

# The 2-colouring problem for $(m, n)$ -mixed graphs with switching is polynomial

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A mixed graph is a set of vertices together with an edge set and an arc set. An  $(m, n)$ -mixed graph  $G$  is a mixed graph whose edges are each assigned one of  $m$  colours, and whose arcs are each assigned one of  $n$  colours. A *switch* at a vertex  $v$  of  $G$  permutes the edge colours, the arc colours, and the arc directions of edges and arcs incident with  $v$ . The group of all allowed switches is  $\Gamma$ .

Let  $k \geq 1$  be a fixed integer and  $\Gamma$  a fixed permutation group. We consider the problem that takes as input an  $(m, n)$ -mixed graph  $G$  and asks if there is a sequence of switches at vertices of  $G$  with respect to  $\Gamma$  so that the resulting  $(m, n)$ -mixed graph admits a homomorphism to an  $(m, n)$ -mixed graph on  $k$  vertices. Our main result establishes that this problem can be solved in polynomial time for  $k \leq 2$ , and is NP-hard for  $k \geq 3$ . This provides a step towards a general dichotomy theorem for the  $\Gamma$ -switchable homomorphism decision problem.

**Keywords:** Graph colouring, edge-coloured homomorphism, reconfiguration, switching

## 1 Introduction

Homomorphisms of graphs (and in general relational systems) are well studied generalizations of vertex colourings (Hell and Nešetřil, 2008). Given a graph (or some generalization)  $G$ , the question of whether  $G$  admits a  $k$ -colouring, can be equivalently rephrased as “does  $G$  admit a homomorphism to a target on  $k$  vertices?”.

In this paper we study homomorphisms of  $(m, n)$ -mixed graphs endowed with a switching operation under some fixed permutation group. (Formal definitions and precise statements of our results are given below.) Our main result is that the 2-colouring problem under these homomorphisms can be solved in polynomial time. As  $k$ -colouring for classical graphs can be encoded within our framework,  $k$ -colouring in our setting is NP-hard for fixed  $k \geq 3$ . That is,  $k$ -colouring for  $(m, n)$ -mixed graphs with a switching operation exhibits a dichotomy analogous to  $k$ -colouring of classical graphs (Garey and Johnson, 1979). Thus, our work may be viewed as a first step towards a dichotomy theorem for homomorphisms of  $(m, n)$ -mixed graphs with a switching operation. We remark that the  $k$ -colouring problem in our setting is not

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obviously a Constraint Satisfaction Problem (Feder and Vardi, 1993; Bulatov, 2017; Zhuk, 2020) nor is membership in NP clear. These ideas are explored further in a companion paper (Brewster et al., 2022).

We begin with the key definitions to state our main result. In this paper, all graphs and all groups are finite.

A *mixed graph* is a triple  $G = (V(G), E(G), A(G))$  consisting of a set of vertices  $V(G)$ , a set of edges  $E(G)$  of unordered pairs of vertices, and a set of arcs  $A(G)$  of ordered pairs of vertices. Given a pair of vertices  $u$  and  $v$ , there is at most one edge, or one arc, but not both, joining them. Further we assume  $G$  is loop-free. We will use  $uv$  to denote an edge or an arc with end points  $u$  and  $v$  where in the latter case the arc is oriented from  $u$  to  $v$ .

Mixed graphs were introduced in (Nešetřil and Raspaud, 2000) as an attempt to unify the theories of homomorphisms of 2-edge coloured graphs and of oriented graphs. Numerous similarities between the two settings have been observed (see for example, (Alon and Marshall, 1998; Raspaud and Sopena, 1994; Kostochka et al., 1997)), whereas, (Sen, 2014) provides examples highlighting key differences.

In this work we study edge and arc coloured generalizations of mixed graphs. Thus, our work may be viewed as a unification of homomorphisms of edge-coloured graphs and of arc-coloured graphs. Let  $m$  and  $n$  be non-negative integers. Denote by  $[m]$  the set  $\{1, 2, \dots, m\}$ . An  $(m, n)$ -mixed graph is a mixed graph  $G = (V(G), E(G), A(G))$  together with functions  $c : E(G) \rightarrow [m]$  and  $d : A(G) \rightarrow [n]$  that assign to each edge one of  $m$  colours, and to each arc one of  $n$  colours respectively. (The colour sets for edges and arcs are disjoint.) The *underlying mixed graph* of  $G$  is  $(V(G), E(G), A(G))$ , i.e., the mixed graph obtained by ignoring edge and arc colours. The *underlying graph* of  $G$  is the graph obtained by ignoring edge and arc colours and arc directions. An  $(m, n)$ -mixed graph is a cycle if its underlying graph is a cycle and similarly for other standard graph theoretic terms such as path, tree, bipartite, etc.

Fundamental to our work is the following definition. An  $(m, n)$ -mixed graph is *monochromatic of colour  $i$*  if either every edge is colour  $i$  and there are no arcs, or every arc is colour  $i$  and there are no edges. While a monochromatic mixed graph with only edges is naturally isomorphic to its underlying graph, we note that we still view the edges as having colour  $i$ .

Let  $G$  and  $H$  be  $(m, n)$ -mixed graphs. A *homomorphism* of  $G$  to  $H$  is a function  $h : V(G) \rightarrow V(H)$  such that if  $uv$  is an edge of colour  $i$  in  $G$ , then  $h(u)h(v)$  is an edge of colour  $i$  of  $H$ , and if  $uv$  is an arc of colour  $j$  in  $G$ , then  $h(u)h(v)$  is an arc of colour  $j$  in  $H$ . We denote the existence of a homomorphism of  $G$  to  $H$  by  $G \rightarrow H$  or  $h : G \rightarrow H$  when the name of the function is required.

