y_{j+3} \equiv_k y_{j+1}$ and $y_j \equiv_k y_{j+2}$. Hence, $y_{j+1} \equiv_k y_i \equiv_k y_{j+2}$ contradicting $|y_{j+1} - y_{j+2}| = 1$.

(3) If R has length at least five, there exists at most one index $i \ge 2$ such that $x_0 \ne x_i$. Proof: Suppose $2 \le i < j$ such that $x_i \ne x_0 \ne x_j$. It follows from (1) that $x_{i+1} = x_0$. Then $j \ne i+1$. It follows from (2) that we may assume without loss of generality that j = i+2. Then $(x_{i+3}, y_{i+3}) \in R$ or i > 2 and hence (x_{i-1}, y_{i-1}) is not a neighbour of $(x_0, y_0) \in R$. We obtain a contradiction by applying (2) to the sequence (x_0, y_0) and $(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}), (x_{i+3}, y_{i+3})$ or to the sequence (x_0, y_0) and $(x_{i-1}, y_{i-1}), (x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$.

- (4) Let $(x_0, y_0), (x_1, y_1), ..., (x_6, y_6)$ be an induced path in \widehat{G}_k for $k \geq 2$. If there exists i such that $x_0 \neq x_i$, then i = 1.

 Proof: If not, then by (3) there is a unique $i \geq 2$ such that $x_0 \neq x_i$. If $i \neq 4$, then there are then at least two indices $r, s \geq 2$ such that |r s| = 1 and (x_r, y_r) and (x_s, y_s) are not adjacent to (x_i, y_i) . Hence, $y_r \equiv_k y_0 \equiv_k y_s$. Since $x_r = x_0 = x_s$ we infer that $|y_s y_r| = 1$ contradicting $y_s \equiv y_r \mod k$. We now assume i = 4. Hence, $x_0 = x_2 = x_3 = x_5 = x_6$. We now consider the path $(x'_0, y'_0) = (x_6, y_6), (x'_1, y'_1) = (x_5, y_5), ..., (x'_5, y'_5) = (x_1, y_1), (x'_6, y'_6) = (x_0, y_0)$. Since $x'_0 = x'_4$ we apply the previous argument (as in the case $i_0 \neq 4$) to this path and we obtain a contradiction.
- (5) Let $(x_0, y_0), (x_1, y_1), ..., (x_6, y_6)$ be an induced path in \widehat{G}_k for $k \ge 2$. Then $x_0 = x_1 = ... = x_6$. *Proof:* It follows from (4) that $x_0 = x_2 = ... = x_6$. By symmetry, $x_6 = x_4 = x_3 = x_2 = x_1 = x_0$. Therefore, $x_0 = x_1 = ... = x_6$ as required.

The value six is best possible: The sequence (0,0),(1,1),(0,k),(0,k+1),(1,2k),(0,2k+1) yields an induced path in \widehat{G}_k which is not included in a component of \widehat{G}_k .

Claim 3: The graph \widehat{Q}_k is path-minimal.

Proof: Let Y be a subset of $V(\widehat{Q}_k)$ such that the graph induced by \widehat{Q}_k on Y contains finite induced paths of unbounded length. From Claim 2 above, each induced path of length at least six is included in some component of \widehat{G}_k , hence in a component of \widehat{Q}_k . Since the components of \widehat{Q}_k are finite, there is an infinite sequence of integers $(i_n)_{n\in\mathbb{N}}$ such that $\widehat{Q}_k \upharpoonright Y \cap X_{i_n}$ contains a path on a subset Z_{i_n} of length n. According to Lemma 15, \widehat{Q}_k embeds in $\widehat{Q}_k \upharpoonright \bigcup_{i_n \in \mathbb{N}} Z_{i_n}$.

Claim 4: For each $k \ge 2$, $Age(\widehat{Q}_k) = Age(\widehat{G}_k)$ and if $k \ne k'$, then $Age(\widehat{G}_k)$ and $Age(\widehat{G}_{k'})$ are incomparable with respect to set inclusion.

Proof: The first part is obvious. For the second part, let R be a path of length $m \ge \max\{k+1,6\}$ in a component of \widehat{G}_k . Let x be a vertex in another component. The trace on R of the neighborhood of x is a union of paths, each of length k-1; two such paths consecutive in R being separated by a single vertex. Then $\widehat{G}_k \upharpoonright R \cup \{x\}$ cannot be embedded in $\widehat{G}_{k'}$. Indeed, otherwise, by Claim 2, the path R of length m will be mapped in a component of $\widehat{G}_{k'}$ on a path R' and the element x will be mapped on an element x' in another component. Hence $\widehat{G}_k \upharpoonright R \cup \{x\}$ will be isomorphic to $\widehat{G}_{k'} \upharpoonright R' \cup \{x'\}$. But the trace on R' of the neighborhood of x' is a union of paths, each of length k'-1 and two such paths consecutive in R' are separated by a single vertex. Since $k \neq k'$ this is not possible.

Sequences other than $u := (u(n))_{n \in \mathbb{N}}$, where u(n) is the residue of n modulo k, will produce pathminimal graphs with different ages. This is the subject of the next two subsections.

П

4.3 A general construction of path-minimal graphs with incomparable ages

Our aim in this subsection is to show that there is an operation \star on $L \coloneqq \{0,1\}$ and 2^{\aleph_0} uniformly recurrent words in such a way that the ages of the graphs $\widehat{Q}_{(u,\star)}$ given in Proposition 17 are pairwise incomparable w.r.t. inclusion. Since $\widehat{Q}_{(u,\star)}$ and $\widehat{G}_{(u,\star)}$ have the same age, we only need to consider $\widehat{G}_{(u,\star)}$. We restrict ourselves to $L \coloneqq \{0,1\}$ while several arguments extend to general alphabets. We leave the general case to further investigations.

The proof has two parts, developed in the next two subsections. We show first that long paths in $\widehat{G}_{(u,\star)}$ are contained in components (Lemma 20). Next, we use that fact to prove that well-chosen words provide distinct ages (Proposition 21).

