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Multivalued forbidden numbers of two-rowed configurations – the missing cases

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The present paper considers extremal combinatorics questions in the language of matrices. An s-matrix is a matrix with entries in $\{0, 1, \ldots, s-1\}$. An s-matrix is simple if it has no repeated columns. A matrix F is a configuration in a matrix A, denoted $F \prec A$, if it is a row/column permutation of a submatrix of A. $\mathrm{Avoid}(m, s, F)$ is the set of m-rowed, simple s-matrices not containing a configuration of F and $\mathrm{forb}(m, s, F) = \max\{|A|\colon A \in \mathrm{Avoid}(m, s, F)\}$. Dillon and Sali initiated the systematic study of $\mathrm{forb}(m, s, F)$ for 2-matrices F, and computed $\mathrm{forb}(m, s, F)$ for all 2-rowed F when $s > 3$. In this paper we tackle the remaining cases when $s = 3$. In particular, we determine the asymptotics of $\mathrm{forb}(m, 3, p \cdot K_2) - \mathrm{forb}(m, 3, p \cdot I_2)$ for $p > 3$, where K_2 is the 2×4 simple 2-matrix and I_2 is the 2×2 identity matrix, as well as the exact values of $\mathrm{forb}(m, 3, F)$ for many 2-rowed 2-matrices F.

Keywords: Forbidden configurations, (0,1)- matrices, s-matrices, extremal set theory

1 Introduction

Several questions of extremal combinatorics can be conveniently expressed in the language of $(0, 1)$-matrices. A matrix A is said to be *simple* if it contains no repeated columns. There is a natural correspondence between columns of an m-rowed $(0, 1)$-matrix and subsets of $[m]$. We consider an extremal set theory problem in matrix terminology as follows. Let $|A|$ be the number of columns of A. For a given matrix F, we say F is a *configuration* in A, denoted $F \prec A$, if there is a submatrix of A which can be obtained by permuting the rows and columns of F. Define

$$\mathrm{Avoid}(m, F) = \{A\colon A \text{ is an } m\text{-rowed } (0,1)\text{-matrix and } F \not\prec A\},$$

$$\mathrm{forb}(m, F) = \max\{|A| : A \in \mathrm{Avoid}(m, F)\}.$$

We say a matrix $A \in \mathrm{Avoid}(m, F)$ is *extremal* if $|A| = \mathrm{forb}(m, F)$, and let

$$\mathrm{ext}(m, F) = \{A \in \mathrm{Avoid}(m, F)\colon |A| = \mathrm{forb}(m, F)\}.$$

The foremost theorem in forbidden configurations is Theorem 1.1 below. Here, K_k denotes the $k \times 2^k$ matrix containing all possible $0, 1$-columns.

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Theorem 1.1. *Sauer (1972); Shelah (1972); Vapnik and Chervonenkis (1971)*

$$\mathrm{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}.$$

Further interesting results and research problems can be found in the survey paper Anstee and Sali (2025). However, our main interest in the present paper is the generalization of these concepts to s-matrices, which have entries in $\{0, 1, \ldots, s-1\}$. Such matrices can be thought of as s-coloured set systems or as representations of collections of functions from a given finite set into $\{0, 1, \ldots, s-1\}$. The concepts of *simple* matrices and *configurations* naturally extend. For a given collection $\mathcal{F}$ of matrices, we denote by $\mathrm{Avoid}(m, s, \mathcal{F})$ the collection of m-rowed, simple s-matrices that avoid every matrix $F \in \mathcal{F}$. The main extremal function in the study of forbidden configurations is

$$\mathrm{forb}(m, s, \mathcal{F}) = \max\{|A|\colon A \in \mathrm{Avoid}(m, s, \mathcal{F})\},$$

which we call the *forbidden number*, and we similarly define

$$\mathrm{ext}(m, s, F) = \{A \in \mathrm{Avoid}(m, s, F)\colon |A| = \mathrm{forb}(m, s, F)\}.$$

Alon Alon (1983) and Steele Steele (1978) gave a generalization of Theorem 1.1 to s-matrices, with the bound given being an exponential function of m. This is a special case of the following more general phenomenon proved by Füredi and Sali Füredi and Sali (2012). A matrix is called an (i, j)-*matrix* if its entries are all from the set $\{i, j\}$.

Theorem 1.2. *Let $\mathcal{F}$ be a family of s-matrices. If for every pair of distinct $i, j \in \{0, 1, \ldots, s-1\}$ there is an (i, j)-matrix in $\mathcal{F}$, then $\mathrm{forb}(m, s, \mathcal{F}) = O(m^k)$ for some positive integer k. If $\mathcal{F}$ has no (i, j)-matrix for some distinct $i, j \in \{0, 1, \ldots, s-1\}$, then $\mathrm{forb}(m, s, \mathcal{F}) = \Omega(2^m)$.*

Several papers Anstee et al. (2023), Anstee and Lu (2014), Ellis et al. (2020) considered special sets of forbidden configurations for which the forbidden numbers are of polynomial magnitudes. The systematic investigation of exact exponential bounds when forbidding $(0, 1)$-configurations was started in Dillon and Sali (2021). They provided exact formulas for the forbidden numbers $\mathrm{forb}(m, s, F)$ of many $(0, 1)$-matrices F, including all 2-rowed matrices when $s > 3$. Along the way, they exposed some interesting qualitative differences between the cases $s = 2$, $s = 3$, and $s > 3$.

Some important matrices include I_k, the $k \times k$ identity matrix and K_k, the $k \times 2^k$ matrix of all possible (0,1)-columns on k rows. We use the following notation for general 2-rowed configurations:

$$F(a, b, c, d) = \begin{bmatrix} \overbrace{00\cdots0}^{a} & \overbrace{11\cdots1}^{b} & \overbrace{00\cdots0}^{c} & \overbrace{11\cdots1}^{d} \\ 00\cdots0 & 00\cdots0 & 11\cdots1 & 11\cdots1 \end{bmatrix}.$$

For example, $F(1, 1, 1, 1) = K_2$. Let $p \cdot F$ denote the concatenation of p copies of F, so that in particular $p \cdot K_2$ and $F(p, p, p, p)$ are the same as configurations, while $p \cdot I_2$ is the same as $F(0, p, p, 0)$.

When $s > 3$, the forbidden numbers $\mathrm{forb}(m, s, F(a, b, c, d))$ for all 2-rowed configurations were determined exactly in Dillon and Sali (2021). Under mild conditions similar to Lemma 1.3 below, it was shown that $\mathrm{forb}(m, s, F(a, b, c, d)) = \mathrm{forb}(m, s, F(a, p, p, d))$, where $p = \min\{b, c\}$. In particular, it turned out that $\mathrm{forb}(m, s, p \cdot K_2) = \mathrm{forb}(m, s, p \cdot I_2)$, which implies that $\mathrm{forb}(m, s, p \cdot K_2) = \mathrm{forb}(m, s, F(a, p, p, d))$ if $s > 3$ and $a, d \leq p$.

For general 2-rowed configurations $F(a, b, c, d)$, the case when $s = 3$ is harder than when $s > 3$. In particular, it was shown in Dillon and Sali (2021) that $\mathrm{forb}(m, 3, p \cdot K_2) > \mathrm{forb}(m, 3, p \cdot I_2)$ if $p \geq 3$, exhibiting a different behaviour to the $s > 3$ case. More generally, $\mathrm{forb}(m, 3, F(a, b, c, d))$ were determined in Dillon and Sali (2021) only when $\max\{a, d\} \geq \min\{b, c\}$. Our goal in this paper is to study the missing cases when $\max\{a, d\} < \min\{b, c\}$.

The following important reduction lemma was proved in Dillon and Sali (2021).