We now turn our attention to the concept of switching an  $(m, n)$ -mixed graph at a vertex  $v$ . This generalizes the concept of switching edge colours or signs (Brewster and Graves, 2009; Zaslavsky, 1982) (permuting the colour of edges incident at  $v$ ) and pushing digraphs (Klostermeyer and MacGillivray, 2004) (reversing the direction of arcs incident at  $v$ ). Let  $\Gamma \leq S_m \times S_n \times S_2^n$  be a permutation group. An element of  $\Gamma$  will act on edge colours, arc colours, and arc directions. Specifically, the element is an ordered  $(n+2)$ -tuple  $\pi = (\alpha, \beta, \gamma_1, \gamma_2, \dots, \gamma_n)$  where  $\alpha$  acts on the edge colours,  $\beta$  acts on the arc colours, and  $\gamma_i$  acts on the arc direction of arcs of colour  $i$ . For the remainder of the paper,  $\Gamma$  will be a permutation group as described here.

Let  $G$  be a  $(m, n)$ -mixed graph, and  $\pi = (\alpha, \beta, \gamma_1, \gamma_2, \dots, \gamma_n) \in \Gamma$ . Define  $G^{(v, \pi)}$  as the  $(m, n)$ -mixed graph arising from  $G$  by *switching at vertex  $v$  with respect to  $\pi$*  as follows. Replace each edge  $vw$  of colour  $i$  by an edge  $vw$  of colour  $\alpha(i)$ . Replace each arc  $a$  of colour  $i$  incident at  $v$  (i.e.,  $a = vx$  or  $a = xv$ ) with an arc of colour  $\beta(i)$  and orientation  $\gamma_i(a)$ . Note,  $\gamma_i(a) \in \{vx, xv\}$ .

Given a sequence of ordered pairs from  $V(G) \times \Gamma$ , say  $\Sigma = (v_1, \pi_1)(v_2, \pi_2) \dots (v_k, \pi_k)$ , we define

switching  $G$  with respect to the sequence  $\Sigma$  as follows:

$$G^\Sigma = (G)^{(v_1, \pi_1)(v_2, \pi_2) \dots (v_k, \pi_k)} = (G^{(v_1, \pi_1)})^{(v_2, \pi_2)(v_3, \pi_3) \dots (v_k, \pi_k)}.$$

Note if we let  $\Sigma^{-1} = (v_k, \pi_k^{-1}) \dots (v_1, \pi_1^{-1})$ , then  $G^{\Sigma\Sigma^{-1}} = G^{\Sigma^{-1}\Sigma} = G$ .

Given a subset of vertices,  $X \subseteq V(G)$ , we can switch at each vertex of  $X$  with respect to a permutation  $\pi \in \Gamma$ , the result of which we denote by  $G^{(X, \pi)}$ . This operation is well defined independently of the order in which we switch. If  $uv$  is an edge or arc with one end in  $X$ , say  $u$ , then we simply switch at  $u$  with respect to  $\pi$ . Suppose both ends of  $uv$  are in  $X$ . If  $uv$  is an edge of colour  $i$ , then after switching at each vertex of  $X$ , the edge will have colour  $\alpha^2(i)$ . If  $uv$  is an arc, then after switching the colour will be  $\beta^2(i)$  and the direction will be  $\gamma_{\beta(i)}\gamma_i(uv)$ .

Two  $(m, n)$ -mixed graphs  $G$  and  $G'$  with the same underlying graph are  $\Gamma$ -switch equivalent if there exists a sequence of switches  $\Sigma$  such that  $G^\Sigma = G'$ . We may simply say *switch equivalent* when  $\Gamma$  is clear from context. Note since  $V(G) = V(G')$ , we are viewing both  $(m, n)$ -mixed graphs as labelled and thus are not considering equivalence under switching followed by an automorphism. Such an extension of equivalence is possible but unnecessary in this work. Since  $\Gamma$  is a group, the following proposition is immediate.

**Proposition 1.1.**  $\Gamma$ -switch equivalence is an equivalence relation on the set of (labelled)  $(m, n)$ -mixed graphs.

We are now ready to define switching homomorphisms. Our definition naturally builds on push homomorphisms of digraphs (Klostermeyer and MacGillivray, 2004) and homomorphisms of signed graphs, introduced by (Guenin, 2005) and developed further by (Naserasr et al., 2015). See (Naserasr et al., 2021) for a recent survey on signed graph homomorphisms. Let  $G$  and  $H$  be  $(m, n)$ -mixed graphs. A  $\Gamma$ -switchable homomorphism of  $G$  to  $H$  is a sequence of switches  $\Sigma$  together with a homomorphism  $G^\Sigma \rightarrow H$ . We denote the existence of such a homomorphism by  $G \rightarrow_\Gamma H$ , or  $f : G \rightarrow_\Gamma H$  when we wish to name the mapping. Observe the notation  $G \rightarrow H$  refers to a homomorphism of  $(m, n)$ -mixed graphs without switching, and  $G \rightarrow_\Gamma H$  refers to switching  $G$  followed by a homomorphism of (the resulting)  $(m, n)$ -mixed graphs.

A useful fact is the following. If  $G \rightarrow_\Gamma H$ , then  $G \rightarrow_\Gamma H^{(v, \pi)}$  for any  $v \in V(H)$  and any  $\pi \in \Gamma$ . To see this let  $\Sigma$  be a sequence of switches such that  $f : G^\Sigma \rightarrow H$ . Let  $X = f^{-1}(v) \subseteq V(G^\Sigma)$ . It is easy to see the same vertex mapping  $f : V(G) \rightarrow V(H)$  defines a homomorphism  $(G^\Sigma)^{(X, \pi)} \rightarrow H^{(v, \pi)}$ . As a result of this observation, we have two immediate corollaries. First,  $\Gamma$ -switchable homomorphisms compose. Second, when studying the question “does  $G$  admit a  $\Gamma$ -switchable homomorphism to  $H$ ?” we are free to replace  $H$  with any  $H'$  switch equivalent to  $H$ .

For (classical) graphs,  $G$  is  $k$ -colourable if and only if it admits a homomorphism to a graph  $H$  of order  $k$ . Analogously, we say an  $(m, n)$ -mixed graph  $G$  is  $\Gamma$ -switchable  $k$ -colourable, if there is an  $(m, n)$ -mixed graph  $H$  of order  $k$  such that  $G \rightarrow_\Gamma H$ . The corresponding decision problem is defined as follows. Let  $k \geq 1$  be a fixed integer and  $\Gamma \leq S_m \times S_n \times S_2^n$  be a fixed group. We define the following decision problem.