4.3.1 Long paths are contained in components

Let $L:=\{0,1\}$. The *first projection* is the operation defined by $i\star j=i$ for all i,j, while the *second projection* is defined by $i\star j=j$ for all i,j. The *Boolean sum* is the operation \dotplus on $\{0,1\}$ such that $0\dotplus 0=1\dotplus 1=0$ and $0\dotplus 1=1\dotplus 0=1$. This operation is also the sum modulo two (cf. Section 4.2). Note that the term Boolean Sum is also used for the inclusive or in Boolean Algebra in which case 1+1=1. If \star is an operation on $\{0,1\}$, its *dual*, \star^d , is the operation defined by $i\star^d j:=j\star i$. Let $\overline{\star}$ be the operation on $\{0,1\}$ defined by $i\overline{\star}j:=(i\dotplus 1)\star(j\dotplus 1)$. The Boolean sum is invariant under the two transformations defined above (i.e. $(\dotplus)^d=\dotplus$ and $\dotplus=\dotplus$). The dual of the first projection is the second, while $i\overline{\star}j=i\dotplus 1$ if \star is the first projection and $i\overline{\star}j=j\dotplus 1$ if \star is the second.

Let $u := (u_n)_{n \in \mathbb{N}}$ be a 0-1 sequence; let $u \dotplus 1$ be the sequence defined by $u \dotplus 1(n) := u(n) \dotplus 1$.

Lemma 19 Let
$$u$$
 be a 0-1 sequence. Then $\widehat{G}_{(u,\star)} = \widehat{G}_{(u+1,\overline{\star})}$ and $Age(\widehat{G}_{(u,\star)}) = Age(\widehat{G}_{(u,\star^d)})$.

Proof: The verification of the first equality is immediate. For the second, observe that from Corollary 16, $Age(\widehat{G}_{(u,\star)}) := Age(\widehat{P_u \cdot_\star \omega}) = Age(\widehat{P_u \cdot_\star \omega})$. Since by Lemma 13, $\widehat{P_u \cdot_\star \omega}^d = \widehat{P_u \cdot_\star d} \omega = \widehat{G}_{(u,\star^d)}$, the second equality follows.

This fact suggests that we consider operation \star and \star' as equivalent for each $\star' \in \{\star, \star^d, \overline{\star}, (\overline{\star})^d\}$.

Let u be a uniformly recurrent and non constant word on L. For $i \in \{0, 1\}$ there is a maximum, denoted by $l_i(u)$, of the length of factors of i's in u. Let $l(u) := \max\{l_0(u), l_1(u)\}$ and set $\varphi(u) := 2(l(u)+1)+1$ if u is not the periodic word 010101...; otherwise set $\varphi(u) := 6$. Note that $\varphi(u) = \varphi(u+1)$.

Lemma 20 Let \star be the Boolean sum or an operation equivalent to the second projection. If u is any uniformly recurrent and non constant word, then every path of length at least $\varphi(u)$ in $\widehat{G}_{(u,\star)}$ is contained in one component.

Proof: Set $G := \widehat{G}_{(u,\star)}$ and for $i \in \mathbb{N}$, set $X_i := \{i\} \times \mathbb{N}$. Throughout this proof, R denotes a finite induced path of G, V(R) denotes its set of vertices and E(R) its set of edges. We let $V_0(R)$, respectively $V_1(R)$, denote the vertices of R labelled 0, respectively labelled 1, in $G_{(u,\star)}$.

(•) \star is the Boolean sum. We observe first that since $\widehat{G}_{(u,\star)} = \widehat{G}_{(u\dot{+}1,\star)}$, R is a path of $\widehat{G}_{(u\dot{+}1,\star)}$. Since $\varphi(u) = \varphi(u\dot{+}1)$, we may suppose that the values of u are defined up to complementation. Also, and since R is connected $V_0(R) \neq \emptyset \neq V_1(R)$.

Claim 1: Every induced path, regardless of its length, is contained in at most three components of G. *Proof of Claim 1:* If R is contained in at most two components, then we are done. Next, suppose that

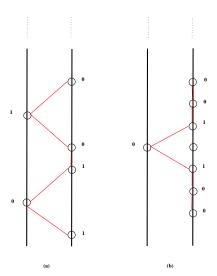


Fig. 2: Two examples of paths not contained in one component of $\widehat{G}_{(u,\star)}$ if \star is the Boolean sum. The thick vertical lines represent components of $\widehat{G}_{(u,\star)}$.

R is contained in at least three components of G.

Sub Claim: Up to the complement of u, there are three vertices a, b, c of R contained in three distinct components of G such that $\ell(a) = 1$, $\ell(b) = 0$ and $\ell(c) = 0$.

Proof of Sub Claim: Let X be a component meeting R in a vertex a labeled 1. If all vertices of R not in X have the same label 0 then pick b and c not in X and in two distinct components. If all vertices of R not in X have the same label 1, then since R is connected there must be a vertex a' in X labeled 0, then choose a' and pick b and c not in X and in two distinct components. Replacing a by its complement a+1, the vertices a', b, c satisfy the conclusion of the claim. Else, there exists a component A meeting A in a vertex A labeled 0. If all the remaining vertices of A not in A of A not in A of the vertices A and in two distinct components and not in A both labeled 0. Else if there exists a vertex A of A not in A of A which is labeled 1, then replacing A by its complement, A in the vertices A, A, A, A with A is A is a vertex A. Satisfy the conclusion of the claim. This completes the proof of the sub claim.

Denote by X_a , X_b and X_c the components of G containing a, b and c, respectively. We claim that $V(R) \subseteq X_a \cup X_b \cup X_c$. Indeed, let $x \notin X_a \cup X_b \cup X_c$. If $\ell(x) = 1$, then x is adjacent to both b and c and therefore the graph induced on $\{a,b,c,x\}$ is a 4-cycle. Else if $\ell(x) = 0$, then x is adjacent to a and hence a has degree at least three. Hence, x cannot be a vertex of x. This proves x is adjacent to x and therefore x is contained in three components of x. This completes the proof of Claim 1.

Claim 2: Every path of length at least five is contained in at most two components of G.

Proof of Claim 2: We prove Claim 2 by induction on the length of the path contained in G. For the basis case we prove that every path of length five is contained in at most two components of G. In fact, we will prove that every path that contains the vertices a,b,c defined in the Sub Claim of Claim 1 will have length at most four.