Lemma 1.3. *Dillon and Sali (2021) Let $p = \min\{b, c\}$. If $2^{m-2} \geq (\max\{a, b, c, d\} - 1)m^2$ and $p \geq 1$, then*

$$\mathrm{forb}(m, 3, F(a, b, c, d)) = \mathrm{forb}(m, 3, F(a, p, p, d)).$$

Note that by Lemma 1.3, we may assume without loss of generality that $b = c = p$ provided that m is large. It was also proved in Dillon and Sali (2021) that $\mathrm{forb}(m, 3, F(p, p, p, p)) = \mathrm{forb}(m, 3, p \cdot K_2) = 2^m + m2^{m-1} + (p-1)\binom{m}{2}$. In what follows, we investigate the forbidden numbers $\mathrm{forb}(m, 3, F(a, p, p, d))$ for various parameters $a, d < p$.

In Section 2, we begin by introducing some terminologies and notations. Then, we prove first a generalization of a lemma in Dillon and Sali (2021), which roughly says that any matrix in $\mathrm{Avoid}(m, 3, F(a, p, p, d))$ that contains many columns must satisfy some stability-type structural properties. In Section 2.1, we determine the difference between $\mathrm{forb}(m, 3, p \cdot K_2)$ and $\mathrm{forb}(m, 3, p \cdot I_2)$ asymptotically. Throughout the paper we let $r = \lceil \log_2(p-1) \rceil$, so $2^{r-1} < p - 1 \leq 2^r$. Our first, maybe surprising, observation is that the difference above does not depend on m.

Proposition 1.4. *For all $p \geq 2$, $\mathrm{forb}(m, 3, p \cdot K_2) - \mathrm{forb}(m, 3, p \cdot I_2)$ is independent of m as long as $m \geq 2r + 2$.*

This allows us to define g_p to be the common value of $\mathrm{forb}(m, 3, p \cdot K_2) - \mathrm{forb}(m, 3, p \cdot I_2)$ for $m \geq 2r+2$, and we prove the following asymptotic result for g_p.

Theorem 1.5.

$$g_p \sim \frac{1}{2} p (\log_2 p)^2.$$

In Section 2.2, we compute $\mathrm{forb}(m, 3, F(a, p, p, d))$ for all $p \leq 4$ and $\max\{a, d\} < p$, which all turn out to be equal to $\mathrm{forb}(m, 3, p \cdot I_2)$.

This no longer holds for larger values of p, but in Section 2.3 we have further exact results for many cases. Let

$$N_r = \binom{\lceil \frac{r}{2} \rceil + 1}{2} + \binom{\lfloor \frac{r}{2} \rfloor + 1}{2}.$$

The next theorem determines $\mathrm{forb}(m, 3, F(p-q, p, p, p-q))$ for a wide range of values p and q.

Theorem 1.6. *For all $m \geq 2N_r + 2$ and $0 \leq q \leq p - 1$,*

$$\mathrm{forb}(m, 3, F(p-q, p, p, p-q)) = \mathrm{forb}(m, 3, p \cdot K_2) - qN_r$$

if one of the following holds

- $p \geq 2^{r-1} + 2q + 1$,
- $p \leq 2^{r-1} + 2q$, $qN_r \leq 2^{r-3}$ *and* $\frac{1}{2}(\lfloor \frac{r}{2} \rfloor + 1)(p - 1 - 2^{r-1}) \geq q$.

In particular, this is true for all fixed constants q and all p sufficiently large.

With a more careful analysis of the proof of Theorem 1.6, we prove the following result.

Corollary 1.7. *For all $m \geq 2N_r + 2$ and $p \geq 2$,*

$$\mathrm{forb}(m, 3, F(p-1, p, p, p-1)) = \mathrm{forb}(m, 3, p \cdot K_2) - N_r.$$

Determining the exact forbidden number in the remaining cases when q is relatively large might be difficult, given that the exact value of $\mathrm{forb}(m, 3, p \cdot I_2)$ is not known. In particular, the following result shows that the exact formula in Theorem 1.6 may not hold if those conditions are not satisfied.

Proposition 1.8. *If $m \geq 2N_r + 2$, $0 \leq q \leq p - 1$ and $d := q - (\lceil \frac{r}{2} \rceil + 1)(p - 1 - 2^{r-1}) > 0$, then*

$$\mathrm{forb}(m, 3, F(p-q, p, p, p-q)) \geq \mathrm{forb}(m, 3, p \cdot K_2) - qN_r + \left\lfloor \frac{r}{2} \right\rfloor d.$$

We also have the following more general lower bound.

Proposition 1.9. *For every $0 \leq q_0, q_1 \leq p-1$, every $0 \leq r_1, r_2 \leq r$ satisfying $r_1 + r_2 = r$, and every $m \geq 2\binom{r_1+1}{2} + 2\binom{r_2+1}{2} + 2$, we have*

$$\mathrm{forb}(m, 3, F(p-q_0, p, p, p-q_1)) \geq \mathrm{forb}(m, 3, p \cdot K_2) - \min\{q_0, q_1\}\binom{r_1+1}{2} - \max\{q_0, q_1\}\binom{r_2+1}{2}.$$

For fixed p, q_0, q_1, we may optimize the right-hand side by viewing r_1 as the variable and using $r_2 = r - r_1$. That is, assuming $q_0 \leq q_1$, we minimize $q_0\binom{r_1+1}{2} + q_1\binom{r-r_1+1}{2}$. Easy calculation shows that if $q_1 = \alpha q_0$ for some $\alpha \geq 1$, then the minimum is achieved when r_1 is the closest integer to $\frac{(2r+1)\alpha-1}{2(\alpha+1)}$, as long as this fraction is at most r, which is equivalent to $\alpha \leq 2r+1$. Otherwise, $r_1 = r$ is optimal. Note that if $q_0 = q_1$ and so $\alpha = 1$, then the minimum is achieved when $r_1 = \lceil \frac{r}{2} \rceil, r_2 = \lfloor \frac{r}{2} \rfloor$. The resulting lower bound in this case is tight in many cases by Theorem 1.6. We suspect that a more involved version of this proof could potentially show that after optimization, the general lower bound for $\mathrm{forb}(m, 3, F(p-q_0, p, p, p-q_1))$ given by Proposition 1.9 is also tight in several cases.

2 Results

Recall that from Lemma 1.3, we may assume that $b = c$, as long as m is sufficiently large. From now on, we will assume $b = c = p \geq 3$, $0 \leq a, d \leq p-1$ unless stated otherwise. Generalizing some ideas from Dillon and Sali (2021), we now make some definitions to help better understand the structure of matrices in $\mathrm{Avoid}(m, 3, F(a, p, p, d))$.

Definition 2.1. *Let $A \in \mathrm{Avoid}(m, 3, F(a, p, p, d))$. To each pair (i, j) with $1 \leq i < j \leq m$, associate a vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ so that if $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then rows i, j contain at most $p-1$ copies of $\begin{bmatrix} x \\ y \end{bmatrix}$, if $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then rows i, j contain at most $a-1$ copies of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and if $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then rows i, j contain at most $d-1$ copies of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Such a vector must exist as $F(a, p, p, d) \not\leftarrow A$, and if multiple choices are possible, we pick one arbitrarily. In all cases, we say a column v of A has a mark at the pair (i, j) if $v_i = x$ and $v_j = y$.*

Let B be the matrix consisting of all columns of A with no mark, and let C be the matrix consisting of all columns of A with at least one mark. We call $A = \begin{bmatrix} B \mid C \end{bmatrix}$ the standard decomposition of A.

We also associate to A a directed graph T on $[m]$ as follows: for each pair $1 \leq i < j \leq m$, if $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is associated with (i, j), add the directed edge ij; if $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is associated with (i, j), add the directed edge ji; otherwise do nothing. Let N denote the number of pairs (i, j) that are associated with either $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We say that these pairs form 0-non-edges and 1-non-edges in T, respectively.

Using the notations above, we have some immediate observations. It follows from the definition that B is in $\mathrm{Avoid}(m, 3, K_2)$ so $|B| \leq 2^m + m2^{m-1}$. We also have $|C| \leq (p-1)(\binom{m}{2} - N) + N(\max\{a, d\} - 1) = (p-1)\binom{m}{2} - N(p - \max\{a, d\})$, since every column in C contains at least one mark.