$\Gamma$ -SWITCHABLE  $k$ -COL

INPUT: An  $(m, n)$ -mixed graph  $G$ .

QUESTION: Is  $G$   $\Gamma$ -switchable  $k$ -colourable?

Our main result is the following dichotomy result for  $\Gamma$ -SWITCHABLE  $k$ -COL.

**Theorem 1.2.** *Let  $k \geq 1$  be an integer and  $\Gamma \leq S_m \times S_n \times S_2^n$  be a group. If  $k \leq 2$ , then  $\Gamma$ -SWITCHABLE  $k$ -COL is solvable in polynomial time. If  $k \geq 3$ , then  $\Gamma$ -SWITCHABLE  $k$ -COL is NP-hard.*

The NP-hardness half of the dichotomy is immediate.

**Proposition 1.3.** *For  $k \geq 3$ ,  $\Gamma$ -SWITCHABLE  $k$ -COL is NP-hard.*

**Proof:** Let  $G$  be an instance of  $k$ -colouring (for classical graphs). Let  $G'$  be the  $(m, n)$ -mixed graph obtained from  $G$  by assigning each edge colour 1. If  $G$  is  $k$ -colourable, then clearly  $G'$  is  $k$ -colourable. (Assign all edges in  $G'$  and  $K_k$  the colour 1 and use the same mapping.) Conversely, if  $G'$  is  $k$ -colourable, then the  $\Gamma$ -switchable homomorphism induces a homomorphism of the underlying graphs showing  $G$  is  $k$ -colourable.  $\square$

For an Abelian group we remark that if  $G$  and  $G'$  are switch equivalent, then there is a sequence of switches  $\Sigma$  of length at most  $|V(G)|$  so that  $G^\Sigma = G'$ . (This is discussed in more detail below.) Thus when  $\Gamma$  is Abelian,  $\Gamma$ -SWITCHABLE  $k$ -COL is in NP, and we can conclude for  $k \geq 3$ , the problem is NP-complete. The situation for non-Abelian groups is more complicated and is studied further in (Brewster et al., 2022).

It is trivial to decide if an  $(m, n)$ -mixed graph is 1-colourable. Thus to complete the proof we settle the case  $k = 2$ . Results are known when  $\Gamma$  belongs to certain families of groups (Duffy et al., 2021; Leclerc et al., 2021). The remainder of the paper establishes the problem is polynomial time solvable for all groups  $\Gamma$ .

We conclude the introduction with a remark on the general homomorphism problem. Let  $H$  be a fixed  $(m, n)$ -mixed graph and  $\Gamma$  a fixed permutation group.

$\Gamma$ -HOM- $H$

INPUT: An  $(m, n)$ -mixed graph  $G$ .

QUESTION: Does  $G$  admit a  $\Gamma$ -switchable homomorphism to  $H$ ?

The complexity of  $\Gamma$ -HOM- $H$  has been investigated for the same families of groups as  $\Gamma$ -switchable  $k$ -colouring in (Duffy et al., 2021; Leclerc et al., 2021). The following theorem is an immediate corollary to our main result. We remark that similar polynomial complexity results have been proved for push homomorphisms (Klostermeyer and MacGillivray, 2004) and for signed graphs (Brewster et al., 2017).

**Theorem 1.4.** *Let  $H$  be a 2-colourable  $(m, n)$ -mixed graph, then  $\Gamma$ -HOM- $H$  is polynomial time solvable.*

## 2 Restriction to $m$ -edge coloured graphs

If a non-trivial  $(m, n)$ -mixed graph  $G$  is 2-colourable, then the target of order 2 to which  $G$  maps must be a monochromatic  $K_2$  or a monochromatic tournament  $T_2$ . In the former case  $G$  must have only edges and in the latter only arcs. Moreover, the underlying graph of  $G$  must be bipartite as a 2-colouring of  $G$  induces a 2-colouring of the underlying graph.

In this section we focus on the case where  $G$  has only edges and is bipartite. For ease of notation, and to align with the existing literature, we will refer to  $G$  as an  $m$ -edge coloured graph. Recall we use  $[m]$  as the set of edge colours, and in this case we may restrict  $\Gamma$  to be a subgroup of  $S_m$ . We let  $H$  be the  $m$ -edge coloured  $K_2$  with its single edge of colour  $i$ , and denote  $H$  by  $K_2^i$ .

We begin with some key observations. Let  $G$  be an  $m$ -edge coloured graph. If  $G \rightarrow_{\Gamma} K_2^i$ , then every colour appearing on an edge of  $G$  must belong to the orbit of  $i$  under  $\Gamma$ ; otherwise,  $G$  is a no instance. Therefore, we make the assumption that  $\Gamma$  acts transitively on  $[m]$ . Under this assumption  $K_2^i$  is switch equivalent to  $K_2^j$  for any  $j \in [m]$ . Thus we have the following proposition.

**Proposition 2.1.** *Fix  $i \in [m]$ . Let  $G$  be a bipartite  $m$ -edge coloured graph. The following are equivalent.*

- (1)  $G \rightarrow_{\Gamma} K_2^i$ ,
- (2)  $G \rightarrow_{\Gamma} K_2^j$  for any  $j \in [m]$ ,
- (3)  $G$  can be switched to be monochromatic of some colour  $j$ .

**Proof:** The implication (1)  $\Rightarrow$  (2) follows from the fact that  $K_2^i \rightarrow_{\Gamma} K_2^j$  for any  $j \in [m]$  by the transitive action of  $\Gamma$ . The implication (2)  $\Rightarrow$  (3) is trivial. Suppose  $G$  can be switched to be monochromatic of some colour  $j$ . Let  $G$  have the bipartition  $X \cup Y$ . Since  $\Gamma$  is transitive, there is  $\pi \in \Gamma$  such that  $\pi(j) = i$ . Then  $G^{(X, \pi)}$  is monochromatic of colour  $i$  implying  $G \rightarrow_{\Gamma} K_2^i$ .  $\square$

We have reduced the problem of determining whether an  $m$ -edge coloured graph  $G$  is 2-colourable to testing if  $G$  is bipartite and can be switched to be monochromatic of some colour  $j$ .