Suppose that c is not an end vertex of R. Let $c' \in R$ be the neighbor of c distinct of a. Then $\ell(c') = 1$ and $c' \in X_b$. Indeed, if $\ell(c') = 0$ then $c' \in X_c$ and since $\ell(a) = 1$, $\{a, c'\}$ is an edge, which is impossible. Thus $\ell(c') = 1$. If $c' \notin X_b$, then $\{b, c'\}$ is an edge, thus $\{a, c, c', b\}$ forms a 4-cycle. This is impossible. Similarly, if b is not an end vertex, and if b' is the neighbor of b in R distinct from a then $\ell(b') = 1$ and b' is in X_c . Now, if there are a' and b' as above, the path c', c, a, b, b' cannot be extended. Suppose not and let $x \notin \{a, b, c, b', c'\}$ a vertex that extends the path c', c, a, b, b'. If $\ell(x) = 0$, then $x \in X_a$ because otherwise a would have degree three in a which is not possible. But then $a \in A$ induces a 6-cyle. Else if $a \in A$ induces a 6-cyle. Else if $a \in A$ induces a 6-cyle. Else if $a \in A$ is the neighbor of $a \in A$ induces a 6-cyle. Else if $a \in A$ is the neighbor of $a \in A$ induces a 6-cyle. Else if $a \in A$ is the neighbor of $a \in A$ induces a 6-cyle. Else if $a \in A$ is the neighbor of $a \in A$ induces a 6-cyle. Else if $a \in A$ is the neighbor of $a \in A$ induces a 6-cyle. Else if $a \in A$ is the neighbor of $a \in A$.

Now, suppose that b is an end vertex. Then c is not an end vertex because R has length five. Let $c' \in R$ be the neighbor of c distinct of a. Then, as proved in the previous paragraph, $\ell(c') = 1$ and $c' \in X_b$. Let d be the neighbor of c' in R distinct from c. Then $\ell(d) = 0$ and $d \in X_a$. Indeed, suppose $\ell(d) = 1$, then d is adjacent to c or to d which is impossible. It follows then that $d \in X_a$ because otherwise d would be adjacent to d, making d of degree three in d which is impossible. We now prove that d cannot be extended. Indeed, let d for d is adjacent to d or d and this is impossible. Else if d is adjacent to d or d and this is impossible.

For the inductive step, let k > 5 be the length of $R := x_0, ..., x_k$; assume that every path of length k-1 is contained in at most two components of G. By the induction hypothesis, the path $x_1, ..., x_k$ is contained in at most two components X_n and X_m of G. If m = n or $x_0 \in X_m \cup X_n$, then R is contained in at most two components of G and we are done. So we may assume that $m \neq n$ and $x_0 \notin X_m \cup X_n$. We argue to a contradiction. We may assume that $x_1 \in X_n$. Since the path $x_0, ..., x_{k-1}$ has a smaller length than that of R, it is contained in two components of G, namely X_n and the component containing x_0 . Thus $V(R) \cap X_m = \{x_k\}$. Next, we may assume without loss of generality that $\ell(x_0) = 0$ (otherwise, replace u by u + 1). Then $\ell(x_k) = 0$ (this is because $\{x_0, x_k\}$ is not an edge and x_0 and x_k are in different components) and $\ell(x_1) = 1$ (this is because $\{x_0, x_1\}$ is an edge and x_0 and x_1 are in different components). From $\ell(x_1) = 1$ and $\ell(x_k) = 0$ we get that $\{x_1, x_k\}$ is an edge hence k = 2, a contradiction. This completes the proof of Claim 2.

We should mention that the bound in Claim 2 is sharp. For an example see Figure 3.

Claim 3: If R is contained in two distinct components X_m and X_n of G such that $|V(R) \cap X_m| = 1$ or $|V(R) \cap X_n| = 1$, then the length of R is less than $\varphi(u)$.

Proof of Claim 3: Let R be a path that is contained in G so that $|V(R) \cap X_m| = 1$. Let x be such that $V(R) \cap X_m \coloneqq \{x\}$. Without loss of generality we may assume that $\ell(x) = 0$ (otherwise, replace u by $u \dotplus 1$). We suppose that the length of R is at least $\varphi(u)$ and we argue to a contradiction. Since the degree of x in R is at most two we infer that the graph induced by $V(R) \setminus \{x\}$ on X_m is the direct sum of at most two paths R_1 and R_2 where R_1 or R_2 could be empty. Hence $|V(R)| = |V(R_1)| + |V(R)| + 1$. Since the length of R is at least $\varphi(u) \coloneqq 2(l(u) + 1) + 1$, we infer that R_1 or R_2 has length at least l(u) + 1, say R_1 has length at least l(u) + 1. From x is adjacent to exactly one vertex y of R_1 and $\ell(x) = 0$ we deduce that $\ell(y) = 1$ and all other vertices z of R_1 verify $\ell(z) = 0$. Hence u has a factor of zero's of cardinality l(u) + 1 which is impossible. This completes the proof of Claim 3.

Claim 4: If R is contained in two distinct components X_m and X_n of G such that $|V(R) \cap X_m| \ge 2$ and $|V(R) \cap X_n| \ge 2$, then the length of R is at most five.

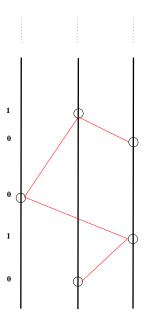


Fig. 3: An example of a path of length four contained in three components in the case \star is the Boolean sum. The thick vertical lines represent components of $\widehat{G}_{(u,\star)}$.

Proof of Claim 4: Since R is connected we infer that, up to a permutation of m and n, there are vertices x and y such that $x \in V(R) \cap X_m$, $\ell(x) = 0$, and $y \in V(R) \cap X_n$, $\ell(y) = 1$. From $\ell(y) = 1$ we deduce that $|V_0(R) \cap X_m| \le 2$. Now suppose for a contradiction that all vertices of $V(R) \cap X_m$ are labeled 0. From $|V_0(R) \cap X_m| \le 2$ and our assumption $|V(R) \cap X_m| \ge 2$ we deduce that $|V(R) \cap X_m| = 2$. In particular, y has degree two in R. Therefore, no vertex $v \in V(R) \cap X_n$ distinct from y is joined to y and furthermore can be labeled 1 because otherwise the graph induced on $\{y,v\} \cup (V(R) \cap X_m)$ is a 4-cycle. But then there is no subpath of R that connects y and v (such a path will have to contain some vertex labelled 1 and there are none). This proves that not all vertices of $V(R) \cap X_m$ can be labeled 0. By symmetry we obtain that not all vertices of $V(R) \cap X_n$ can be labeled 1. Let $t \in V(R) \cap X_m$ be labeled 1 and let $z \in V(R) \cap X_n$ be labeled 0.