The following key lemma, generalizing one from Dillon and Sali (2021), shows that if $|B|$ is large, then the associated directed graph T is transitive.

Lemma 2.2. *If $|B| > 2^m + m2^{m-1} - 2^{m-3}$, then T is transitive.*

Proof: Assume T is not transitive, then there exists a triple (i, j, k) of distinct vertices such that ij, jk are edges of T but either ki is an edge of T or there is no edge between k and i. For a column v in B, define its *support* to be the set of row indices i such that v_i is either 0 or 1. For any set $X \subset [m]$ of rows of B, let $B(X)$ be the matrix

consisting of columns of B with support X. Then its restriction $B(X)|_X$ of $B(X)$ to the rows in X is simple and is in $\mathrm{Avoid}(|X|, 2, K_2)$. Thus, if $\{i,j,k\} \not\subset X$, then $|B(X)| = |B(X)|_X| \leq \mathrm{forb}(|X|, 2, K_2) = |X|+1$ by Theorem 1.1.

If, however, $\{i,j,k\} \subset X$, then consider $B(X)|_{\{i,j,k\}}$. If ij, jk, ki are all edges of T, the only possible columns in $B(X)|_{\{i,j,k\}}$ are $\begin{bmatrix}0\\0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$. So $B(X)|_{X\setminus\{i,j\}}$ is simple and avoids K_2. Thus $|B(X)| = \left|B(X)|_{X\setminus\{i,j\}}\right| \leq \mathrm{forb}(|X|-2, 2, K_2) = |X|-1$. If ij, jk are edges of T but there is no edge between i and k, there are two cases. If rows i,k are associated with $\begin{bmatrix}0\\0\end{bmatrix}$, then any column in $B(X)|_{\{i,j,k\}}$ has a 1 in entry i so $B(X)|_{X\setminus\{i\}}$ is simple and avoids K_2. If rows i,k are associated with $\begin{bmatrix}1\\1\end{bmatrix}$, then any column in $B(X)|_{\{i,j,k\}}$ has a 0 in entry k so $B(X)|_{X\setminus\{k\}}$ is simple and avoids K_2. In both cases, we have $|B(X)| \leq \mathrm{forb}(|X|-1, 2, K_2) = |X|$.

Therefore, if ij, jk, ki are all edges of T, by counting columns in B according to their support, we have

$$\begin{aligned}
|B| &\leq \sum_{\substack{X\subset[m]\\ \{i,j,k\}\not\subset X}} (|X|+1) + \sum_{\substack{X\subset[m]\\ \{i,j,k\}\subset X}} (|X|-1)\\
&= \sum_{X\subset[m]} (|X|+1) - 2\sum_{\substack{X\subset[m]\\ \{i,j,k\}\subset X}} 1\\
&= \sum_{j=0}^{m}\binom{m}{j}(j+1) - 2\sum_{j=0}^{m-3}\binom{m-3}{j} = 2^m + m2^{m-1} - 2^{m-2}.
\end{aligned}$$

A similar calculation shows that $|B| \leq 2^m + m2^{m-1} - 2^{m-3}$ if ij, jk are edges of T but neither ik nor ki is. Both contradict the assumption that $|B| > 2^m + m2^{m-1} - 2^{m-3}$, which completes the proof. □

Note that if $|A| > 2^m + m2^{m-1} - 2^{m-3} + (p-1)\binom{m}{2}$, then as $|C| \leq (p-1)\binom{m}{2}$, we must have $|B| > 2^m + m2^{m-1} - 2^{m-3}$. Therefore, by Lemma 2.2, T is transitive, and hence the directed comparability graph of a finite poset P. Thus, by taking a linear extension of P, we can rearrange the rows of A so that no pair $1 \leq i < j \leq m$ is associated with $\begin{bmatrix}1\\0\end{bmatrix}$. In particular, unless (i,j) is associated with $\begin{bmatrix}0\\0\end{bmatrix}$ or $\begin{bmatrix}1\\1\end{bmatrix}$, there is no occurrence of 0 above 1 in rows i, j in B.

2.1 Asymptotics of $\mathrm{forb}(m, 3, p\cdot K_2) - \mathrm{forb}(m, 3, p\cdot I_2)$

By Dillon and Sali (2021), if $2^{m-2} \geq p-1$, then $\mathrm{forb}(m, 3, p\cdot K_2) = 2^m + m2^{m-1} + (p-1)\binom{m}{2}$, but the exact value of $\mathrm{forb}(m, 3, p\cdot I_2)$ is not known for general p. In this subsection, we determine asymptotically the difference between $\mathrm{forb}(m, 3, p\cdot K_2)$ and $\mathrm{forb}(m, 3, p\cdot I_2)$.

We first give a lower bound construction that uses only 0-mark and 1-mark columns.

Lemma 2.3. $\mathrm{forb}(m, 3, p\cdot I_2) \geq 2^m + m2^{m-1} + (p-1)\left(\binom{m}{2} - \binom{r+1}{2}\right) + (r-1)2^r + 1$.

Proof: Let every pair $i < j$ of rows be associated with $\begin{bmatrix}0\\1\end{bmatrix}$, so a mark in a column corresponds exactly to an instance of 0 above 1.

If a column v contains a unique mark at (i,j), then $v_\ell \in \{1,2\}$ for all $1 \leq \ell < i$, $v_\ell = 2$ for all $i < \ell < j$, and $v_\ell \in \{0,2\}$ for $j < \ell \leq m$. Thus, there are exactly $2^{m-1+i-j}$ such columns, and we can include $\min\{p-1, 2^{m-1+i-j}\}$ of them. Let $r = \lceil \log_2(p-1)\rceil$, then $2^{m-1+i-j} < p-1$ if and only if $i \in [r]$ and $m-r+i \leq j \leq m$. Therefore, the number of 1-mark columns we can include is

$$(p-1)\left(\binom{m}{2} - \binom{r+1}{2}\right) + \sum_{i=1}^{r}\sum_{j=m-r+i}^{m} 2^{m-1+i-j} = (p-1)\left(\binom{m}{2} - \binom{r+1}{2}\right) + (r-1)2^r + 1.$$

Since there are $2^m + m2^{m-1}$ columns with no 0 above 1, and we can always include all of them, the result follows. □

If $A \in \text{ext}(m, 3, p \cdot I_2)$, let $A = \begin{bmatrix} B \mid C \end{bmatrix}$ be the standard decomposition. Since $|A| \geq 2^m + m2^{m-1} + (p-1)\left(\binom{m}{2} - \binom{r+1}{2}\right) + (r-1)2^r + 1$ by Lemma 2.3, and $|C| \leq (p-1)\binom{m}{2}$ from definition, we have that $|B| > 2^m + m2^{m-1} - 2^{m-3}$ as long as $m \geq 2r+2$. Hence, by Lemma 2.2, we can suitably permute the rows of A so that every pair $i < j$ of rows is associated with $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Note that in this case the directed graph T is actually a tournament as there are no non-edges.

Motivated by Lemma 2.3 above, we say a pair of rows $i < j$ is *abundant* if there are at least $p-1$ possible columns with exactly one mark at (i, j), or equivalently if $2^{m-1+i-j} \geq p-1$. A pair of rows $i < j$ is called *scarce* if it is not abundant. It follows that if $r = \lceil \log_2(p-1) \rceil$, then there are $\binom{r+1}{2}$ scarce pairs of rows, which are exactly (i, j) for $i \in [r]$ and $m - r + i \leq j \leq m$. The next lemma shows that we can always assume the maximum number of 1-mark columns are present in A.