In the case of signed graphs (two edge colours),  $G$  can be switched to be monochromatic of colour  $j$  if and only if each cycle of  $G$  can be switched to be a monochromatic cycle of colour  $j$  (Zaslavsky, 1982). We shall show the same result holds for bipartite  $m$ -edge coloured graphs. However, for our setting the question of when a cycle can be switched to be monochromatic is more complicated. Hence, we begin by characterizing when an  $m$ -edge coloured even cycle can be made monochromatic. To this end, let  $G$  be a  $m$ -edge coloured cycle of length  $2k$  on vertices  $v_0, v_1, \dots, v_{2k-1}, v_0$ . By switching at  $v_1$ , the edge  $v_0v_1$  can be made colour  $i$ . Next by switching at  $v_2$ , the edge  $v_1v_2$  can be made colour  $i$ . Continuing, we see that  $G$  can be switched so that all edges except  $v_{2k-1}v_0$  are colour  $i$ . For  $i, j \in [m]$ , we say the cycle  $G$  is *nearly monochromatic of colours  $(i, j)$*  if  $G$  has  $2k - 1$  edges of colour  $i$  and 1 edge of colour  $j$ . Thus the problem of determining if an even cycle can be switched to be monochromatic is reduced to the problem of determining if a nearly monochromatic cycle of length  $2k$  can be switched to be monochromatic.

Let  $G$  be a cycle of length  $2k$  that is nearly monochromatic of colours  $(i, j)$ . We define a relation on  $[m]$  by  $j \sim_{2k} i$  if  $G$  is  $\Gamma$ -switch equivalent to a monochromatic  $C_{2k}$  of colour  $i$  or equivalently  $G \rightarrow_{\Gamma} K_2^i$ .

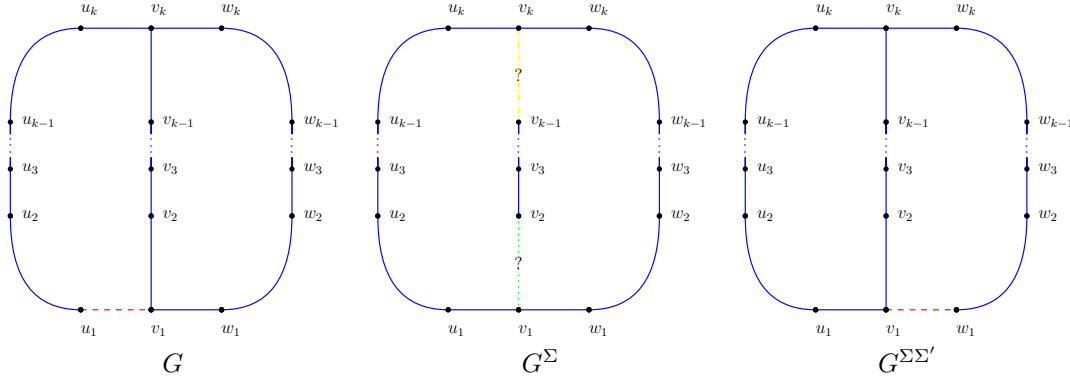
As the definition suggests, the relation is an equivalence relation.

**Lemma 2.2.** *The relation  $\sim_{2k}$  is an equivalence relation.*

**Proof:** The relation is trivially reflexive.

To see  $\sim_{2k}$  is symmetric, assume  $j \sim_{2k} i$ . Let  $G$  be a cycle of length  $2k$  that is nearly monochromatic of colour  $(j, i)$ . Label the vertices of the cycle in the natural order as  $v_0, v_1, \dots, v_{2k-1}, v_0$  where  $v_0v_{2k-1}$  is the unique edge of colour  $i$ . Suppose  $\pi(j) = i$ . Let  $\Sigma = (v_1, \pi), (v_3, \pi), \dots, (v_{2k-3}, \pi)$ . Then  $G^{\Sigma}$  is nearly monochromatic of colour  $(i, j)$ , with edge  $v_{2k-2}v_{2k-1}$  being the unique edge of colour  $j$ . By assumption there is a sequence of switches, say  $\Sigma'$ , so that  $G^{\Sigma\Sigma'}$  is monochromatic of colour  $i$ , giving  $G \rightarrow_{\Gamma} K_2^i$ . Thus,  $G \rightarrow_{\Gamma} K_2^j$  by Proposition 2.1. That is,  $G$  can be made monochromatic of colour  $j$  or  $i \sim_{2k} j$ .

To prove  $\sim_{2k}$  is transitive, suppose  $i \sim_{2k} j$  and  $j \sim_{2k} l$ . Let  $G, G'$ , and  $G''$  be  $m$ -edge coloured cycles of length  $2k$  each with the vertices  $v_0, v_1, \dots, v_{2k-1}$ . (Technically, we are considering three distinct edge



**Fig. 1:** Switching of the theta graph in Theorem 2.3. Solid blue edges are colour  $i$  and dashed red edges are colour  $j$ .

colourings of the same underlying graph.) Suppose  $G, G'$ , and  $G''$  are nearly monochromatic of colours  $(j, i)$ ,  $(l, j)$ , and  $(l, i)$  respectively. There are  $2k - 1$  edges of colour  $j$  in  $G$  with edge  $v_0v_{2k-1}$  of colour  $i$  in  $G$ . Similarly there are  $2k - 1$  edges of colour  $l$  in  $G'$  with edge  $v_0v_{2k-1}$  of colour  $j$  in  $G'$  and  $2k - 1$  edges of colour  $l$  with edge  $v_0v_{2k-1}$  of colour  $i$  in  $G''$ . We shall show  $G''$  can be switched to be monochromatic of colour  $l$ .

By hypothesis, there is a sequence  $\Sigma'$  such that  $G''^{\Sigma'}$  is monochromatic of colour  $l$ . In particular, under  $\Sigma'$  all edges of colour  $l$  remain colour  $l$ , and the edge  $v_0v_{2k-1}$  changes from  $j$  to  $l$ . Thus, if we apply  $\Sigma'$  to  $G''$  the edges of colour  $l$  remain colour  $l$  and the product of those switches at  $v_0$  and  $v_{2k-1}$  changes  $v_0v_{2k-1}$  from colour  $i$  to colour  $\sigma(i)$  for some  $\sigma \in \Gamma$ . We observe by the fact that  $G''^{\Sigma'}$  is monochromatic,  $\sigma(j) = l$ .