Next, we prove that $|V_0(R)| \le 3$. Suppose for a contradiction that there are distinct vertices $r,s \in V(R)$ labeled 0 and such that $\{r,s\} \cap \{x,z\} = \varnothing$. Then $\{r,s\} \notin X_m$ and $\{r,s\} \notin X_n$ because otherwise y or t would have degree at least three which is not possible. Say $r \in X_m$ and $s \in X_n$ and note that t and y have degree two. We prove that $V(R) = \{r,s,t,x,y,z\}$. This leads to a contradiction since $\{r,y,x\}$ and $\{s,t,z\}$ are then nontrivial connected components of R. For the proof, suppose that $q \notin \{r,s,t,x,y,z\}$ is a vertex of G. If $\ell(q) = 0$, then q is adjacent to t if t if t in t is adjacent to t if t in t

 $|V(R)| \le 6$. This proves that the length of R is at most five, as claimed. This completes the proof of Claim 4.

The conclusion of the lemma in the case \star is the Boolean sum is obtained as follows. Let R be an induced path of length at least $\varphi(u)$ in G. It follows from Claim 2 and $\varphi(u) \geq 6$ that R is contained in at most two components of G. It follows from Claims 3 and 4 that R cannot be included in two distinct components of G. Hence, R is contained in one component of G as required.

(•) \star is the second projection, that is $i \star j = j$ for all i, j. We note that since R is connected, there exist $n' < n, x' \in V(R) \cap X_{n'}, x \in V(R) \cap X_n$ such that $\{x', x\} \in E(R)$. Since $\{x', x\} \in E(R), \ell(x) = 1$. Since \star is the second projection, x is adjacent to all vertices of $V(R) \setminus X_n$. Hence, $|V(R) \setminus X_n| \le 2$ because otherwise the degree of x in R would be at least three which is not possible.

Claim 1: $|V(R) \setminus X_n| = 1$. Hence, $V(R) \setminus X_n = \{x'\}$.

Suppose that $|V(R) \setminus X_n| = 2$. If $y \in V(R) \cap X_n$ distinct from x, then y cannot be adjacent to x (this is because x has degree two in R). Furthermore, $\ell(y) = 0$ because otherwise y would be adjacent to all vertices of $V(R) \setminus X_n$ and hence R would have a 4-cycle which is not possible. From that, y is not adjacent to any vertex of R which is impossible. Consequently, $|V(R) \cap X_n| = 1$. It follows that |V(R)| = 3. Due to $\varphi(u) \ge 5$, we have $|V(R)| \ge 6$, a contradiction. This completes the proof of Claim 1.

Note that since every vertex of $V(R) \setminus X_n$ is adjacent to every vertex of $V_1(R) \cap X_n$ we must have $|V_1(R) \cap X_n| \le 2$.

Case 1. $|V_1(R) \cap X_n| = 1$.

In this case, the length of R is at most $l_0(u) + 1$.

Indeed, we have $V_1(R) \cap X_n = \{x\}$. Hence x is the only neighbour of x'. Thus x' is an end vertex of R, the remaining vertices of R are in X_n , all vertices distinct from x being labelled 0. So the length of R is $l_0 + 1$ as claimed. Since $\varphi(u) > l_0(u) + 1$ this case is not possible.

Case 2. $|V_1(R) \cap X_n| = 2$.

In this case, the length of R is at most $2(l_0(u) + 1)$.

Indeed, the vertex x' is linked to the two vertices of $V_1(R)$, thus, $R \setminus \{x'\}$ has two connected components, each one made of a path in X_n labelled 0 except an end vertex belonging to $V_1(R)$. Thus the length of R is $2(\ell_0(u) + 1)$. Since $\varphi(u) > 2(\ell_0(u) + 1)$ this case is not possible.

This proof is quite different from the original one. We thank one of the referees of this paper for suggesting an other approach that eventually lead us to the proof given above.

Remark 2 Recall that $G_{\star}\omega^d$ and $G_{\star}\omega$ have the same age (Corollary 16). If the operation \star is symmetric, then $G_{\star}\omega^d$ and $G_{\star}\omega^d$ are equimorphic and the corresponding $\widehat{Q}_{(u,\star)}$'s are path-minimal. On the other hand, if \star is the second projection the corresponding $\widehat{Q}_{(u,\star)}$'s are path-minimal but not equimorpic.

4.3.2 Incomparable ages

We complete the proof of Theorem 1 by proving:

Proposition 21 Let \star be the Boolean sum or the second projection on $\{0,1\}$. If u and u' are two uniformly recurrent words which are not constant and the factor sets Fac(u) and Fac(u') are nonequivalent in the sense of Definition 1 then $Age(\widehat{G}_{(u,\star)})$ and $Age(\widehat{G}_{(u',\star)})$ are incomparable w.r.t. set inclusion.

Let $G_{(u,\star)}$ be the lexicographical sum of copies of P_u and $G := \widehat{G}_{(u,\star)}$ be the resulting graph on $\mathbb{N} \times \mathbb{N}$. We associate to this graph G a set $\mathcal{A}_G(u)$ of words over the alphabet $\{0,1\}$ defined as follows. A word $w := w_0 \cdots w_n \cdots w_{m-1}$ belongs to \mathcal{A}_G if there is a path $(j,v_0),\ldots,(j,v_{m-1})$ in $\{j\} \times \mathbb{N}$ and some $(i,v) \in \mathbb{N} \times \mathbb{N}$ with $j \neq i$ such that $w_n = 1$ if and only if the pair $\{(j,v_n),(i,v)\}$ is an edge of G for all n < m.

Taking account of the definition of $G_{(u,\star)}$ the condition on w amounts to say that either $w_n = u(v) \star u(v_n)$ for all n < m if j < i or $w_n = u(v_n) \star u(v)$ if i < j.

