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*Separating Layered Treewidth and Row Treewidth**

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Layered treewidth and row treewidth are recently introduced graph parameters that have been key ingredients in the solution of several well-known open problems. In particular, the *layered treewidth* of a graph G is the minimum integer k such that G has a tree-decomposition and a layering such that each bag has at most k vertices in each layer. The *row treewidth* of G is the minimum integer k such that G is isomorphic to a subgraph of $H \boxtimes P$ for some graph H of treewidth at most k and for some path P. It follows from the definitions that the layered treewidth of a graph is at most its row treewidth plus 1. Moreover, a minor-closed class has bounded layered treewidth if and only if it has bounded row treewidth. However, it has been open whether row treewidth is bounded by a function of layered treewidth. This paper answers this question in the negative. In particular, for every integer k we describe a graph with layered treewidth 1 and row treewidth k. We also prove an analogous result for layered pathwidth and row pathwidth.

Keywords: treewidth, layered treewidth, row treewidth

1 Introduction

Treewidth is a graph parameter that measures how similar a given graph is to a tree; it is of fundamental importance in structural and algorithmic graph theory; see the surveys [3, 23, 32].

Layered treewidth is a variant of treewidth introduced independently by Dujmović, Morin, and Wood [15] and Shahrokhi [37]; see Section 2 for the definition. A key property is that layered treewidth is bounded on planar graphs but treewidth is not. In particular, planar graphs have layered treewidth at most 3 [15], but the $n \times n$ grid graph has treewidth n. Layered treewidth has been used in upper bounds on several graph parameters including queue-number [13, 15], stack number [13], boxicity [36], clustered chromatic number [28], generalised colouring numbers [26], asymptotic dimension [4], as well for results in intersection graph theory [37].

Row treewidth[(i)] is a refinement of layered treewidth, implicitly introduced by Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [14], who proved that planar graphs have row treewidth at most 8. This result

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(i) The name "row treewidth" is an original contribution of the present paper.

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and its generalisations have been the key to solving several open problems, regarding queue-number [14], nonrepetitive chromatic number [12], universal graphs [5, 11, 21], centred colouring [7], graph drawing [14, 31], and vertex ranking [6].

Layered and row treewidth are closely related in that the layered treewidth of a graph is at most its row treewidth plus 1, and a minor-closed class has bounded layered treewidth if and only if it has bounded row treewidth. However, a fundamental open problem is whether row treewidth is bounded by a function of layered treewidth. This paper answers this question in the negative.

Theorem 1. *For each $k \in \mathbb{N}$ there is a graph with layered treewidth 1 and row treewidth k.*

This result is proved in Section 3. In Section 4 we use a key lemma from the proof of Theorem 1 to prove a result about the relationship between layered treewidth and queue-number.

Layered pathwidth is a graph parameter analogous to layered treewidth, first studied by Bannister, Devanny, Dujmović, Eppstein, and Wood [1] and Dujmović, Eppstein, Joret, Morin, and Wood [10]. Row pathwidth is defined in a similar way to row treewidth. We prove the following analogue of Theorem 1 for layered pathwidth and row pathwidth.

Theorem 2. *For each $k \in \mathbb{N}$ there is a tree with layered pathwidth 1 and row pathwidth k.*

2 Definitions

We consider finite, undirected, simple graphs and use standard graph theory terminology [8].

Minors

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. A graph class $\mathcal{G}$ is *minor-closed* if for every graph $G \in \mathcal{G}$, every minor of G is in $\mathcal{G}$. A graph class $\mathcal{G}$ is *proper* if some graph is not in $\mathcal{G}$. A *K_t model* in a graph G is a set $\{X_1, \dots, X_t\}$ of pairwise-disjoint connected subgraphs in G, such that there is an edge of G between X_i and X_j for all distinct $i, j \in \{1, \dots, t\}$. Clearly, K_t is a minor of G if and only if G contains a K_t model.

Treewidth and Pathwidth

A *tree decomposition* $\mathcal{T}$ of a graph G is a collection $(B_x : x \in V(T))$ of subsets of $V(G)$ called *bags* indexed by the nodes of a tree T such that (i) for each $v \in V(G)$, the induced subgraph $T[\{x \in V(T) : v \in B_x\}]$ is nonempty and connected; and (ii) for each edge $vw \in E(G)$, there exists $x \in V(T)$ such that $\{v, w\} \subseteq B_x$. The *width* of a tree-decomposition is the size of its largest bag, minus 1. The *treewidth* $\text{tw}(G)$ of a graph G is the minimum width of any tree-decomposition of G. If $\mathcal{P} = (B_x : x \in V(P))$ is a tree-decomposition of G and P is a path, then $\mathcal{P}$ is a *path-decomposition* of G. The *pathwidth* $\text{pw}(G)$ of a graph G is the minimum width of any path-decomposition of G. For each $k \in \mathbb{N}$, the graphs with treewidth at most k form a minor-closed class, as do the graphs with pathwidth at most k. Note that $\text{tw}(K_n) = \text{pw}(K_n) = n - 1$.