We now construct a modified inverse of  $\Sigma'$ . Let  $\Sigma''$  be the subsequence of  $\Sigma'$  consisting of the switches only at  $v_0$  or  $v_{2k-1}$ . That is,  $\Sigma''$  is a subsequence  $(v_{s_0}, \pi_0), (v_{s_1}, \pi_1), \dots, (v_{s_t}, \pi_t)$  where each  $v_{s_r} \in \{v_0, v_{2k-1}\}$ . Let  $X$  (respectively  $Y$ ) be the set of vertices of  $G''$  with even (respectively odd) subscripts. Starting with  $G''^{\Sigma'}$  apply the following sequence of switches. For  $r = t, t - 1, \dots, 0$ , if  $v_{s_r} = v_0$ , then apply the switch  $(X, \pi_r^{-1})$ ; otherwise,  $v_{s_r} = v_{2k-1}$  and apply the switch  $(Y, \pi_r^{-1})$ . The net effect is to apply  $\sigma^{-1}$  to each edge of  $G''^{\Sigma'}$ . Thus each edge of colour  $l$  switches to  $j$  and the edge  $v_0v_{2k-1}$  of colour  $\sigma(i)$  becomes colour  $i$ . That is, we can switch  $G''$  to be  $G$ . By hypothesis,  $G$  can be switched to be monochromatic of colour  $j$ . By Proposition 2.1, the resulting  $m$ -edge coloured graph can be switched to be monochromatic of colour  $l$ , i.e.,  $i \sim_{2k} l$ , as required.  $\square$

We denote the equivalence classes with respect to  $\sim_{2k}$  by  $[i]_{\Gamma}^{2k} = \{j | j \sim_{2k} i\}$ . We now show that these classes are independent of cycle length (for even length cycles).

**Theorem 2.3.** *Let  $\Gamma \leq S_m$  and  $i \in [m]$ . Then  $[i]_{\Gamma}^{2l} = [i]_{\Gamma}^{2k}$  for all  $l, k \in \{2, 3, \dots\}$ .*

**Proof:** Let  $i \in [m]$  and let  $k$  be an integer  $k \geq 2$ . We show  $[i]_{\Gamma}^4 = [i]_{\Gamma}^{2k}$  from which the result follows.

Suppose  $j \in [i]_{\Gamma}^4$ . Let  $G$  be a cycle of length  $2k$  and  $H$  a cycle of length 4 where both are nearly monochromatic of colours  $(i, j)$ . Since  $G \rightarrow H$  and by hypothesis,  $H \rightarrow_{\Gamma} K_2^i$ , we have  $G \rightarrow_{\Gamma} K_2^i$  and thus  $j \in [i]_{\Gamma}^{2k}$ .

Conversely, suppose  $j \in [i]_{\Gamma}^{2k+2}$ . We will show  $j \in [i]_{\Gamma}^{2k}$  from which we can conclude by induction that  $j \in [i]_{\Gamma}^4$ . Let  $G$  be the  $m$ -edge coloured graph constructed as follows. Let  $v_1, v_2, \dots, v_k; u_1, u_2, \dots, u_k;$  and  $w_1, w_2, \dots, w_k$  be three disjoint paths of length  $k-1$ . Join  $v_1$  to both  $u_1$  and  $w_1$ , and  $v_k$  to both  $u_k$  and  $w_k$ . Each edge is colour  $i$  with the exception of  $v_1u_1$  which is colour  $j$ . (Thus,  $G$  is the  $\theta$ -graph with path lengths  $k+1, k-1, k+1$ .) Denote the cycles  $u_1, \dots, u_k, v_k, \dots, v_1, u_1$  and  $w_1, \dots, w_k, v_k, \dots, v_1, w_1$  by  $C_1$  and  $C_2$  respectively. Observe both have length  $2k$ ,  $C_1$  is nearly monochromatic of colours  $(i, j)$  and  $C_2$  is monochromatic of colour  $i$ . Finally, let  $C_3$  be the cycle  $u_1, \dots, u_k, v_k, w_k, \dots, w_1, v_1, u_1$ . The cycle  $C_3$  has length  $2k+2$  and is nearly monochromatic of colours  $(i, j)$ . See Figure 1.

By assumption there exists a sequence of switches  $\Sigma$  (acting on the vertices of  $C_3$ ) such that in  $G^{\Sigma}$  the cycle  $C_3$  is monochromatic of colour  $i$ . We note that  $v_1v_2$  and  $v_{k-1}v_k$  might not be of colour  $i$  in  $G^{\Sigma}$ .

There is an automorphism  $\varphi$  of the underlying graph  $G$  that fixes each  $v_l$ ,  $l = 1, 2, \dots, k$ , and interchanges each  $u_l$  with  $w_l$ . We apply  $\Sigma^{-1}$  to  $\varphi(G^{\Sigma})$  as follows. Let  $\Sigma'$  be the sequence obtained from  $\Sigma$  by reversing the order of the sequence, replacing each permutation with its inverse permutation and replacing all switches on vertices  $u_l$  with switches on  $w_l$  and vice versa. (Switches on  $v_1$  and  $v_k$  are applied to  $v_1$  and  $v_k$  respectively.) Then in  $G^{\Sigma\Sigma'}$  we see that  $C_1$  is monochromatic of colour  $i$ . Therefore  $[i]_{\Gamma}^{2k} \supseteq [i]_{\Gamma}^{2k+2}$  for all  $k \geq 2$ . We conclude  $[i]_{\Gamma}^4 = [i]_{\Gamma}^{2k}$  for all  $k \geq 2$ .  $\square$

As the equivalence classes depend only on the group and not the length of the cycle, we henceforth denote these classes as  $[i]_{\Gamma}$ . If  $j \in [i]_{\Gamma}$ , we say  $i$  can be  $\Gamma$ -substituted for  $j$ ; that is, the single edge of colour  $j$  in the cycle can be switched to colour  $i$ . We call  $[i]_{\Gamma}$  the  $\Gamma$ -substitution class for  $i$ .