Let $u := (u(n))_{n \in I}$ be a word on a general alphabet L, the index set I being an interval of \mathbb{Z} . For $i \in L$ set $i \star u := (i \star u(n))_{n \in I}$. For a set U of words set $i \star U := \{i \star u : u \in U\}$ and if X is a subset of L set $X \star U := \bigcup_{i \in X} i \star U$. Similarly define $u \star i$, $U \star i$, $U \star X$.

With these notations we have immediately:

Lemma 22 $A_G(u)$ is the set union of $L \star Fac(u)$, $Fac(u) \star L$ and their duals.

Specializing to $L = \{0, 1\}$, if \star is the Boolean sum then $L \star Fac(u) = Fac(u) \star L = Fac(u) \cup Fac(u+1)$, while $L \star Fac(u) = Fac(u)$ and $Fac(u) \star L = Fac(0) \cup Fac(1)$ if \star is the second projection.

Proof of Proposition 21. The proof goes in two steps.

- 1. If $Age(\widehat{G}_{(u,\star)}) \subseteq Age(\widehat{G}_{(u',\star)})$ then $A_G(u) \subseteq A_G(u')$.
- 2. If $A_G(u) \subseteq A_G(u')$ then Fac(u) and Fac(u') are equivalent.

The proof of Item (1) is easy. Let $w \in \mathcal{A}_G(u)$. Let H be a path in a component and a vertex (i,v) in $G_{(u,\star)}$ witnessing this fact. Since the component are infinite this path is included in a path H' long enough so that Lemma 20 ensures that in any embedding of $\widehat{G}_{(u,\star)} \upharpoonright H' \cup \{(i,v)\}$ in $\widehat{G}_{(u',\star)}$ the image of H' is mapped into a component. Since necessarily (i,v) is mapped into another component, from this embedding we get that $w \in \mathcal{A}_G(u')$.

The proof of Item (2) goes as follows. Suppose that $A_G(u) \subseteq A'_G(u')$. Then, by Lemma 22, Fac(u) is included in the set union of $L \star Fac(u')$, $Fac(u') \star L$ and their dual. Since Fac(u) is an ideal included in a finite union of initial segments, it is included in one of them. Hence it is included in a set of the form Fac(v) equivalent to Fac(u') or equal to $Fac(\mathbf{0})$ or to $Fac(\mathbf{1})$. According to Theorem 5, since v is uniformly recurrent, Fac(v) is Jónsson. Hence Fac(v) = Fac(u). Since u is non constant, v is distinct from $\mathbf{0}$ and $\mathbf{1}$, hence Fac(v) is equivalent to Fac(u').

4.4 Primality and path-minimality

Let G := (V, E) be a graph. If G is path-minimal then, by definition, G is embeddable in $G_{\uparrow V'}$ provided that this restriction contains finite induced paths of unbounded length. And, in this case, $G_{\uparrow V'}$ is path-minimal too. In some cases, as presented in Theorem 23 and 24, G is prime; in other cases, e.g. in Theorem 25, no graph equimorphic to G is prime.

Theorem 23 The graphs \widehat{Q}_k , for $k \geq 2$, are prime.

Proof: Let M be a nontrivial module in \widehat{Q}_k .

• M meets each component in at most one vertex.

Indeed, suppose that M contains at least two vertices, say a and b, of some component X_i of G. Then, it contains X_i . Indeed, first, it contains the vertices of X_i exterior to the path joining a and b; since $X_i \coloneqq \{(i,j): j < i+5\}$ it contains the vertices on that path. Next, we prove that M = X. For that we prove that M contains X_k for all $k \ne i$. Let $a \coloneqq (i,0), b \coloneqq (i,1), c \coloneqq (k,0), d \coloneqq (k,1)$. By construction $d \sim b$ and $d \sim a$. Hence by the same token, $c, d \in M$. Since M is a module containing a and b it contains c. Similarly $c \not\sim a$ and $c \sim b$. Hence by the same token, $d \in M$. Since M contains two elements of X_k it contains all of it. Thus M = X. Contradicting the fact that M is nontrivial.

• M is a singleton.

Indeed, suppose that M contains at least two elements $a \coloneqq (i,n)$ and $b \coloneqq (j,m)$. According to the previous item, $i \ne j$. Without loss of generality we may suppose that i > j, so that $|X_i| \ge 6$. We claim that there is some $a' \coloneqq (i,n')$ distinct of a such that a' is not linked to a and b in the same way. Since M is a module, this implies that a belongs to M and this contradicts the previous item. In order to prove our claim, set $v \coloneqq 1$ if $n \ne k$ m and v = 0 otherwise. Suppose that v = 0, that is $a \ne b$. Since $|X_i| \ge 6$, one of the sets $\{n+1,n+2,n+3\}$ and $\{n-1,n-2,n-3\}$ is included in X_i . Suppose that $\{n+1,n+2,n+3\} \subseteq X_i$. If k > 2 then for $n' \coloneqq n+2$, $a' \in X_i$ and $a' \ne a$ but $a' \sim b$. If k = 2 then the same conclusion holds for $n' \coloneqq n+3$. Suppose that v = 1, that is $a \sim b$. If $n+1 \equiv_k m$, set $n' \coloneqq n+1$. If not, then either $n+2 \not\equiv_k m$ in which case set $n' \coloneqq n+2$, or $n+2 \equiv_k m$, in which case $n+3 \not\equiv_k m$ and we may set $n' \coloneqq n+3$.

Remark 3 \widehat{Q}_k , $k \ge 2$, is not minimal prime in the sense of Definition 5.

Indeed, with similar arguments as in the proof of Theorem 23, the restriction of \widehat{Q}_k to $\mathbb{N} \times \{0, 1, 2, 3, 4\}$ is also prime and hence \widehat{Q}_k is not minimal prime in the sense of Definition 5.

Theorem 24 There are 2^{\aleph_0} path-minimal prime graphs whose ages are pairwise incomparable and wqo.

Proof: Let \star be the Boolean sum. Let u be a uniformly recurrent and non constant word, let $G_{(u,\star)}$ be the labelled graph on $\mathbb{N} \times \mathbb{N}$ associated to u, let ℓ be the labelling and G be the restriction of $\widehat{G}_{(u,\star)}$ to the set $X := \{(n,m) : m \leq n + \varphi(u)\}$. Containing only paths of finite unbounded length, this graph is path-minimal. We show in an almost identical way as in Theorem 23 that this graph is prime.