Layered Treewidth and Pathwidth

A *layering* $\mathcal{L}$ of a graph G is a partition of $V(G)$ into a sequence of sets $(L_0, L_1, \dots)$ such that for any edge $vw \in E(G)$, if $v \in L_i$ and $w \in L_j$ then $|i - j| \leqslant 1$. For example, if r is a vertex in a connected graph G, and L_i is the set of vertices at distance i from r in G, then $(L_0, L_1, \dots)$ is the *breadth-first* layering of G rooted at r. A *layered tree-decomposition* of a graph G consists of a pair

$(\mathcal{L},\mathcal{T})$ where $\mathcal{L}=(L_0,L_1,\dots)$ is a layering of G and $\mathcal{T}=(B_x:x\in V(T))$ is a tree-decomposition of G. The (layered) *width* of $(\mathcal{L},\mathcal{T})$ is the size of the largest intersection between a layer and a bag; that is, $\max\{|B_x\cap L_i| : x\in V(T), i\in\mathbb{N}_0\}$. The *layered treewidth* $\operatorname{ltw}(G)$ of G is the minimum width of any layered tree-decomposition of G.

This definition was introduced independently by Shahrokhi [37] and Dujmović et al. [15]. The latter authors proved that $\operatorname{ltw}(G)\leqslant 3$ for every planar graph G; more generally that $\operatorname{ltw}(G)\leqslant 2g+3$ for every graph G of Euler genus g; and that a minor-closed class $\mathcal{G}$ has bounded layered treewidth if and only if some apex graph[ii] is not in $\mathcal{G}$. For an arbitrary proper minor-closed class $\mathcal{G}$, Dujmović et al. [15] showed that every graph in $\mathcal{G}$ has a tree-decomposition in which each bag has a bounded set of vertices whose deletion leaves a subgraph with bounded layered treewidth. This version of Robertson and Seymour's Graph Minor Structure Theorem has proved to be very useful [4, 27–29].

If $(\mathcal{L},\mathcal{P})$ is a layered tree-decomposition of G and $\mathcal{P}$ is a path-decomposition, then $(\mathcal{L},\mathcal{P})$ is a *layered path-decomposition* of G. The *layered pathwidth* of G is the minimum width of any layered path-decomposition of G. This parameter was introduced by Bannister et al. [1], who proved that every outerplanar graph has layered pathwidth at most 2 (amongst other examples).

Row Treewidth and Pathwidth

The *cartesian product* of graphs A and B, denoted by $A\,\square\,B$, is the graph with vertex set $V(A)\times V(B)$, where distinct vertices $(v,x),(w,y)\in V(A)\times V(B)$ are adjacent if: $v=w$ and $xy\in E(B)$; or $x=y$ and $vw\in E(A)$. The *direct product* of A and B, denoted by $A\times B$, is the graph with vertex set $V(A)\times V(B)$, where distinct vertices $(v,x),(w,y)\in V(A)\times V(B)$ are adjacent if $vw\in E(A)$ and $xy\in E(B)$. The *strong product* of A and B, denoted by $A\boxtimes B$, is the union of $A\,\square\,B$ and $A\times B$.

The *row treewidth* $\operatorname{rtw}(G)$ of a graph G is the minimum treewidth of a graph H such that G is isomorphic to a subgraph of $H\boxtimes P_\infty$, where P_∞ is the 1-way infinite path. This definition is implicit in the work of Dujmović et al. [14] who proved that $\operatorname{rtw}(G)\leqslant 8$ for every planar graph G; more generally, that $\operatorname{rtw}(G)\leqslant 2g+9$ for every graph G of Euler genus g; and that a minor-closed class $\mathcal{G}$ has bounded row treewidth if and only if some apex graph is not in $\mathcal{G}$. For an arbitrary minor-closed class $\mathcal{G}$, Dujmović et al. [14] showed that every graph in $\mathcal{G}$ has a tree-decomposition in which each bag has a bounded set of vertices whose deletion leaves a subgraph with bounded row treewidth.