For a fixed  $m$  and  $\Gamma$ ,  $[i]_{\Gamma}$  can be computed in constant time as there is a constant number of  $m$ -edge coloured 4-cycles, and a constant number of (single) switches that can be applied to these cycles, from which the equivalence classes can be computed using the transitive closure.

**Theorem 2.4.** *Let  $G$  be an  $m$ -edge coloured  $C_{2k}$ . It can be determined in polynomial time whether there is a  $\Gamma$ -switchable homomorphism of  $G$  to  $K_2^i$ .*

**Proof:** As described above, we can switch  $G$  to be nearly monochromatic of colours  $(i, j)$ , for some  $j$ . Then  $G \rightarrow_{\Gamma} K_2^i$  if and only if  $j \in [i]_{\Gamma}$ . Testing this condition can be done in constant time.  $\square$

We now show the  $\Gamma$ -HOM- $K_2^i$  problem is polynomial time solvable. This is accomplished by showing the problem of determining whether a given  $m$ -edge coloured bipartite graph can be made monochromatic of colour  $i$  is polynomial time solvable.

We begin with the following observation that trees can always be made monochromatic.

**Lemma 2.5.** *Let  $T$  be a  $m$ -edge coloured tree, then for any  $\Gamma$ ,  $T \rightarrow_{\Gamma} K_2^i$ .*

**Proof:** Let  $T$  be a  $m$ -edge coloured tree. Let  $v_1, v_2, \dots, v_{|T|}$  be an ordering of  $V(T)$  rooted at  $v_1$  where each parent appears before all of its children (for example DFS or BFS will work). For each  $k = 2, \dots, |T|$ , switch at  $v_k$  so that the edge from  $v_k$  to its parent in the depth first search ordering has colour  $i$ . We observe that if the subtree  $T[v_1, \dots, v_{k-1}]$  is monochromatic of colour  $i$ , then after switching at  $v_k$ , so is the subtree  $T[v_1, \dots, v_k]$ .  $\square$

Let  $G$  and  $H$  be  $m$ -edge coloured graphs such that  $H$  is a subgraph of  $G$ . A *retraction* from  $G$  to  $H$ , is a homomorphism  $r : G \rightarrow H$  such that  $r(x) = x$  for all  $x \in V(H)$ . We use the following result of Hell.

**Theorem 2.6** (Hell (1972)). *Let  $G$  be a bipartite graph. Suppose  $P$  is a shortest path from  $u$  to  $v$  in  $G$ . Then  $G$  admits a retraction to  $P$ .*

We now show, for general  $m$ -edge coloured graphs  $G$ , testing if  $G \rightarrow_{\Gamma} K_2^i$  comes down to testing if each cycle admits a  $\Gamma$ -switchable homomorphism to  $K_2^i$ . To this end define  $\mathcal{C}(G)$  to be the set of cycles in an  $m$ -edge coloured graph  $G$ , and  $\mathcal{F}_{\Gamma}$  to be the collection of cycles  $C$  such that  $C \not\rightarrow_{\Gamma} K_2^i$ .

**Theorem 2.7.** *Let  $G$  be a connected  $m$ -edge coloured graph and  $\Gamma$  a transitive group acting on  $[m]$ . Suppose  $i \in [m]$ . The following are equivalent.*

- (1)  $G \rightarrow_{\Gamma} K_2^i$ .
- (2) For all cycles  $C \in \mathcal{C}(G)$ ,  $C \rightarrow_{\Gamma} K_2^i$ .
- (3)  $G$  is bipartite and for any spanning tree  $T$  of  $G$ , there is a switching sequence  $\Sigma$  such that in  $G^{\Sigma}$ ,  $T$  is monochromatic of colour  $i$  and for each cotree edge the colour  $i$  can be  $\Gamma$ -substituted for the colour of the cotree edge.
- (4) For all cycles  $C \in \mathcal{F}_{\Gamma}$ ,  $C \not\rightarrow_{\Gamma} G$ .

**Proof:** We first prove the equivalence of statements (1), (2), and (3).

(1)  $\Rightarrow$  (2) is trivially true.

(2)  $\Rightarrow$  (3). We first observe that  $G$  must be bipartite as all cycles in the underlying graph map to  $K_2$ . Let  $T$  be a spanning tree in  $G$  and let  $\Sigma$  be the switching sequence constructed as in the proof of Lemma 2.5. Then  $T$  is monochromatic of colour  $i$  in  $G^{\Sigma}$ . Let  $e$  be a cotree edge of colour  $j$ . The fundamental cycle  $C_e$  in  $T + e$  is nearly monochromatic of colours  $(i, j)$ . By hypothesis  $C \rightarrow_{\Gamma} K_2^i$ . Hence,  $i$   $\Gamma$ -substitutes for  $j$ .

(3)  $\Rightarrow$  (1). As above, let  $T$  be a spanning tree that is monochromatic of colour  $i$  in  $G^{\Sigma}$ . Let  $e_1, e_2, \dots, e_k$  be an enumeration of the cotree edges of  $T$ . By hypothesis for each cotree edge  $e_t$ , its colour, say  $j$  (in  $G^{\Sigma}$ ), belongs to  $[i]_{\Gamma}$ .

Let  $T + \{e_1, \dots, e_t\}$  be the subgraph of  $G^{\Sigma}$  induced by the edges of  $E(T) \cup \{e_1, \dots, e_t\}$ . By Lemma 2.5,  $T \rightarrow_{\Gamma} K_2^i$ . Suppose  $T + \{e_1, \dots, e_{t-1}\} \rightarrow_{\Gamma} K_2^i$ . Let  $e_t = uv$  have colour  $j$ . Let  $P$  be a shortest path from  $u$  to  $v$  in  $T + \{e_1, \dots, e_{t-1}\}$ . By (Hell, 1972), there is a retraction  $r : T + \{e_1, \dots, e_{t-1}\} \rightarrow P$  with  $r(u) = u$  and  $r(v) = v$ . Adding the edge  $e_t$  shows  $T + \{e_1, \dots, e_t\} \rightarrow_{\Gamma} P + e_t$  where  $P + e_t$  is a nearly monochromatic cycle of colours  $(i, j)$ . By assumption  $i$   $\Gamma$ -substitutes for  $j$ , so  $P + e_t \rightarrow_{\Gamma} K_2^i$  and by composition  $T + \{e_1, \dots, e_t\} \rightarrow_{\Gamma} K_2^i$ . By induction,  $G \rightarrow_{\Gamma} K_2^i$ .