Let M be a nontrivial module in G.

• M meets each component in at most one vertex. Indeed, suppose that M contains at least two vertices, say a and b, of some component X_i of \widehat{Q}_k . Then it contains X_i . Indeed, note first that it contains the vertices in X_i exterior to the path joining a to b, next note that it contains the vertices of X_i on that path (since $X_i := \{(i,j): j < i+5\}$). Next we prove that M = X. Let $c := (j,k) \notin X_i$, that is $j \neq i$. Since $|X_i| \geq \varphi(u)$, X_i contains a := (i,n) and b := (i,m) with $u(n) \neq u(m)$, hence $\ell(a) \dotplus \ell(c) = u(n) \dotplus u(k) \neq u(m) \dotplus u(k) = \ell(b) \dotplus \ell(c)$. That means that c is not linked to a in the same way as to b. Since M is a module, it contains c. Thus M = X. A contradiction.

• M is a singleton.

Indeed, suppose that M contains at least two elements a := (i, n) and b := (j, m). According to the previous item, $i \neq j$. We claim that there is some c := (i, k) distinct of a such that c is not linked to a and b in the same way. Since M is a module, this implies that c belongs to M and this contradicts the previous item. In order to prove our claim, set $v := \ell(a) \star \ell(b)$. Suppose that v = 0, that is a not linked to b. Since $|X_i| \geq \varphi(u)$ we may find k with $|k - i| \geq 2$ and $\ell(c) \neq \ell(a)$, proving our claim in that case. We do similarly if v = 1.

Theorem 25 There are 2^{\aleph_0} path-minimal non prime graphs whose ages are pairwise incomparable and wqo and such that any graph equimorphic to these graphs is also nonprime.

Proof: Let \star to be the second projection, i.e., $i \star j = j$ for all $i, j \in \{0, 1\}$. Consider the graph $\widehat{G}_{(u, \star)}$ for a uniformly recurrent word u. For $i \in \mathbb{N}$, set $X_i \coloneqq \{i\} \times P_u$. Let $k \in \mathbb{N}$. We claim that $M_k \coloneqq \cup_{j=0}^k X_i$ is a module in $\widehat{G}_{(u, \star)}$. Indeed, if $x \notin M_k$, then either $\ell(x) = 0$, in which case x is not adjacent to any vertex of M_k , or $\ell(x) = 1$, in which case x is adjacent to every vertex of M. Now let G be a graph equimorphic to $\widehat{Q}_{(u, \star)}$. Clearly the trace of M_k on G is a module in G and if is k large enough, M_k is nontrivial proving that G is not prime. According to Proposition 17 this graph is path-minimal. According to Proposition 21 (applied to the second projection) there are 2^{\aleph_0} graphs of that form with pairwise incomparable ages. \square

Remark 4 We get the same conclusion by replacing the second projection by the first. Indeed, in that case, the sets $\mathbb{N} \setminus M_k$ are modules of $\widehat{G}_{(u,\star)}$.

5 Proof of Theorem 2

Let G := (V, E) be a graph and let $F \subseteq V$. We define a binary relation \equiv_F on $V \setminus F$ as follows. For $v, v' \in V \setminus F$ set

$$v \equiv_F v'$$
 if for all $f \in F$ $(\{f, v\} \in E \Leftrightarrow \{f, v'\} \in E)$.

It is easily seen that \equiv_F defines an equivalence relation on $V \setminus F$. Let $\tau_G(F)$ be the partition of $V \setminus F$ induced by \equiv_F . Each equivalence class B is completely determined by a subset of F, namely, the subset of $f \in F$ such that $\{f,v\} \in E$ for every $v \in B$. Hence, the number of equivalence classes of \equiv_F is at most $2^{|F|}$.

Lemma 26 Let G := (V, E) be a graph, $F \subseteq V$ and $B \in \tau_G(F)$. Then the equivalence \equiv_B induces a partition of F in at most two subsets A and A'. In particular, $G_{\uparrow A \cup B}$ is the direct or complete sum of $G_{\uparrow A}$ and $G_{\uparrow B}$.

Proof: Let A be the set of $u \in F$ such that $\{u, x\} \in E$ for some $x \in B$. Let $u' \in A$. Since B is an equivalence class of \equiv_F we have $\{u', x'\} \in E$ for all $x' \in B$. Furthermore, if $u' \in F$ we have $u \equiv_B u'$ if $u' \in A$. Hence A is an element of the partition induced by \equiv_B and its complement in F is also an element of the partition induced by \equiv_B . This proves the first assumption of the lemma. The rest is a consequence. \square

Paths in the incomparability graph of a poset have a special form given the following lemma which is an improvement of I.2.2 Lemme, p.5 of Pouzet (1978a).

Lemma 27 Let x, y be two vertices of a poset P with x < y. If x_0, \ldots, x_n is an induced path from x to y in the incomparability graph of P then $x_i < x_j$ for all $j \ge i + 2$.

Proof: We proceed by induction on n. If $n \le 2$ the property holds trivially. Suppose $n \ge 3$. Taking out x_0 , induction applies to x_1, \ldots, x_n . Similarly, taking out x_n , induction applies to x_0, \ldots, x_{n-1} . Since the path from x_0 to x_n is induced, x_0 is comparable to every x_j with $j \ge 2$ and x_n is comparable to every x_j with j < n-1. In particular, since $n \ge 3$, x_0 is comparable to x_{n-1} . Necessarily, $x_0 < x_{n-1}$. Otherwise, $x_{n-1} < x_0$ and then by transitivity $x_{n-1} < x_n$ which is impossible since $\{x_{n-1}, x_n\}$ is an edge of the incomparability graph. Thus, we may apply induction to the path x_0, \ldots, x_{n-1} and get $x_0 < x_j$ for every j > 2. Similarly, we get $x_1 < x_n$ and via the induction applied to the path from x_1 to x_n , we obtain $x_j < x_n$ for j < n-1. The stated result follows.

If $P := (V, \leq)$ is a poset, a subset X of V is *order convex* or *convex* if for all $x, y \in X$, $[x, y] := \{z : x \leq z \leq y\} \subseteq X$.

Lemma 28 Let $P := (V, \leq)$ be a poset. If a convex subset C of V contains the end vertices of an induced path of length n + 3, then it contains an induced subpath of length n.