Lemma 2.4. *There exists $A \in \text{ext}(m, 3, p \cdot I_2)$ containing $2^m + m2^{m-1}$ 0-mark columns and $(p-1)\left(\binom{m}{2} - \binom{r+1}{2}\right) + (r-1)2^r + 1$ 1-mark columns. More precisely, for every abundant pair (i, j) of rows, A contains exactly $p-1$ columns with their only mark at (i, j), and for every scarce pair (i, j) of rows, A contains all columns with their only mark at (i, j).*

Proof: We can always take all $2^m + m2^{m-1}$ columns with no mark. Let $A \in \text{ext}(m, 3, p \cdot I_2)$ maximize the number of 1-mark columns. Assume for some pair $i < j$ of rows, A does not contain $\min\{2^{m-1+i-j}, p-1\}$ columns with their unique marks at (i, j). Then there exists a column v with its only mark at (i, j) that is not in A. There are two cases. If A contains less than $\min\{2^{m-1+i-j}, p-1\}$ marks at (i, j), $\begin{bmatrix} A \mid v \end{bmatrix}$ is then in $\text{Avoid}(m, 3, p \cdot I_2)$ with more columns than A, contradiction. If A contains exactly $\min\{2^{m-1+i-j}, p-1\}$ marks at (i, j), then A contains a column u with at least two marks including a mark at (i, j). Remove column u and add column v to get a new matrix A'. Then A' is in $\text{ext}(m, 3, p \cdot I_2)$ with one more 1-mark column than A, contradiction. Thus, for all pairs $i < j$ of rows, A contains $\min\{2^{m-1+i-j}, p-1\}$ columns with their unique marks at (i, j), which add up to $(p-1)\left(\binom{m}{2} - \binom{r+1}{2}\right) + (r-1)2^r + 1$ 1-mark columns in total. □

We can now prove Proposition 1.4, which shows that $\text{forb}(m, 3, p \cdot K_2) - \text{forb}(m, 3, p \cdot I_2)$ is the same constant for all $m \geq 2r+2$.

Proof Proposition 1.4: Since $\text{forb}(m, 3, p \cdot K_2) = 2^m + m2^{m-1} + (p-1)\binom{m}{2}$ by Dillon and Sali (2021), it suffices to show that for all $m \geq 2r+2$

$$\begin{aligned} \text{forb}(m+1, 3, p \cdot I_2) - \text{forb}(m, 3, p \cdot I_2) &= \text{forb}(m+1, 3, p \cdot K_2) - \text{forb}(m, 3, p \cdot K_2) \\ &= 2^{m-1}(m+4) + (p-1)m. \end{aligned}$$

First we prove that $\text{forb}(m+1, 3, p \cdot I_2) - \text{forb}(m, 3, p \cdot I_2) \geq 2^{m-1}(m+4) + (p-1)m$. Let $A \in \text{ext}(m, 3, p \cdot I_2)$. From above and by Lemma 2.2 we can assume A contains at most $p-1$ instances of 0 above 1 in each pair $i < j$ of rows. A must contain all $2^m + m2^{m-1}$ columns with no mark. Let S denote the number of columns in A with at least one mark.

Consider the following matrix A' with $m+1$ rows. First take all $2^{m+1} + (m+1)2^m$ columns with no 0 above 1. Then, fix an index $r+1 \leq k \leq m+1-r$, which exists as $m \geq 2r+2$. For every column v in A with at least one mark, define a new column v' with $m+1$ rows by $v'_i = v_i$ if $i < k$, $v'_k = 2$ and $v'_i = v_{i-1}$ if $k < i \leq m+1$. The new columns v' obtained in this way are all distinct and have marks only on row set $\{1, 2, \ldots, k-1, k+1, \ldots m+1\}$ corresponding to the marks in the original columns. Add them all to A'. Moreover, for each $1 \leq j < k$, the pair (j, k) is abundant so we can find $p-1$ columns with their unique marks at (j, k). Similarly, for each $k < j \leq m+1$, the pair (k, j) is abundant so we can find $p-1$ columns with their unique marks at (k, j). Add all such $(p-1)m$ columns to A'. Then for any two rows $i < j$ of A', there are at most $p-1$ columns with 0 above 1 by construction, so $A' \in \text{Avoid}(m+1, 3, p \cdot I_2)$, which implies $\text{forb}(m+1, 3, p \cdot I_2) \geq |A'| = 2^{m+1} + (m+1)2^m + S + (p-1)m$. Therefore,

$$\text{forb}(m+1, 3, p \cdot I_2) - \text{forb}(m, 3, p \cdot I_2)$$

$$\geq \left(2^{m+1} + (m+1)2^m + S + (p-1)m\right) - \left(2^m + m2^{m-1} + S\right)$$
$$= 2^{m-1}(m+4) + (p-1)m.$$

Now we prove that $\mathrm{forb}(m+1, 3, p \cdot I_2) - \mathrm{forb}(m, 3, p \cdot I_2) \leq 2^{m-1}(m+4) + (p-1)m$. Let $A \in \mathrm{ext}(m+1, 3, p \cdot I_2)$. We can assume that A has all $2^{m+1} + (m+1)2^m$ columns with no mark. By Lemma 2.4, we can also assume A has the maximum number $(p-1)\left(\binom{m+1}{2} - \binom{r+1}{2}\right) + (r-1)2^r + 1$ of 1-mark columns.

Fix a row index k with $r+1 \leq k \leq m+1-r$. Let S be the number of columns in A with at least two marks and let v be any one of them. For any abundant pair (i, j), every column with a mark at (i, j) is a 1-mark column by Lemma 2.4, so all marks in v correspond to scarce pairs. In particular, if v has a mark at (i, j), then $i \leq r < k < m+2-r \leq j$, which implies that $v_k = 2$, as otherwise v will have a mark at abundant pair (i, k) or (k, j). It follows that all columns in A with at least two marks remain distinct and do not lose any mark after removing row k. Now we construct an m-rowed matrix A' by taking all $2^m + m2^{m-1}$ columns with no mark, $(p-1)(\binom{m}{2} - \binom{r+1}{2}) + (r-1)2^r + 1$ columns with exactly one mark, as well as all columns obtained by removing the row k entries from all columns in A with at least two marks. Then $A' \in \mathrm{Avoid}(m, 3, p \cdot I_2)$, so we have

$$\begin{aligned}\mathrm{forb}(m, 3, p \cdot I_2) \geq |A'| &= \left(2^m + m2^{m-1}\right) + (p-1)\left(\binom{m}{2} - \binom{r+1}{2}\right) + (r-1)2^r + 1 + S \\ &= \mathrm{forb}(m+1, 3, p \cdot I_2) - 2^{m-1}(m+4) - (p-1)m.\end{aligned}$$

This completes the proof. □

This allows us to define g_p to be the common value of $\mathrm{forb}(m, 3, p \cdot K_2) - \mathrm{forb}(m, 3, p \cdot I_2)$ for all $m \geq 2r+2$. By Dillon and Sali (2021), we have $g_1 = g_2 = 0, g_3 = 1, g_4 = 2$ and $g_5 = 5$. In the remaining part of this section, we prove Theorem 1.5, which establishes the asymptotic formula $g_p \sim \frac{1}{2}p(\log_2 p)^2$. As will be seen later, it follows from Lemma 2.3 that $g_p \leq \frac{1}{2}p(\log_2 p)^2 + o(p(\log_2 p)^2)$. To prove the reverse inequality, we use the following two results.