Finally, we show (1) and (4) are equivalent. If there is  $C \in \mathcal{F}_{\Gamma}$  such that  $C \rightarrow_{\Gamma} G$ , then  $G \not\rightarrow_{\Gamma} K_2^i$ . Conversely, if  $G \not\rightarrow_{\Gamma} K_2^i$ , then by (2), there is a cycle  $C$  in  $G$  such that  $C \not\rightarrow_{\Gamma} K_2^i$ . In particular,  $C \in \mathcal{F}_{\Gamma}$  and the inclusion map gives  $C \rightarrow_{\Gamma} G$ .  $\square$

Given an  $m$ -edge coloured graph  $G$ , it is easy to test condition (3) for each component. Checking  $G$  is bipartite and the switching of a spanning forest can be done in linear time in  $|E(G)|$ . The look up for each cotree edge requires constant time.

However, the theorem actually gives us a certifying algorithm which we now outline (under the assumption  $G$  is connected). First test if  $G$  is bipartite. If it is not, then we discover an odd cycle certifying a no instance. Otherwise construct a spanning tree, and switch so that the tree is monochromatic of colour  $i$ . Either the colour of each cotree edge belongs to  $[i]_{\Gamma}$  or we discover a cotree edge that does not. In the latter case we have a cycle of  $C \in \mathcal{F}_{\Gamma}$  that certifies  $G$  is a no instance.

Thus assume all cotree edges have colours in  $[i]_{\Gamma}$ . The proof of Theorem 2.7 provides an algorithm for switching  $G$  to be monochromatic of colour  $i$  through lifting the switching of the retract  $P + e_t$  to all of



$G$ . We show how using a similar idea with  $C_4$  also works and gives a clearer bound on the running time. Let  $j$  be the colour of a cotree edge, say  $uv$ . Recall  $j \in [i]_\Gamma^4$ . Let  $H$  be a  $C_4$  with vertices labelled as  $v_0, v_1, v_2, v_3$  and edges coloured as  $v_0v_3$  is colour  $j$  and all other edges are colour  $i$ . Let  $\Sigma$  be a switching sequence so that  $H^\Sigma$  is monochromatic of colour  $i$ . Let  $X$  (respectively  $Y$ ) be the set of vertices of  $G$  in the same part of the bipartition as  $u$  (respectively  $v$ ). For each  $(v_i, \pi_i)$  in  $\Sigma$  we apply the same switch  $\pi_i$  in  $G$  at  $u$  if  $v_i = v_0$ ; at  $X \setminus \{u\}$  if  $v_i = v_2$ ; at  $v$  if  $v_i = v_3$ ; and at  $Y \setminus \{v\}$  if  $v_i = v_1$ . At the end of applying all switches in  $\Sigma$ , edges in  $G$  that were of colour  $i$  remain colour  $i$ , and the cotree edge  $uv$  switches from  $j$  to  $i$ . As  $|\Sigma|$  is constant (in  $|\Gamma|$ ), this switching sequence for  $uv$  requires  $O(|V(G)|)$  switches. In this manner the concatenation of  $|E(G)| - |V(G)| + 1$  such switching sequences (together with the switches required to make  $T$  monochromatic) switch  $G$  to be monochromatic of colour  $i$ . This sequence together with the bipartition of  $G$  certifies that  $G \rightarrow_\Gamma K_2^i$ . We have the following.

**Corollary 2.8.** *The problem  $\Gamma$ -HOM- $K_2^i$  is polynomial time solvable by a certifying algorithm.*

### 3 General $(m, n)$ -coloured graphs

In this section we show the  $\Gamma$ -SWITCHABLE  $k$ -COL problem is polynomial time solvable. As noted above, a general  $(m, n)$ -mixed graph  $G$  is 2-colourable if it only has edges and for some edge colour  $i$ ,  $G \rightarrow_\Gamma K_2^i$  or it only has arcs and for some arc colour  $i$ ,  $G \rightarrow_\Gamma T_2^i$ . Having established the  $\Gamma$ -HOM- $K_2^i$  problem is polynomial time solvable, we now show  $\Gamma$ -HOM- $T_2^i$  polynomially reduces to  $\Gamma$ -HOM- $K_2^i$ . This establishes the polynomial time result of Theorem 1.2 which we restate.

**Theorem 3.1.** *The  $\Gamma$ -SWITCHABLE 2-COL problem is polynomial time solvable.*

**Proof:** Let  $G$  be an instance of  $\Gamma$ -SWITCHABLE 2-COL, i.e., an  $(m, n)$ -mixed graph. If  $G$  is not bipartite, we can answer No. If  $G$  has both edges and arcs, then we can answer No. If  $G$  only has edges, then by Corollary 2.8 we can choose any edge colour  $i$  (we still assume  $\Gamma$  is transitive) and test  $G \rightarrow_\Gamma K_2^i$  in polynomial time.

Thus assume  $G$  is bipartite with bipartition  $(A, B)$  and has only arcs. Analogous to Section 2, we can view  $\Gamma$  as acting transitively on the  $n$  arc colours. If  $\Gamma$  does not allow any arc colours to switch direction, i.e., for all  $\pi \in \Gamma$ ,  $\gamma_i(uv) = uv$  for all  $i$ , then  $G$  must have all its arcs from say  $A$  to  $B$ ; otherwise, we can say No. At this point  $G$  may be viewed as an  $n$ -edge coloured graph. (We can ignore the fixed arc directions.) We apply the results of Section 2.

Finally, we may assume  $G$  is bipartite, with only arcs, and  $\Gamma$  acts transitively on arc colours and directions. That is, for any arc colours  $i$  and  $j$ ,  $\Gamma$  contains a permutation  $\pi_1$  (respectively  $\pi_2$ ) that takes an arc  $uv$  of colour  $i$  to an arc  $uv$  (respectively  $vu$ ) of colour  $j$ .