It follows from the definitions that for every graph G,

$$\operatorname{ltw}(G)\leqslant\operatorname{rtw}(G)+1.$$

To see this, suppose that $G\subseteq H\boxtimes P_\infty$ where $\operatorname{tw}(H)=\operatorname{rtw}(G)$. Let $\mathcal{T}=(B_x:x\in V(T))$ be a minimum-width tree-decomposition of H. Assume $V(P_\infty)=\mathbb{N}$. For each $b\in\mathbb{N}$, let $L_b:=\{(a,b)\in V(G):a\in V(H)\}$. So $\mathcal{L}=(L_1,L_2,\dots)$ is a layering of G. For each $x\in V(T)$, let $B'_x:=\{(a,b)\in V(G):a\in V(H)\cap B_x, b\in\mathbb{N}\}$. So $\mathcal{T}':=(B'_x:x\in V(T))$ is a tree-decomposition of G. Note that $|B'_x\cap L_b|\leqslant|B_x|\leqslant\operatorname{tw}(H)+1$. Thus $(\mathcal{L},\mathcal{T})$ is a layered tree-decomposition of G with width at most $\operatorname{tw}(H)+1=\operatorname{rtw}(G)+1$.

Define the *row pathwidth* $\operatorname{rpw}(G)$ of a graph G to be the minimum pathwidth of a graph H such that G is isomorphic to a subgraph of $H\boxtimes P_\infty$. Then $\operatorname{lpw}(G)\leqslant\operatorname{rpw}(G)+1$ for every graph G. As explained above, "bounded layered treewidth" and "bounded row treewidth" coincide for minor-closed classes. However, this is not the case for their pathwidth analogues. Dujmović et al. [10] proved that a

[ii] A graph G is *apex* if $G-v$ is planar for some vertex v. A graph G is an *apex-forest* if $G-v$ is a forest for some vertex v.

minor-closed class $\mathcal{G}$ has bounded layered pathwidth if and only if some apex-forest is not in $\mathcal{G}$. However, forests are a minor-closed class excluding some apex-forest (namely, K_3), but forests have unbounded row pathwidth by Theorem 2. In fact, Robertson and Seymour [33] proved that for every fixed forest T, the class of T-minor-free graphs has bounded pathwidth (see [2] for a tight bound). It follows that a minor-closed class $\mathcal{G}$ has bounded row pathwidth if and only if $\mathcal{G}$ has bounded pathwidth if and only if some tree is not in $\mathcal{G}$.

3 Separating Layered Treewidth and Row Treewidth

This section proves Theorem 1. Subdivisions play a key role. A *subdivision* of a graph G is a graph G' obtained by replacing each edge vw of G with a path P_{vw} from v to w whose internal vertices have degree 2. If each P_{vw} has exactly s internal vertices, then G' is the *s-subdivision* of G. If each P_{vw} has at most s internal vertices, then G' is a *$(\leqslant s)$-subdivision* of G. It is well known and easily proved that $\mathrm{tw}(G') = \mathrm{tw}(G)$ for every subdivision G' of G.

The proof of Theorem 1 is based on two lemmas. The first shows that subdivisions efficiently reduce layered treewidth (Lemma 3). The second shows that subdivisions do not efficiently reduce row treewidth (Lemma 6). The theorem quickly follows.

3.1 Subdivisions Efficiently Reduce Layered Treewidth

Lemma 3. *For every graph G with layered treewidth $k \in \mathbb{N}$,*

(a) *there exists a $(\leqslant 2k-2)$-subdivision G' of G of layered treewidth 1; and*

(b) *if any subdivision G' of G has layered treewidth at most c, then some edge of G is subdivided at least $k/c - 1$ times in G'.*

Proof: Let $(B_x : x \in V(T))$ be a tree-decomposition of G and let $(V_0, V_1, \dots)$ be a layering of G, such that $|B_x \cap V_i| \leqslant k$ for each $x \in V(T)$ and $i \in \mathbb{N}_0$.