Lemma 2.5. *Let $m \geq 2r+2$. Suppose $b, c \geq 1$ and $b + c \leq r+1$. Then there are at most $r2^r$ m-rowed $0, 1, 2$-columns with exactly bc marks formed by c 0's above b 1's, all corresponding to scarce pairs of rows.*

Proof: Let the positions of the c 0's be $i_1 < \cdots < i_c$ and the positions of the b 1's be $j_1 < \cdots < j_b$. All bc marks correspond to scarce pairs if and only if $j_1 - i_c \geq m - r$. The index i_c must take values in $\{1, \ldots, r\}$, so there are at most r possible choices of i_c. Once i_c is fixed, the immediately following $(m-r-1)$ positions are all equal to '2', since all the marked pairs of rows are scarce. This leaves r positions still undetermined. There are at most 2^r possibilities to choose which of these positions are equal to '2', and once this choice is made, the whole column is uniquely determined, since the values of b and c are fixed. This shows that there are at most $r2^r$ such columns. □

Remark 2.6. *With tedious calculations a better upper bound of such columns can be shown, namely there are at most $2^r(r-b-c+1) + 2^{b+c-2}$ $0, 1, 2$-columns with exactly bc marks formed by c 0's above b 1's, all corresponding to scarce pairs of rows.*

Proposition 2.7. *Let $k \geq 2$ be fixed, then we have $g_p \geq \frac{(p-1)k}{k+1}\binom{r+1}{2} - r2^r \sum_{i=1}^{k} \tau(i)$, where $\tau(n)$ is the number of divisors of n.*

Proof: Let $A \in \mathrm{ext}(m, 3, p \cdot I_2)$. As before we can assume A contains at most $p-1$ instances of 0 above 1 in every pair of rows, and contains the maximum number of 1-mark columns. For $j \geq 0$, let a_j be the number of j-mark columns in A. Note that if $j \geq 2$ and v is a j-mark column, then every mark in v corresponds to a scarce pair of rows. From before, we have $a_0 = 2^m + m2^{m-1}$ and $a_1 = (p-1)\left(\binom{m}{2} - \binom{r+1}{2}\right) + (r-1)2^r + 1$. Since there are at most $p-1$ marks at each pair of rows, we have $\sum_{i=1}^{\infty} ia_i \leq (p-1)\binom{m}{2}$. Let $\sum_{i=1}^{\infty} ia_i = (p-1)\binom{m}{2} - S$.

Let $m \geq 2r+2$, $2 \leq i \leq k$, and consider an m-rowed $0,1,2$-column v with exactly i marks, all of which are scarce. Since $m \geq 2r+2$, there does not exist $x < y < x' < y'$ such that both (x,y) and (x',y') are scarce. So there exists b, c with $bc = i$ such that all i marks of v are formed as c 0's above b 1's. Thus Lemma 2.5 shows there are at most $\tau(i)r2^r$ such columns, and so $a_i \leq \tau(i)r2^r$.

We have

$$\sum_{i=k+1}^{\infty} a_i \leq \sum_{i=k+1}^{\infty} \frac{1}{k+1} i a_i = \frac{1}{k+1}\left((p-1)\binom{m}{2} - S - \sum_{i=1}^{k} i a_i\right)$$

and therefore

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{\infty} a_i \leq \frac{1}{k+1}\left((p-1)\binom{m}{2} - S\right) + \sum_{i=1}^{k}\left(1 - \frac{i}{k+1}\right) a_i.$$

It follows that

$$\begin{aligned}
g_p &= \operatorname{forb}(m,3,p\cdot K_2) - \operatorname{forb}(m,3,p\cdot I_2) = \operatorname{forb}(m,3,p\cdot K_2) - |A| \\
&= (p-1)\binom{m}{2} - \sum_{i=1}^{\infty} a_i \\
&\geq \frac{(p-1)k}{k+1}\binom{r+1}{2} + \frac{S}{k+1} - \frac{k}{k+1}((r-1)2^r+1) - \sum_{i=2}^{k}\left(1-\frac{i}{k+1}\right) a_i \\
&\geq \frac{(p-1)k}{k+1}\binom{r+1}{2} - r2^r - \sum_{i=2}^{k} a_i \\
&\geq \frac{(p-1)k}{k+1}\binom{r+1}{2} - r2^r \sum_{i=1}^{k} \tau(i).
\end{aligned}$$

□

Proof of Theorem 1.5: By Lemma 2.3, $\operatorname{forb}(m,3,p\cdot I_2) \geq 2^m + m2^{m-1} + (p-1)(\binom{m}{2} - \binom{r+1}{2}) + (r-1)2^r + 1$. Hence,

$$\begin{aligned}
g_p &\leq \operatorname{forb}(m,3,p\cdot K_2) - 2^m - m2^{m-1} - (p-1)\left(\binom{m}{2} - \binom{r+1}{2}\right) - (r-1)2^r - 1 \\
&= (p-1)\binom{r+1}{2} - (r-1)2^r - 1 = \frac{1}{2}p(\log_2 p)^2 + o(p(\log_2 p)^2).
\end{aligned}$$

On the other hand, Proposition 2.7 implies that for every fixed $k \geq 2$,

$$g_p \geq \frac{(p-1)k}{k+1}\binom{r+1}{2} - r2^r \sum_{i=1}^{k} \tau(i) = \left(1 - \frac{1}{k+1}\right)\frac{1}{2}p(\log_2 p)^2 + o(p(\log_2 p)^2).$$

Combining these two bounds and letting $k \to \infty$, we get $g_p \sim \frac{1}{2}p(\log_2 p)^2$. □

2.2 $p \leq 4$

In this subsection, we compute the exact values of $\operatorname{forb}(m,3,F(a,b,c,d))$ when $\max\{a,d\} < \min\{b,c\} = p \leq 4$. If $p = 1,2$, then $\operatorname{forb}(m,3,p\cdot I_2) = \operatorname{forb}(m,3,p\cdot K_2)$ by Dillon and Sali (2021). Then, for large m, Lemma 1.3 implies that $\operatorname{forb}(m,3,p\cdot I_2) \leq \operatorname{forb}(m,3,F(a,b,c,d)) = \operatorname{forb}(m,3,F(a,p,p,d)) \leq \operatorname{forb}(m,3,p\cdot K_2)$, so $\operatorname{forb}(m,3,F(a,b,c,d)) = \operatorname{forb}(m,3,p\cdot I_2) = \operatorname{forb}(m,3,p\cdot K_2) = 2^m + m2^{m-1} + (p-1)\binom{m}{2}$.

Next, we treat the cases when $p = 3$ and $p = 4$.

Lemma 2.8. *For $p \geq 3$, let $1 \leq k \leq \min\{p, g_p\}$. If $\min\{b,c\} = p$, $\max\{a,d\} \leq p-k$, $m \geq 2r+2$ and $2^{m-2} \geq (\max\{b,c\}-1)m^2$, then*

$$\mathrm{forb}(m,3,F(a,b,c,d)) \leq \mathrm{forb}(m,3,p\cdot I_2) + g_p - k = \mathrm{forb}(m,3,p\cdot K_2) - k.$$

Proof: The conditions ensure that Lemma 1.3 applies, so we can assume that $b = c = p$, and that $\mathrm{forb}(m,3,p\cdot I_2) = \mathrm{forb}(m,3,p\cdot K_2) - g_p$ by Proposition 1.4. Let $A \in \mathrm{Avoid}(m,3,F(a,p,p,d))$ and let $A = \begin{bmatrix} B \mid C \end{bmatrix}$ be a standard decomposition. Recall from the definition that N is the number of pairs of rows that are associated with either $\begin{bmatrix}0\\0\end{bmatrix}$ or $\begin{bmatrix}1\\1\end{bmatrix}$. If $N = 0$, then $A \in \mathrm{Avoid}(m,3,p\cdot I_2)$ so $|A| \leq \mathrm{forb}(m,3,p\cdot I_2) = \mathrm{forb}(m,3,p\cdot K_2) - g_p$. If $N \geq 1$, then $|A| \leq |B| + |C| \leq 2^m + m2^{m-1} + (p-1)\binom{m}{2} - N(p-(p-k)) \leq 2^m + m2^{m-1} + (p-1)\binom{m}{2} - k = \mathrm{forb}(m,3,p\cdot K_2) - k$. □

Corollary 2.9. *If $\min\{b,c\} = 3$, $\max\{a,d\} \leq 2$, $m \geq 4$ and $2^{m-2} \geq (\max\{b,c\}-1)m^2$, then*

$$\mathrm{forb}(m,3,F(a,b,c,d)) = \mathrm{forb}(m,3,3\cdot I_2) = \mathrm{forb}(m,3,3\cdot K_2) - 1.$$

Proof: Set $p = 3, k = 1$ in Lemma 2.8 and note that $g_3 = 1$. □

Proposition 2.10. *If $\min\{b,c\} = 4$, $\max\{a,d\} \leq 3$, $m \geq 6$ and $2^{m-2} \geq (\max\{b,c\}-1)m^2$, then*

$$\mathrm{forb}(m,3,F(a,b,c,d)) = \mathrm{forb}(m,3,4\cdot I_2) = \mathrm{forb}(m,3,4\cdot K_2) - 2.$$

Proof: The conditions ensure that Lemma 1.3 applies so we can assume that $b = c = 4$. We have that $4\cdot I_2 \prec F(a,4,4,d)$ and $g_4 = 2$ by Dillon and Sali (2021), so $2^m + m2^{m-1} + 3\binom{m}{2} - 2 = \mathrm{forb}(m,3,4\cdot I_2) \leq \mathrm{forb}(m,3,F(a,4,4,d))$. It suffices to show the reverse inequality.