Proof: Let $x_0, ..., x_{n+3}$ be an induced path of length n+3 with $x_0, x_{n+3} \in C$. It follows from Lemma 27 that $x_0 \le x_i \le x_{n+1}$ if $2 \le i \le n+3$. Hence the path $x_2, ..., x_{n+1}$ is in C.

Let $P := (V, \leq)$ be a poset. For $v \in V$, denote by $D(v) := \{u \in V : u < v\}$, $U(v) := \{u \in V : u > v\}$ and $Inc(v) := \{u \in V : u \text{ is incomparable to } v\}$.

Lemma 29 If G is the incomparability graph of a poset $P = (V, \leq)$, then for each subset F of V, each element of $\tau_G(F)$ is convex.

Proof: Observe first that for each $u \in V$, the sets U(v) and D(v) and Inc(v) are convex. Next, observe that the intersection of convex subsets is convex. To conclude that an element of $\tau_G(F)$ is convex it is enough to prove that it is the intersection of convex sets. For that, let $v \notin F$. For each $y \in F$ set $V_v(y)$ to be Inc(y) if $v \mid y$, U(y) if y < v, or D(y) if v < y. We claim that the equivalence class containing $v \in B$ is convex since intersection of convex sets. Next, if $v' \in F$, $v' \in F$.

Lemma 30 Let n, k be nonnegative integers and $f_k(n) = k \cdot (n+4)$. If the incomparability graph of a poset P contains an induced path C of length at least $f_k(n)$, then for every covering of P by at most k convex sets, one convex set contains a subpath of C of length n.

Proof: We start with the following claim.

Claim 1 If the vertices of a path of length m are coloured with k colors, then there are two vertices of the same color at distance at least $\lfloor \frac{m}{k} \rfloor$.

Proof of Claim 1 Let $v_0, v_1, ..., v_m$ be a path, $l := \lfloor \frac{m}{k} \rfloor$ and $w_0 := v_0, w_1 := v_l, w_2 := v_{2l}, ..., w_k := v_{kl}$. The colors of these k+1 vertices cannot be all different, so two vertices w_i, w_j with i < j have the same color. Their distance is at least l. \square

If we have a path of length at least $f_k(n) = k \cdot (n+4)$, there are two vertices x, y on this path at distance at least n+4. The induced path is $x = x_i, x_{i+1}, x_{i+2}, ..., x_{i+n+2}, x_{i+n+3}, x_{i+n+4}$. According to the proof of Lemma 28, all vertices $x_{i+2}, ..., x_{i+n+2}$ are in the convex hull of $\{x_i, x_{i+n+4}\}$.

Lemma 31 Let $P := (V, \leq)$ be a poset. If Inc(P) contains induced finite paths of unbounded length, then for every integer n there is an induced path C of length n and a subset V' of $V \setminus V(C)$ such that:

- (1) V' is convex and contains induced paths of Inc(P) of unbounded finite lengths.
- (2) Either all vertices $x \in V(C)$ and $y \in V'$ are adjacent, or nonadjacent in Inc(P).

Proof: Since Inc(P) contains induced paths of unbounded length, it contains a path C' of length $n' \ge 2 \cdot (n+4)$. Let F := V(C'). Since Inc(P) contains induced paths of unbounded lengths and F is finite, the same holds for $V \setminus F$. The partition of $V \setminus F$ in equivalence classes with respect to \exists_F has at most $2^{n'+1}$ equivalence classes. According to Lemma 29 each block is convex. According to Lemma 30 some of these equivalence classes contain paths of unbounded length. According to Lemma 26, the equivalence $\exists_{V'}$ on F induces a partition in at most two sets. According to Lemma 30, one of these two sets contains a subpath C of C' of length C', C' we get the required conclusion. \Box We prove Theorem 2

as follows. With Lemma 31 we build a sequence $(C'_n, V'_n)_{n \in \mathbb{N}}$ where C'_n is an induced path of length n of V'_{n-1} , with $V'_{-1} = V$ and such that V'_n is convex and all its elements are either all adjacent or nonadjacent to C'_n . This does not prevent the situation that at some stage C'_n is nonadjacent to all elements of V'_n and some other C'_n adjacent to all. But at least either there will be infinitely many in the first case, in which case we obtain a direct sum of paths of arbitrarily large lengths, or not, in which case we obtain a complete sum.

6 Proof of Theorem 3

The result is a consequence of the following lemma.

Lemma 32 The supremum of the diameters of the connected components of a graph G is infinite if and only if G embeds a direct sum of isometric paths of arbitrarily large lengths.

Proof: Let us associate to each connected component C of G the supremum of the lengths of the isometric paths in C. If all these suprema are finite we are done (in each connected component we can select a path of maximal length). Otherwise, for some connected component the supremum is infinite. We can assume that G is the graph induced on this connected component. Let V be its set of vertices. If X is a subset of V it has a diameter $\delta(X)$. If this diameter is finite, the diameter of $B_G(X,1)$ is also finite (at most $\delta(X)+2$), and hence $B_G(X,m)$ is finite for any every $m \ge 1$. The diameter of $V \setminus B_G(X,m)$ is infinite (indeed if U,W are two subsets the diameter of $U \cup W$ is at most $\delta(U)+\delta(W)+d(U,W)$). Choose an integer m arbitrarily. Take k < m and two elements x,y in $V \setminus B_G(X,m)$ at distance k and a path k of length k. If a vertex of this path is in k in the complement of k in the foreign in the sum of the other paths already constructed.

Acknowledgements

Results of this paper have been presented by the second author to ALGOS 2020 Algebras, Graphs and Ordered Sets, Loria, Nancy, August 26th to 28th 2020. The authors are pleased to thank the organizers for their warm welcome.

The authors are grateful to two anonymous referees for their careful reading of the manuscript and their many comments and suggestions.