First we prove (a). We may assume that $B_x \cap V_i$ is a clique for each $x \in V(T)$ and $i \in \mathbb{N}_0$. So $G[V_i]$ is a chordal graph with no $(k+1)$-clique, which is therefore k-colourable. Let $c : V(G) \to \{0, 1, \dots, k-1\}$ be a function such that c is a proper k-colouring of $G[V_i]$ for each $i \in \mathbb{N}_0$. Thus for each $x \in V(T)$ and $i \in \mathbb{N}_0$, and for all distinct vertices $v, w \in B_x \cap V_i$, we have $c(v) \neq c(w)$. Let $L_{ki+j} := \{v \in V_i : c(v) = j\}$ for $i \in \mathbb{N}_0$ and $j \in \{0, 1, \dots, k-1\}$. Let G' be obtained from G as follows. Consider each edge $e = vw$ of G. Say $v \in V_i$ and $w \in V_{i'}$, and $v \in L_a$ and $w \in L_{a'}$. Without loss of generality, $a < a'$. Then $a' \leqslant k(i'+1) - 1 \leqslant k(i+2) - 1 \leqslant a + 2k - 1$. Replace e by the path $(v, s_{e,a+1}, s_{e,a+2}, \dots, s_{e,a'-1}, w)$ in G'. The number of division vertices is $a' - 1 - a \leqslant 2k - 2$. Put the division vertex $s_{e,b}$ in L_b for each $b \in \{a+1, \dots, a'-1\}$. So $(L_1, L_2, \dots)$ is a layering of G'. Some bag B_x contains v and w. Add a leaf node to T adjacent to x with corresponding bag $\{v, s_{e,a+1}, s_{e,a+2}, \dots, s_{e,a'-1}, w\}$. We obtain a tree-decomposition of G' with at most one vertex in each layer and in each bag. Hence $\mathrm{ltw}(G') = 1$.

We now prove (b). Suppose that some $(\leqslant r)$-subdivision G' of G has $\mathrm{ltw}(G') \leqslant c$. Let $(B_x : x \in V(T))$ be a tree-decomposition of G' and let $(V_0, V_1, \dots)$ be a layering of G' such that $|B_x \cap V_i| \leqslant c$ for each $x \in V(T)$ and $i \in \mathbb{N}_0$. Orient each edge of G arbitrarily. For each oriented edge vw of G and for each division vertex z of vw, let $\alpha(z) := v$. For each node $x \in V(T)$, let C_x be obtained from B_x by replacing each division vertex $z \in B_x$ by $\alpha(z)$. Observe that $(C_x : x \in V(T))$ is a tree-decomposition of G. For $j \in \mathbb{N}_0$, let $L_j := V(G) \cap (V_{j(r+1)} \cup V_{j(r+1)+1} \cup \dots \cup V_{(j+1)(r+1)-1})$. Consider an edge vw of G with $v \in V_i$ and $w \in V_{i'}$ and $i \leqslant i'$. Then $i' \leqslant i+r+1$ since vw is subdivided at most r times. Say $v \in L_j$

and $w \in L_{j'}$. By definition, $j(r+1) \leqslant i \leqslant (j+1)(r+1) - 1$ and $j'(r+1) \leqslant i' \leqslant (j'+1)(r+1) - 1$. Hence $j'(r+1) \leqslant i' \leqslant i + r + 1 \leqslant (j+1)(r+1) - 1 + (r+1) = (j+1)(r+1) + r$, implying $j' \leqslant j + 1$. That is, $(L_0, L_1, \dots)$ is a layering of G. Each layer L_j contains at most $c(r+1)$ vertices in each bag C_x. Thus $k = \operatorname{ltw}(G) \leqslant c(r+1)$, implying $r \geqslant k/c - 1$. □

Note that the proof of Lemma 3(a) is easily adapted to show that if $s_e \geqslant 2k-2$ for each edge $e \in E(G)$ and G' is the subdivision of G in which each edge e is subdivided s_e times, then $\operatorname{ltw}(G') \leqslant 2$. We omit these straightforward details.

Bannister et al. [1] proved that $\operatorname{lpw}(G) \leqslant \lceil(\operatorname{pw}(G) + 1)/2\rceil$. An analogous proof shows that $\operatorname{ltw}(G) \leqslant \lceil(\operatorname{tw}(G) + 1)/2\rceil$. Lemma 3(a) then implies:

Corollary 4. *Every graph with treewidth k has a $(\leqslant k)$-subdivision with layered treewidth 1.*

We remark that Corollary 4 is tight, up to a small constant factor:

Observation 5. *Let G' be any $(\leqslant s)$-subdivision of the complete graph K_{k+1}. Then $\operatorname{ltw}(G') \geqslant (k+1)/(2s+3)$. In particular, $\operatorname{ltw}(G') > 1$ if $s < (k-2)/2$.*

Proof: Let $(\mathcal{L}, \mathcal{T})$ be a layered tree-decomposition of G'. Since K_{k+1} has radius 1, G' has radius at most $s+1$ and therefore $|\mathcal{L}| \leqslant 2s+3$. Since K_{k+1} has treewidth k, so does G'. Thus $\mathcal{T}$ has at least one bag B_x of size at least $k+1$. By the pigeonhole principle, $|B_x \cap L| \geqslant (k+1)/|\mathcal{L}| \geqslant (k+1)/(2s+3)$ for at least one $L \in \mathcal{L}$. □

3.2 Subdivisions do not Efficiently Reduce Row Treewidth

This section proves the following result.