Let $A \in \mathrm{Avoid}(m,3,F(a,4,4,d))$, and let $A = \begin{bmatrix} B \mid C \end{bmatrix}$ be a standard decomposition. If $N = 0$, then $A \in \mathrm{Avoid}(m,3,4\cdot I_2)$ so $|A| \leq \mathrm{forb}(m,3,4\cdot I_2)$. If $N \geq 2$, then $|A| \leq |B| + |C| \leq (2^m + m2^{m-1}) + (3\binom{m}{2} - N) \leq \mathrm{forb}(m,3,4\cdot I_2)$.

If $N = 1$, $|A| \leq |B| + |C| \leq (2^m + m2^{m-1}) + (3\binom{m}{2} - 1) = \mathrm{forb}(m,3,4\cdot I_2) + 1$. Assume for a contradiction that equality holds. Then $|B| = 2^m + m2^{m-1}$, so T is transitive by Lemma 2.2. Let $i' < j'$ be the unique pair of rows associated to $\begin{bmatrix}0\\0\end{bmatrix}$ or $\begin{bmatrix}1\\1\end{bmatrix}$. Without loss of generality we can assume it is associated with $\begin{bmatrix}0\\0\end{bmatrix}$. We can rearrange the rows of A so that there is no 0 above 1 in B for every pair of rows except possibly for rows i', j'. For equality to hold we also need $|C| = 3(\binom{m}{2} - 1) + 2 = 3\binom{m}{2} - 1$. This means that every column in C has exactly one mark, and for every $i < j$ with $(i,j) \neq (i',j')$ there are 3 columns in C with a mark at (i,j), and there are 2 columns in C with a mark at (i',j'). There are four cases.

If $1 < i' < j' < m$, then for a column v to have exactly one mark at $(1,m)$, we need $v_k = 2$ for all $2 \leq k \leq m-1$, which leaves only 1 choice for such v, contradiction. If $i' = 1, j' = m$, then for a column v to have exactly one mark at $(1,m-1)$, we need $v_k = 2$ for all $2 \leq k \leq m-2$. So there are only 2 choices for such v, which contradicts the above. If $i' = 1, j' < m$, then for a column v to have exactly one mark at $(1,m)$, we need $v_k = 2$ for all $2 \leq k \leq m-1$ except that $v_{j'}$ could be either 1 or 2, which gives only 2 choice for such v, contradiction. If $i' > 1, j' = m$, then for a column v to have exactly one mark at $(1,m)$, we need $v_k = 2$ for all $2 \leq k \leq m-1$ except that $v_{i'}$ could be either 0 or 2, which gives only 2 choice for such v, contradiction. In all cases we have a contradiction so equality cannot hold and $|A| \leq \mathrm{forb}(m,3,4\cdot I_2)$. Thus $\mathrm{forb}(m,3,F(a,4,4,d)) \leq \mathrm{forb}(m,3,4\cdot I_2)$, and this completes the proof. □

Up to now, we have shown that for $p \leq 4$, $\mathrm{forb}(m,3,F(a,b,c,d)) = \mathrm{forb}(m,3,p\cdot I_2) = \mathrm{forb}(m,3,p\cdot K_2) - g_p$ as long as $\max\{a,d\} < p = \min\{b,c\}$ and Lemma 1.3 applies. This need not be true for larger p as will be seen in the next subsection.

2.3 $F(a,p,p,d)$

Throughout this section we assume that $a, d \leq p$ and $A \in \mathrm{ext}(m, 3, F(a,p,p,d))$.

2.3.1 Structure of $A \in \mathrm{ext}(m, 3, F(a,p,p,d))$

Let $A = \begin{bmatrix} B \mid C \end{bmatrix}$ be a standard decomposition of A as defined in Definition 2.1. Note that $|A| \geq \mathrm{forb}(m, 3, p \cdot I_2) = \mathrm{forb}(m, 3, p \cdot K_2) - g_p = 2^m + m2^{m-1} + (p-1)\binom{m}{2} - g_p$ and $|C| \leq (p-1)\binom{m}{2}$. Thus, for all $m \geq 2N_r + 2$, we have $|B| > 2^m + m2^{m-1} - 2^{m-3}$, so by Lemma 2.2 we may assume that no pair $i < j$ of rows is associated with $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Lemma 2.11. *Let $1 \leq i < j < k \leq m$.*

- *if j, k forms a 0-non-edge, then one of i, j and i, k also forms a 0-non-edge,*
- *if i, j forms a 1-non-edge, then one of i, k and j, k also forms a 1-non-edge.*

Proof: The proof is similar to the one of Lemma 2.2. First assume that $i < j < k$ and j, k forms a 0-non-edge, and neither i, j nor i, k form a 0-non-edge. For any subset $X \subset [m]$ of rows, let $B(X)$ be the matrix consisting of columns of B with support X. Note that $B(X)|_X$ is simple and it is in $\mathrm{Avoid}(|X|, 2, K_2)$, hence $|B(X)|_X| \leq |X| + 1$ if $\{i, j, k\} \not\subseteq X$. For $\{i, j, k\} \subseteq X$ we show below that $|B(X)|_X| \leq |X|$, so then a similar calculation as in Lemma 2.2 shows $|B| \leq 2^m + m2^{m-1} - 2^{m-3}$, a contradiction.

(i) Pair (i, j) is assigned $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(ia) Pair (i, k) is assigned $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. In this case entries in row i of $B(X)$ must all be 1, since $i = 0$ would imply both $j = 0$ and $k = 0$, which contradicts the assumption that j, k forms a 0-non-edge. Therefore, $B(X)|_{X\setminus\{i\}}$ is simple.

(ib) Pair (i, k) is assigned $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. In this case the only possible columns of $B(X)|_{\{i,j,k\}}$ are $\begin{array}{c|cc} i & 0 & 1 \\ j & 0 & 1 \\ k & 1 & 0 \end{array}$, hence $B(X)|_{X\setminus\{i\}}$ is simple.

(ii) Pair (i, j) is assigned $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(iia) Pair (i, k) is assigned $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Now, the only possible columns of $B(X)|_{\{i,j,k\}}$ are $\begin{array}{c|cc} i & 0 & 1 \\ j & 1 & 0 \\ k & 0 & 1 \end{array}$, hence $B(X)|_{X\setminus\{i\}}$ is simple.

(iib) Pair (i, k) is assigned $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Here, row i of $B(X)$ cannot contain any entry 1, hence $B(X)|_{X\setminus\{i\}}$ is simple.