References

- J.-P. Allouche and J. Shallit. *Automatic sequences*. Cambridge University Press, Cambridge, 2003. ISBN 0-521-82332-3. doi: 10.1017/CBO9780511546563. Theory, applications, generalizations.
- R. Assous and M. Pouzet. Jónsson posets. *Algebra Universalis*, 79(3):Paper No. 74, 26, 2018. doi: 10.1007/s00012-018-0551-7.
- V. Berthé and M. Rigo, editors. Combinatorics, automata and number theory, volume 135 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2010. ISBN 978-0-521-51597-9. doi: 10.1017/CBO9780511777653.
- F. Buckley and F. Harary. On longest induced paths in graphs. Chinese Quart. J. Math., 3(3):61–65, 1988.
- A. Ehrenfeucht, T. Harju, and G. Rozenberg. *The theory of 2-structures*. World Scientific Publishing Co., Inc., River Edge, NJ, 1999. ISBN 981-02-4042-2. doi: 10.1142/4197. A framework for decomposition and transformation of graphs.
- N. P. Fogg. Substitutions in dynamics, arithmetics and combinatorics, volume 1794 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2002. ISBN 3-540-44141-7. doi: 10.1007/b13861. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.
- R. Fraïssé. L'intervalle en théorie des relations; ses généralisations; filtre intervallaire et clôture d'une relation. In *Orders: description and roles* (*L'Arbresle, 1982*), volume 99 of *North-Holland Math. Stud.*, pages 313–341. North-Holland, Amsterdam, 1984.
- R. Fraïssé. *Theory of relations*, volume 145 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, revised edition, 2000. ISBN 0-444-50542-3. With an appendix by Norbert Sauer.
- T. Gallai. A translation of T. Gallai's paper: "Transitiv orientierbare Graphen" [Acta Math. Acad. Sci. Hungar. **18** (1967), 25–66; MR0221974 (36 #5026)]. In *Perfect graphs*, Wiley-Intersci. Ser. Discrete Math. Optim., pages 25–66. Wiley, Chichester, 2001. Translated from the German and with a foreword by Frédéric Maffray and Myriam Preissmann.
- G. Higman. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc.* (3), 2:326–336, 1952. doi: 10.1112/plms/s3-2.1.326.
- P. Ille. A characterization of the indecomposable and infinite graphs. *Glob. J. Pure Appl. Math.*, 1(3): 272–285, 2005.

K. A. Kearnes and G. Oman. Jónsson posets and unary Jónsson algebras. *Algebra Universalis*, 69(2): 101–112, 2013. doi: 10.1007/s00012-013-0222-7.

- D. Kelly. Comparability graphs. In *Graphs and order (Banff, Alta., 1984)*, volume 147 of *NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci.*, pages 3–40. Reidel, Dordrecht, 1985.
- D. Kőnig. Über eine schlussweise aus dem endlichen ins unendliche. *Acta Sci. Math. (Szeged)*, 3:121–130, 1927.
- C. Laflamme, M. Pouzet, N. Sauer, and I. Zaguia. Pairs of orthogonal countable ordinals. *Discrete Math.*, 335:35–44, 2014. doi: 10.1016/j.disc.2014.07.005.
- M. Lothaire. Algebraic combinatorics on words, volume 90 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2002. ISBN 0-521-81220-8. doi: 10.1017/CBO9781107326019. A collective work by Jean Berstel, Dominique Perrin, Patrice Seebold, Julien Cassaigne, Aldo De Luca, Steffano Varricchio, Alain Lascoux, Bernard Leclerc, Jean-Yves Thibon, Veronique Bruyere, Christiane Frougny, Filippo Mignosi, Antonio Restivo, Christophe Reutenauer, Dominique Foata, Guo-Niu Han, Jacques Desarmenien, Volker Diekert, Tero Harju, Juhani Karhumaki and Wojciech Plandowski, With a preface by Berstel and Perrin.
- E. C. Milner and M. Pouzet. Antichain decompositions of a partially ordered set. In *Sets*, *graphs and numbers (Budapest, 1991)*, volume 60 of *Colloq. Math. Soc. János Bolyai*, pages 469–498. North-Holland, Amsterdam, 1992.
- M. Morse and G. A. Hedlund. Symbolic dynamics II. Sturmian trajectories. *Amer. J. Math.*, 62:1–42, 1940. doi: 10.2307/2371431.
- D. Oudrar. Sur l'énumération de structures discrètes: une approche par la théorie des relations. PhD thesis, Université d'Alger USTHB à Bab Ezzouar, 2015. URL ArXiv:1604.05839.
- N. Polat. Graphs without isometric rays and invariant subgraph properties. I. *J. Graph Theory*, 27(2): 99–109, 1998. doi: 10.1002/(SICI)1097-0118(199802)27:2(99::AID-JGT5)3.0.CO;2-A.
- M. Pouzet. Un bel ordre d'abritement et ses rapports avec les bornes d'une multirelation. *C. R. Acad. Sci. Paris Sér. A-B*, 274:A1677–A1680, 1972.
- M. Pouzet. Ensemble ordonné universel recouvert par deux chaines. *J. Combin. Theory Ser. B*, 25(1): 1–25, 1978a. doi: 10.1016/0097-3165(78)90026-2.
- M. Pouzet. Sur la théorie des relations. 1978b. Thèse de doctorat d'État.
- M. Pouzet. Relation minimale pour son âge. *Z. Math. Logik Grundlagen Math.*, 25(4):315–344, 1979. doi: 10.1002/malq.19790251905.
- M. Pouzet and N. Sauer. From well-quasi-ordered sets to better-quasi-ordered sets. *Electron. J. Combin.*, 13(1):Research Paper 101, 27, 2006. URL http://www.combinatorics.org/Volume_13/Abstracts/v13i1r101.html.

- M. Pouzet and I. Zaguia. On minimal prime graphs and posets. *Order*, 26(4):357–375, 2009. doi: 10.1007/s11083-009-9131-y.
- M. Sobrani. Structure d'ordre de la collection des âges de relations. PhD thesis, Université Claude-Bernard, Lyon, 1992.
- M. Sobrani. Sur les âges de relations et quelques aspects homologiques des constructions d+m. *Thèse de doctorat d'état, Université S.M.Ben Abdallah-Fez, Fez*, 2002.
- D. P. Sumner. Graphs indecomposable with respect to the *X*-join. *Discrete Math.*, 6:281–298, 1973. doi: 10.1016/0012-365X(73)90100-3.
- M. E. Watkins. Infinite paths that contain only shortest paths. *J. Combin. Theory Ser. B*, 41(3):341–355, 1986. doi: 10.1016/0095-8956(86)90055-9.