Lemma 6. *For each $k \in \mathbb{N}$ and $s \in \mathbb{N}_0$ there exists a graph G such that for every $(\leqslant s)$-subdivision G' of G,*

$$\operatorname{rtw}(G') \geqslant k \geqslant \operatorname{tw}(G).$$

A weaker version (with a more indirect proof) of Lemma 6 is implied by existing results about the *p-centred chromatic number $\chi_p(G)$* of a graph G; see [7, 9] for the definition (we will not need it). Dubois et al. [9] show that, for each $p \in \mathbb{N}$, there exists an integer $k \in \Theta(\sqrt{p})$ and a treewidth-k graph G, such that if G' is the $6k$-subdivision of G, then

$$\chi_p(G') \geqslant 2^{\Omega(\operatorname{tw}(G))}.$$

On the other hand, Dębski et al. [7] (see also [16]) show that for every graph G and $p \in \mathbb{N}$,

$$\chi_p(G) \leqslant (p+1)p^{\binom{p+\operatorname{rtw}(G)}{\operatorname{rtw}(G)}}.$$

Thus, for the graph G' constructed by Dubois et al. [9],

$$2^{\Omega(\operatorname{tw}(G))} \leqslant \chi_p(G') \leqslant (p+1)p^{\binom{p+\operatorname{rtw}(G')}{\operatorname{rtw}(G')}}.$$

This implies that $\operatorname{rtw}(G') \in \Omega(\operatorname{tw}(G)/\log(\operatorname{tw}(G)))$, as noted in [19].

To prove Lemma 6 it will be convenient to use the language of H-partitions from [14]. An *H-partition* of a graph G is a partition $\mathcal{H} = (B_x : x \in V(H))$ of $V(G)$ indexed by the nodes of some graph H such that for any edge $vw \in E(G)$, if $v \in B_x$ and $w \in B_y$, then $xy \in E(H)$ or $x = y$. A *layered H-partition* $(\mathcal{H}, \mathcal{L})$ of a graph G consists of an H-partition $\mathcal{H}$ and a layering $\mathcal{L}$ of G. The (layered) *width* of $(\mathcal{H}, \mathcal{L})$ is $\max\{|B_x \cap L| : x \in V(H), L \in \mathcal{L}\}$. The *layered width* of an H-partition $\mathcal{H}$ of a graph G is the minimum width, taken over all layerings $\mathcal{L}$ of G, of the width of $(\mathcal{H}, \mathcal{L})$. Dujmović et al. [14] observed that:

Observation 7 ([14])**.** *For all graphs G and H, G is isomorphic to a subgraph of $H \boxtimes P_\infty \boxtimes K_w$ if and only if G has an H-partition of layered width at most w. In particular, $\mathrm{rtw}(G)$ equals the minimum treewidth of a graph H for which G has a layered H-partition of layered width 1.*

This observation and the next lemma (with $w = 1$) implies Lemma 6. Lemma 8 generalises a result of Dujmović et al. [14] who proved the $s = 0$ case.

Lemma 8. *For all $w, k \in \mathbb{N}$ and $s \in \mathbb{N}_0$, there exists a graph $G = G_{s,k,w}$ such that $\mathrm{tw}(G) \leqslant k$, and for any $(\leqslant s)$-subdivision G' of G, and for any graph H and any H-partition of G' with layered width at most w, there is a K_{k+1} minor in H, implying $\mathrm{tw}(H) \geqslant k$ and $\mathrm{rtw}(G) \geqslant k$.*

Say a K_k model $\{Y_1, \dots, Y_k\}$ in a graph G *respects* an H-partition $(A_x : x \in V(H))$ of G if for each $x \in V(H)$ there is at most one value of $i \in \{1, \dots, k\}$ for which $V(Y_i) \cap A_x \neq \emptyset$. For each $i \in \{1, \dots, k\}$, let X_i be the subgraph of H induced by those vertices $x \in V(H)$ such that $V(Y_i) \cap A_x \neq \emptyset$. Then $\{X_1, \dots, X_k\}$ is a K_k model in H. Thus the next lemma implies Lemma 8.