In all cases above, it follows that $|B(X)| = |B(X)|_{X\setminus\{i\}}| \leq \mathrm{forb}(|X| - 1, 2, K_2) = |X|$, as required.
The case when $i < j < k$ and i, j forms a 1-non-edge is similar. □

Lemma 2.12. *If $i < j < k$ and i, k forms a non-edge, then one of i, j and j, k also forms a non-edge.*

Proof: If ij and jk are both directed edges, then ik would also be a directed edge by the transitivity provided by Lemma 2.2, a contradiction. □

Lemma 2.13. *If there are N non-edges, then all the 0-non-edges are in the first $N+1$ rows, and all the 1-non-edges are in the last $N+1$ rows.*

Proof: Let k be the largest number such that there exists $j < k$ with j, k forming a 0-non-edge. Then either $1, j$ or $1, k$ is also a 0-non-edge by Lemma 2.11. If $1, k$ is a 0-non-edge, then by Lemma 2.12, for every $1 < \ell < k$, either $1, \ell$ or ℓ, k is a non-edge. This means that the total number of non-edges is at least $k-1$. If instead $1, j$ is a 0-non-edge, then similarly for $1 < i < j$, one of $1, i$ and i, j is a non-edge, and for every $j < \ell < k$, one of j, ℓ and ℓ, k is a non-edge. Together with the non-edges $1, j$ and j, k we get at least $(j-1)+(k-j) = k-1$ non-edges. In both cases, we conclude that $N \geq k-1$, so $k \leq N+1$. The 1-non-edges case is similar. □

2.3.2 Lower bounds

Similar to Section 2.1, we say a pair (i, j) of rows with $i < j$ is *abundant* if $2^{i-1+m-j} \geq p-1$, and *scarce* otherwise. Equivalently, if $r = \lceil \log_2(p-1) \rceil$, then (i, j) is abundant if $i - 1 + m - j \geq r$ and scarce otherwise.

Lemma 2.14. *Suppose $m \geq \max\{2N+2, 2r+2\}$ and there are N non-edges. Then*

- *for every scare pair $(i, m+1-j)$, if a_i is the number of 0-non-edges (i, k) with $i < k$, and b_j is the number of 1-non-edges $(k, m+1-j)$ with $k < m+1-j$, then the number of columns with their unique marks at $(i, m+1-j)$ is $2^{i+j-2+a_i+b_j}$.*
- *for every abundant pair (i, j) that is not a non-edge, there are at least $p-1$ 1-mark columns, each of whose unique mark is at (i, j).*
- *for every non-edge (i, j), there are at least $p-1$ 1-mark columns, each of whose unique mark is at (i, j).*

Proof: Since $m \geq 2N+2$, the first and last $N+1$ rows of A are disjoint. By Lemma 2.13, all 0-non-edges are within the first $N+1$ rows, and all 1-non-edges are within the last $N+1$ rows.

If $(i, m+1-j)$ is a scarce pair, then $i-1+m-(m+1-j) \leq r-1$, so $(m+1-j)-i \geq m-r \geq \frac{1}{2}m \geq N+1$. In particular, $(i, m+1-j)$ is not a non-edge. Let $i < k_1 < k_2 < \ldots < k_{a_i}$ be such that (i, k_t) is a 0-non-edge for all $t \in [a_i]$, and let $\ell_1 < \ell_2 < \ldots < \ell_{b_j} < m+1-j$ be such that $(\ell_s, m+1-j)$ is a 1-non-edge for all $s \in [b_j]$. By Lemma 2.13, $k_{a_i} \leq N+1 < m-N \leq \ell_1$. If v is a 1-mark column with its unique mark at $(i, m+1-j)$, then we must have $v_i = 0, v_{m+1-j} = 1$ and $v_z = 2$ for all $z \in \{i+1, i+2, \ldots, m+1-j\} \setminus \{k_1, k_2, \ldots, k_{a_i}, \ell_1, \ell_2, \ldots, \ell_{b_j}\}$, while v_z can be 1 or 2 for all $z < i$ and all $z \in \{k_1, k_2, \ldots, k_{a_i}\}$, and v_z can be 0 or 2 for all $z > m+1-j$ and all $z \in \{\ell_1, \ell_2, \ldots, \ell_{b_j}\}$. Moreover, every column v of this form does contain exactly one mark at $(i, m+1-j)$, so there are $2^{i+j-2+a_i+b_j}$ 1-mark columns with their unique mark at $(i, m+1-j)$.

If (i, j) is abundant and not a non-edge, then $i-1+m-j \geq r$. It follows that we can find $0 \leq r_1 \leq i-1$ and $0 \leq r_2 \leq m-j$, such that $r_1 + r_2 = r$. Consider the columns v of the form $v_i = 0, v_j = 1$, $v_z = 1$ or 2 for all $z \leq r_1$, $v_z = 0$ or 2 for all $z \geq m - r_2 + 1$, and $v_z = 2$ for all other z. There are $2^{r_1+r_2} = 2^r \geq p-1$ such columns, and one can verify that each of these contains a unique mark at (i, j).

If (i, j) is a non-edge, without loss of generality a 0-non-edge, consider the columns v of the form $v_i = v_j = 0$, $v_z = 0$ or 2 for all $z \geq m - r + 1$, and $v_z = 2$ for all other z. There are $2^r \geq p-1$ such columns, and one can verify that each of these contains a unique mark at (i, j). □

Lemma 2.15. *If there exist $r_1, r_2 \geq 0$ with $r_1 + r_2 \leq m-2$, such that the 0-non-edges are exactly (i, j) with $1 \leq i < j \leq r_1 + 1$ and the 1-non-edges are exactly (i, j) with $m - r_2 \leq i < j \leq m$, then there are exactly $2^m + m2^{m-1}$ columns with no mark.*

Proof: If $v_z = 1$ or 2 for all $z \leq m - r_2 - 1$ and $v_z = 0$ or 2 for all $z \geq m - r_2$, then v contains no mark and there are 2^m such columns.

If v is not of this form and has no mark, then $v_z = 0$ for some $z \leq m - r_2 - 1$, or $v_z = 1$ for some $z \geq m - r_2$. In the first case, let $i \leq m - r_2 - 1$ be minimal such that $v_i = 0$. If $i \leq r_1 + 1$, then v_z must be 1 or 2 for all other $z \in [r_1 + 1]$ and v_z must be 0 or 2 for all $z > r_1 + 1$, and all of these 2^{m-1} columns have no mark. If $r_1 + 1 < i \leq m - r_2 - 1$, then v_z must be 1 or 2 for all other $z < i$ and v_z must be 0 or 2 for all $z > i$, and all of these 2^{m-1} columns have no mark. If we are not in the first case, then $v_z = 1$ or 2 for all $z \leq m - r_2 - 1$. Let $i \geq m - r_2$ be minimal such that $v_i = 1$. Then v_z must be 0 or 2 for all other $z \geq m - r_2$, and all of these 2^{m-1} columns have no mark. In total, there are $m2^{m-1}$ such columns with no mark. □

Recall that $N_r = \binom{\lfloor \frac{r}{2} \rfloor + 1}{2} + \binom{\lceil \frac{r}{2} \rceil + 1}{2}$. We say that a scarce pair $(i, m + 1 - j)$ is *fixed*, if $2^{i+j-2+a_i+b_j} \geq p - 1$, or equivalently $i + j - 2 + a_i + b_j \geq r$, using the notation of Lemma 2.14.