We now construct a  $(2n)$ -edge coloured graph  $G'$  as follows. Let  $V(G') = V(G)$ . If there is an arc of colour  $i$  from  $u \in A$  to  $v \in B$ , we put an edge  $uv$  of colour  $i^+$  in  $G'$ , and if there is an arc of colour  $i$  from  $v \in B$  to  $u \in A$ , we put an edge  $uv$  of colour  $i^-$  in  $G'$ .

From  $\Gamma$  we construct a new group  $\Gamma' \leq S_{2n}$ . Note that  $\Gamma$  as described above acts on  $(m, n)$ -mixed graphs and  $\Gamma'$  will be naturally restricted to act on  $(2n)$ -edge coloured graphs. Let  $\pi = (\alpha, \beta, \gamma_1, \dots, \gamma_n) \in \Gamma$ . Define  $\pi' \in \Gamma'$  as follows. For each arc colour  $i$ ,

$$\pi'(i^+) = \begin{cases} \beta(i)^+ & \text{if } \gamma_i(uv) = uv \\ \beta(i)^- & \text{if } \gamma_i(uv) = vu \end{cases} \quad \text{and} \quad \pi'(i^-) = \begin{cases} \beta(i)^- & \text{if } \gamma_i(uv) = uv \\ \beta(i)^+ & \text{if } \gamma_i(uv) = vu \end{cases}$$

It can be verified that the mapping  $\pi \rightarrow \pi'$  is a group isomorphism.

The translation of  $G$  to  $G'$  can be expressed as a function  $F(G) = G'$ . It is straightforward to verify  $F$  is a bijection from  $n$ -arc coloured graphs to  $2n$ -edge coloured graphs provided we fix the bipartition  $V(G) = A \cup B$ . Moreover, if  $\pi \in \Gamma$  and  $\pi'$  is the resulting permutation in  $\Gamma'$ , then again it is easy to verify that  $F(G^{(v,\pi)}) = (G')^{(v,\pi')}$  for any  $v$  in  $V(G) = V(G')$ .

Suppose  $G \rightarrow_{\Gamma} T_2^i$ . By the transitivity of  $\Gamma$ , we may assume that  $T_2^i$  has its tail in  $A$ , and thus all arcs in  $G$  can be switched to be colour  $i$  with their tail in  $A$ . The corresponding switches on  $G'$  switch all edges to colour  $i^+$ . That is,  $G' \rightarrow_{\Gamma'} K_2^{i^+}$ . On the other hand, if  $G' \rightarrow_{\Gamma'} K_2^{i^+}$ , then the corresponding switches on  $G$  show that  $G \rightarrow_{\Gamma} T_2^i$  (with the vertices of  $A$  mapping to the tail of  $T_2^i$ ).  $\square$

We conclude this section with a remark on the number of switches required to change the input  $G$  to be monochromatic. There are  $|V(G)| - 1$  switches required to change a spanning tree of  $G$  to be monochromatic of colour  $i$ . To change the cotree edges to colour  $i$  (assuming each is of a colour in  $[i]_{\Gamma}$ ), we claim at most  $c_{\Gamma}|V(G)|$  switches are required where  $c_{\Gamma}$  is a constant depending on  $\Gamma$  and the number of colours ( $m$  and  $n$ ). We argue only for  $m$ -edge coloured graphs, given the reduction above. For (a labelled)  $C_4$ , there are  $m^4$  edge colourings. For each vertex there are  $|\Gamma|$  switches. The *reconfiguration graph*  $\mathcal{C}$  has a vertex for each edge-colouring of  $C_4$  and an edge joining two vertices if there is a single switch that changes one into the other. (The existence of inverses ensures this is an undirected graph.) Thus,  $\mathcal{C}$  has order  $m^4$  and is regular of degree  $|\Gamma|$ . Given  $j \in [i]_{\Gamma}$ , there is a path in  $\mathcal{C}$  from a nearly monochromatic  $C_4$  of colours  $(i, j)$  to a monochromatic  $C_4$  of colour  $i$ . The switches on this path can be lifted to  $G$  so that the spanning tree remains of colour  $i$  and the cotree edge switches to colour  $i$ . The total number of switches is at most  $\max\{\text{diam}(\mathcal{C}')\} \cdot |V(G)|$  where  $\mathcal{C}'$  runs over all components of  $\mathcal{C}$ . Thus we have the following.

**Proposition 3.2.** *Let  $G$  be a  $m$ -edge coloured bipartite graph. Let  $\Gamma$  be a group acting transitively on  $[m]$ . If  $G$  is  $\Gamma$ -switch equivalent to a monochromatic graph, then the sequence  $\Sigma$  of switches which transforms  $G$  to be monochromatic satisfies,*

$$|\Sigma| \leq |V(G)| - 1 + c_{\Gamma}|V(G)|(|E(G)| - |V(G)| + 1)$$

where  $c_{\Gamma}$  depends only on  $\Gamma$  and  $m$ .

In the case that  $\Gamma$  is Abelian, the switches in  $\Sigma$  can be reordered, then combined, so that each vertex is switched only once.

## 4 Conclusion

We have established a dichotomy for the  $\Gamma$ -SWITCHABLE  $k$ -COL problem. This is a step in obtaining a dichotomy theorem for  $\Gamma$ -HOM- $H$  for all  $(m, n)$ -mixed graphs  $H$  and all transitive permutation groups  $\Gamma$ . Work towards a general dichotomy is the focus of our companion paper (Brewster et al., 2022).

If  $\Gamma$  and  $m$  are part of the input to  $\Gamma$ -SWITCHABLE  $k$ -COL, then the analysis above with constant time look-ups for the equivalence classes no longer holds. A naïve computation of the closure of the reconfiguration graph is not possible in polynomial time, but the required computation of the equivalence classes  $[i]_{\Gamma}$  is possible without considering all switches. This is studied further in (Brewster et al., 2022).

Finally, we have restricted our attention to simple  $m$ -edge coloured graphs in this work. A natural next step is to allow parallel edges (of different colours) and loops. We believe with parallel edges and loops the analysis of  $\Gamma$ -HOM- $H$  will be interesting, but non-trivial, similar to the situation for signed graphs (Brewster et al., 2017; Brewster and Siggers, 2018).

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