Lemma 9. *For all $w \in \mathbb{N}$ and $k, s \in \mathbb{N}_0$, there exists a graph $G = G_{s,k,w}$ such that $\mathrm{tw}(G) \leqslant k$, and for any $(\leqslant s)$-subdivision G' of G, and for any graph H and any H-partition $(A_x : x \in V(H))$ of G' with layered width at most w, there is a model $\{Y_1, \dots, Y_{k+1}\}$ of K_{k+1} in G' that respects $(A_x : x \in V(H))$, and for each $i \in \{1, \dots, k+1\}$ we have $|V(Y_i)| \leqslant k(2s+1)+1$ and $V(Y_i) \cap V(G) \neq \emptyset$.*

Proof: The proof is by induction on k. For the base case with $k = 0$, let $G_{s,0,w}$ be the graph with one vertex v and no edges. Any subdivision G' contains a model $Y_1 = G'[\{v\}]$ of K_1 with $|Y_1| \leqslant 1 = 0(2s+1)+1$, and this model trivially respects any H-partition of G', for any graph H.

Now assume $k \geqslant 1$, and that the induction hypothesis holds for $k-1$. Let $Q = G_{s,k-1,w}$ be the graph obtained by induction. Let $N := k^2(2s+1)(4s+5)w+1$. Create the graph G by starting with N disjoint copies $Q_1, \dots, Q_N$ of Q. Next add a vertex v and, for each $i \in \{1, \dots, N\}$ and each $u \in V(Q_i)$, add N internally disjoint paths of length 2 from v to u.

First we show that $\mathrm{tw}(G) \leqslant k$. If $k = 1$ then G is a subdivided star, which has treewidth 1. Now assume that $k \geqslant 2$. Begin with any tree-decomposition $(B_x : x \in V(T))$ of $Q_1 \cup \dots \cup Q_N$ with width at most $(k-1)$. Add v to every bag B_x. For all $i, j \in \{1, \dots, N\}$ and for each vertex $u \in V(Q_i)$, choose a bag B_x that contains u, and attach a leaf node adjacent to x whose bag contains v, u, and the degree-2 vertex of the j-th length-2 path from v to u. Each bag of this decomposition has size at most $\max\{3, k+1\} \leqslant k+1$ and therefore $\mathrm{tw}(G) \leqslant k$.

Let G' be a $(\leqslant s)$-subdivision of G. Let H be a graph and let $(\mathcal{H}, \mathcal{L})$ be a layered H-partition of G' with width at most w, where $\mathcal{H} := (A_x : x \in V(H))$. Since the radius of G is 2, the radius of G' is at most $2s+2$. Therefore $|\mathcal{L}| \leqslant 4s+5$, and $|A_x| \leqslant (4s+5)w$ for each $x \in V(H)$.

Let z be the unique node of H such that $v \in A_z$, and let $Q'_1, \dots, Q'_N$ be the (possibly subdivided) copies of $Q_1, \dots, Q_N$ that appear in G'. So $A_z \cap V(Q'_i) \neq \emptyset$ for at most $(4s+5)w - 1$ values of $i \in \{1, \dots, N\}$. Since $N \geqslant (4s+5)w$, we have $V(Q'_i) \cap A_z = \emptyset$ for some $i \in \{1, \dots, N\}$.

Let H_i be the subgraph of H induced by the nodes $\tau \in V(H)$ such that $A_\tau \cap V(Q'_i) \neq \emptyset$. So $(A_\tau \cap V(Q'_i) : \tau \in V(H_i))$ defines a width-w layered H_i-partition of Q'_i (with respect to layering $\mathcal{L}$).

By induction, there is a K_k model $\{Y_1,\dots,Y_k\}$ in G' that respects $(A_\tau \cap V(Q') : \tau \in V(H_i))$, and for each $j \in \{1,\dots,k\}$ we have $|V(Y_j)| \leqslant (k-1)(2s+1)+1 \leqslant k(2s+1)$ and $V(Y_j) \cap V(G) \neq \emptyset$. Let y_j be a vertex in $V(Y_j) \cap V(G)$. Note that $|V(Y_1 \cup \dots \cup Y_k)| \leqslant k^2(2s+1)$. Let $F := \bigcup\{A_x : A_x \cap V(Y_1 \cup \dots \cup Y_k) \neq \emptyset\}$. So $|F| \leqslant k^2(2s+1)(4s+5)w$.

Since $V(Q'_i) \cap A_z = \emptyset$, we have $v \notin F$. Since $N > |F|$, for each $j \in \{1,\dots,k\}$, for at least one of the N paths between v and y_j added in the construction of G, the corresponding path in G' avoids F. Let P_j be this path, not including y_j. Let $Y_{k+1} := \bigcup\{P_j : j \in \{1,\dots,k\}\}$. So $V(Y_{k+1}) \cap F = \emptyset$. By construction, there is an edge from y_j to Y_{k+1} for each $j \in \{1,\dots,k\}$. So $\{Y_1,\dots,Y_{k+1}\}$ is a K_{k+1} model in G'. Since $\{Y_1,\dots,Y_k\}$ respects $(A_\tau \cap V(Q') : \tau \in V(H_i))$ and $V(Y_{k+1}) \cap F = \emptyset$, it follows that $\{Y_1,\dots,Y_{k+1}\}$ respects $(A_x : x \in V(H))$. By construction, $|V(Y_{k+1})| \leqslant k(2s+1)+1$ and $v \in V(Y_{k+1}) \cap V(G)$. By assumption, for each $j \in \{1,\dots,k\}$ we have $|V(Y_j)| \leqslant (k-1)(2s+1)+1$ and $V(Y_j) \cap V(G) \neq \emptyset$. This shows that $\{Y_1,\dots,Y_{k+1}\}$ is the desired K_{k+1} model in G'. □

We now prove the following strengthening of Theorem 1.

Theorem 10. *For each $k \in \mathbb{N}$ there is a graph G with $\operatorname{ltw}(G) = 1$ and $\operatorname{rtw}(G) = \operatorname{tw}(G) = k$.*

Proof: By Lemma 6 with $s = k$, there is a graph G with $\operatorname{tw}(G) \leqslant k$ such that $\operatorname{rtw}(G') \geqslant k$ for every $(\leqslant k)$-subdivision G' of G. By Corollary 4, there exists a $(\leqslant k)$-subdivision G' of G with $\operatorname{ltw}(G') = 1$. By definition, $\operatorname{rtw}(H) \leqslant \operatorname{tw}(H)$ for every graph H. It is well known and easily proved that $\operatorname{tw}(H) = \operatorname{tw}(H')$ for every subdivision H' of H. Thus $k \leqslant \operatorname{rtw}(G') \leqslant \operatorname{tw}(G') = \operatorname{tw}(G) \leqslant k$, implying $\operatorname{rtw}(G) = \operatorname{tw}(G') = k$. □

See [9, 22] for other examples where $O(\operatorname{tw}(G))$-subdivisions of graphs G are used to prove lower bounds.

4 Queue Layouts and Layered Treewidth

The *queue-number* $\operatorname{qn}(G)$ of a graph G is a well-studied graph parameter introduced by Heath, Leighton and Rosenberg [24, 25]; we omit the definition since we will not need it. Queue-number can be upper bounded in terms of layered treewidth and row treewidth. In particular, Dujmović et al. [15] proved that n-vertex graphs of bounded layered treewidth have queue-number in $O(\log n)$, while Dujmović et al. [14] proved that graphs of bounded row treewidth have bounded queue-number. (Row treewidth was discovered as a tool for proving that planar graphs have bounded queue-number.) This is a prime example of a difference in behaviour between layered treewidth and row treewidth. Nevertheless, it is open whether graphs of bounded layered treewidth have bounded queue-number. We show that the answer to this question depends entirely on the case of layered treewidth 1.

Corollary 11. *Graphs of bounded layered treewidth have bounded queue-number if and only if graphs of layered treewidth 1 have bounded queue-number.*

Proof: The forward implication is immediate. Now assume that every graph of layered treewidth 1 has queue-number at most some constant c. Let G be a graph with layered treewidth k. By Lemma 3, G has a $(\leqslant k)$-subdivision G' with layered treewidth 1. So $\operatorname{qn}(G') \leqslant c$. Dujmović and Wood [18] proved that for every graph H and $(\leqslant s)$-subdivision H' of H, we have $\operatorname{qn}(H) \in O(\operatorname{qn}(H')^{2s})$. This bound was improved to $O(\operatorname{qn}(H')^{s+1})$ in [17]. Thus $\operatorname{qn}(G) \leqslant O(c^{k+1})$, implying that graphs of bounded layered treewidth have bounded queue-number. □

This proof highlights the value of considering the behaviour of a graph parameter on subdivisions. An analogous result holds for nonrepetitive chromatic number $\pi(G)$ (using a result of Nešetřil, Ossona de Mendez, and Wood [30] to bound $\pi(H)$ in terms of $\pi(H')$).

Corollary 12. *Graphs of bounded layered treewidth have bounded nonrepetitive chromatic number if and only if graphs of layered treewidth 1 have bounded nonrepetitive chromatic number.*

5 Separating Layered Pathwidth and Row Pathwidth

Recall that Theorem 2 asserts that for all $k \in \mathbb{N}$ there is a tree T with $\operatorname{lpw}(T) = 1$ and $\operatorname{rpw}(T) \geqslant k$. We first show that this theorem, in fact, follows from results in the literature. Bannister et al. [1] noted that $\operatorname{lpw}(T) = 1$ for every tree T. Dvorák et al. [19] showed that any family $\mathcal{G}$ of graphs with bounded row pathwidth has polynomial growth. The family of complete binary trees does not have polynomial growth, so for each $k \in \mathbb{N}$, there exists a complete binary tree T with $\operatorname{rpw}(T) \geqslant k$. This proves Theorem 2.

We now prove the following stronger result.

Theorem 13. *For every $k \in \mathbb{N}$ there exists a tree T with $\operatorname{pw}(T) = \operatorname{rpw}(T) = k$.*

Let $T_{d,h}$ be the complete d-ary tree of height h. It is folklore that $\operatorname{pw}(T_{d,h}) = h$ for all $d \geqslant 3$ (see [20, 34, 35]). So Observation 7 and Lemma 14 with $\ell = 3$ and $w = 1$ implies Theorem 13.

Lemma 14. *For all $h \in \mathbb{N}_0$ and $w, \ell \in \mathbb{N}$ there exists $d \in \mathbb{N}$ such that for every graph H and every H-partition $(A_x : x \in V(H))$ of $T_{d,h}$ with layered width at most w, the graph H contains a subgraph isomorphic to $T_{\ell,h}$. Moreover, if r is the root vertex of $T_{d,h}$ and $r \in A_z$, then z is the root vertex of a subgraph of H isomorphic to $T_{\ell,h}$.*

Proof: We proceed by induction on h. The case $h = 0$ is trivial. Now assume that $h \geqslant 1$. We may assume that the number of layers is at most the diameter of $T_{d,h}$ plus 1, which equals $2h+1$. So $|A_x| \leqslant w(2h+1)$ for each $x \in V(H)$. Let $r_1, \dots, r_d$ be the children of r in $T_{d,h}$. Let $T^1, \dots, T^d$ be the copies of $T_{d,h-1}$ in $T_{d,h}$, where T^i is rooted at r_i. At most $w(2h + 1) - 1$ of $T^1, \dots, T^d$ intersect A_z. Without loss of generality, $T^1, \dots, T^{d-w(2h+1)+1}$ do not intersect A_z. For each $i \in \{1, \dots, d - w(2h + 1) + 1\}$, let z_i be the vertex of H such that $r_i \in A_{z_i}$. Since $T^i \cap A_z = \emptyset$, we have $z_i \neq z$. Since $rr_i \in E(T_{d,h})$, we have $zz_i \in E(H)$. By induction, $H - z$ contains a subgraph S^i isomorphic to $T_{\ell,h-1}$ rooted at z_i. Let X be the intersection graph of $S^1, \dots, S^{d-w(2h+1)+1}$. If S^i and S^j intersect in node y of H, then T^i and T^j both intersect A_y. Since $|V(S^i)| \leqslant (\ell + 1)^{h-1}$ and $|A_y| \leqslant w(2h + 1)$, X has maximum degree $\Delta(X) \leqslant w(2h + 1)(\ell + 1)^h$. Thus $\chi(X) \leqslant w(2h + 1)(\ell + 1)^h + 1$. For sufficiently large d, we have $|V(X)| > (\ell - 1)\chi(X)$. Thus, in any $\chi(X)$-colouring of X, some colour class has at least ℓ vertices. Without loss of generality, $S^1, \dots, S^\ell$ are pairwise-disjoint. Hence, $S_1 \cup \cdots \cup S_\ell$ along with z forms a subgraph of H isomorphic to $T_{\ell,h}$, rooted at z, as desired. □

We finish with an open problem: what is $\operatorname{rpw}(T_{2,h})$? It follows from a result of Dvorák et al. [19] that $\operatorname{rpw}(T_{2,h}) \in \Omega(\frac{h}{\log h})$. Is this tight, or is $\operatorname{rpw}(T_{2,h}) \in \Omega(h)$? Obviously $\operatorname{rpw}(T_{2,h}) \leqslant \operatorname{pw}(T_{2,h}) = \lceil \frac{h}{2} \rceil$.

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