Proof of Proposition 1.9: Since $\text{forb}(m, 3, F(p - q_0, p, p, p - q_1)) = \text{forb}(m, 3, F(p - q_1, p, p, p - q_0))$ by taking 0,1-complements, we may without loss of generality assume $q_0 \leq q_1$. Consider $\binom{r_1+1}{2}$ 0-non-edges, which are (i, j) for all $1 \leq i < j \leq r_1 + 1$, and $\binom{r_2+1}{2}$ 1-non-edges, which are (i, j) for all $m - r_2 \leq i < j \leq m$. Using $r_1 + r_2 = r$, one can check that all $\binom{r+1}{2}$ scarce pairs are fixed. By Lemma 2.15, there are $2^m + m2^{m-1}$ columns with no mark. Since all scarce pairs are fixed, for each pair (i, j) that is not a non-edge, by Lemma 2.14, we can find $p - 1$ columns, each of whose unique mark is at (i, j). Also by Lemma 2.14, for each 0-non-edge (i, j), we can find $p - q_0 - 1$ columns with their unique marks at (i, j), and for each 1-non-edge (i, j), we can find $p - q_1 - 1$ columns with their unique marks at (i, j). The matrix formed by all these distinct columns avoids $F(p - q_0, p, p, p - q_1)$ as a configuration, so

$$\begin{aligned}\text{forb}(m, 3, F(p - q_0, p, p, p - q_1)) &\geq 2^m + m2^{m-1} + (p - 1)\binom{m}{2} - q_0\binom{r_1 + 1}{2} - q_1\binom{r_2 + 1}{2} \\ &= \text{forb}(m, 3, p \cdot K_2) - q_0\binom{r_1 + 1}{2} - q_1\binom{r_2 + 1}{2},\end{aligned}$$

as required. □

Corollary 2.16. *For every* $0 \leq q \leq p - 1$ *and every* $m \geq 2N_r + 2$*,*

$$\text{forb}(m, 3, F(p - q, p, p, p - q)) \geq \text{forb}(m, 3, p \cdot K_2) - qN_r.$$

Proof: Apply Proposition 1.9 with $q_0 = q_1 = q$ and $r_1 = \lceil \frac{r}{2} \rceil, r_2 = \lfloor \frac{r}{2} \rfloor$. □

Let us call a pair $(i, m + 1 - j)$ of rows *weakly fixed* if, using the notation of Lemma 2.14, we have $a_i + b_j = r + 1 - i - j$.

Proof of Proposition 1.8: From assumption, $d = q - (\lceil \frac{r}{2} \rceil + 1)(p - 1 - 2^{r-1}) > 0$. We may assume $r \geq 2$ so $\lfloor \frac{r}{2} \rfloor > 0$, as otherwise the result follows directly from Corollary 2.16. Consider $\binom{\lceil \frac{r}{2} \rceil + 1}{2}$ 0-non-edges, which are (i, j) for all $1 \leq i < j \leq \lceil \frac{r}{2} \rceil + 1$, and $\binom{\lfloor \frac{r}{2} \rfloor}{2} = \binom{\lfloor \frac{r}{2} \rfloor + 1}{2} - \lfloor \frac{r}{2} \rfloor$ 1-non-edges, which are (i, j) for all $m - \lfloor \frac{r}{2} \rfloor + 1 \leq i < j \leq m$. This corresponds to $r_1 = \lceil \frac{r}{2} \rceil$ and $r_2 = \lfloor \frac{r}{2} \rfloor - 1$ in the language of Proposition 1.9, though as $r_1 + r_2 = r - 1 < r$, not every scarce pair is fixed. It can be checked that the scarce pairs $(i, m + 1 - j)$ are weakly fixed for all $i \in [\lceil \frac{r}{2} \rceil + 1]$ and $j \in [\lfloor \frac{r}{2} \rfloor]$, while all other scarce pairs are fixed. This means that we used $N_r - \lfloor \frac{r}{2} \rfloor$ non-edges to fix $\binom{r+1}{2} - \lfloor \frac{r}{2} \rfloor (\lceil \frac{r}{2} \rceil + 1)$ scarce pairs, and leave the remaining $\lfloor \frac{r}{2} \rfloor (\lceil \frac{r}{2} \rceil + 1)$ scarce pairs weakly fixed. Therefore, $\text{forb}(m, 3, F(p - q, p, p, p - q))$ is at least the number of columns with 0 or 1 mark, which by Lemma 2.14 and Lemma 2.15 is

$$\begin{aligned}&2^m + m2^{m-1} + (p - 1)\binom{m}{2} - q\left(N_r - \left\lfloor \frac{r}{2} \right\rfloor\right) - \left\lfloor \frac{r}{2} \right\rfloor \left(\left\lceil \frac{r}{2} \right\rceil + 1\right)(p - 1 - 2^{r-1}) \\ &= \text{forb}(m, 3, p \cdot K_2) - qN_r + \left\lfloor \frac{r}{2} \right\rfloor \left(q - \left(\left\lceil \frac{r}{2} \right\rceil + 1\right)(p - 1 - 2^{r-1})\right) \\ &= \text{forb}(m, 3, p \cdot K_2) - qN_r + \left\lfloor \frac{r}{2} \right\rfloor d,\end{aligned}$$

as claimed. □

2.3.3 Upper bounds

Lemma 2.17. *If $m \geq 2N_r + 2$, and N non-edges are used to fix M scarce pairs, then the maximum of $M - N$ taken ovar all matrices $A \in \mathrm{ext}(m, 3, F(a, p, p, d))$ is*

$$\max(M - N) = \binom{r+1}{2} - N_r.$$

Proof: Since $M \leq \binom{r+1}{2}$, we may assume that $N \leq N_r$, so that $m \geq 2N + 2$. For every $i \in [r]$, let a_i be the number of 0-non-edges (i, k) with $i < k$, and let b_i be the number of 1-non-edges $(k, m + 1 - i)$ with $k < m + 1 - i$. By definition, for every $i \in [r]$ and $j \in [r + 1 - i]$, the scarce pair $(i, m + 1 - j)$ is fixed if and only if $2^{i+j-2+a_i+b_j} \geq p - 1$, which happens if and only if $a_i + b_j \geq r + 2 - i - j$.

Consider the construction in the proof of Proposition 1.9 with $r_1 = \lceil \frac{r}{2} \rceil$ and $r_2 = \lfloor \frac{r}{2} \rfloor$. Here, $N_r = \binom{\lceil \frac{r}{2} \rceil + 1}{2} + \binom{\lfloor \frac{r}{2} \rfloor + 1}{2}$ non-edges are used to fix all $\binom{r+1}{2}$ scarce pairs, so $M - N$ can achieve the value $\binom{r+1}{2} - N_r$.

To show $\binom{r+1}{2} - N_r$ is the best possible, consider the following equivalent problem on the triangular array T_r of size r, whose (i, j) entry is $T_r(i, j) = r + 2 - i - j$ for all $i \in [r]$ and $j \in [r + 1 - i]$.

$$\begin{array}{ccccc} r & r-1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \iddots & \iddots & \\ 3 & 2 & 1 & & \\ 2 & 1 & & & \\ 1 & & & & \end{array}$$

For every $i \in [r]$, a row operation on row i reduces every entry in row i by 1, and a column operation on column i reduces every entry in column i by 1. If we perform a_i row operations on row i and b_i column operations on column i for all $i \in [r]$, then after all of these operations, the (i, j) entry of T_r is non-positive if and only if $a_i + b_j \geq r + 2 - i - j$, and so if and only if the scarce pair $(i, m + 1 - j)$ is fixed. Therefore, it suffices to show that if N row or column operations turn M entries in T_r non-positive, then $M - N \leq \binom{r+1}{2} - N_r$.

To this end, we use induction on r to show $M - N \leq \binom{r+1}{2} - N_r = \left\lfloor \frac{r^2}{4} \right\rfloor$. The $r = 1, 2$ cases can be checked easily, so we assume $r \geq 3$.

For each $i \in [r]$, let a_i and b_i be the number of operations performed on row i and column i, respectively. Observe that we may assume $a_r = 0$, as otherwise reducing a_r by 1 and increasing b_1 by 1 does not decrease M and leaves N unchanged. Similarly, we may assume $b_r = 0$.

Let M_1 be the number of non-positive integers in the first row or column of T_r after N operations, and let $M' = M - M_1$. Let $N' = \sum_{i=2}^{r-1} a_i + \sum_{i=2}^{r-1} b_j$, so that $N - N' = a_1 + a_r + b_1 + b_r = a_1 + b_1$. From the induction hypothesis, $M' - N' \leq \left\lfloor \frac{(r-2)^2}{4} \right\rfloor$, so we are done if $M_1 - a_1 - b_1 \leq \left\lfloor \frac{r^2}{4} \right\rfloor - \left\lfloor \frac{(r-2)^2}{4} \right\rfloor = r - 1$.

If the top left entry of T_r, which is r, is turned non-positive, then $a_1 + b_1 \geq r$, and so $M_1 - a_1 - b_1 \leq 2r - 1 - r = r - 1$ and we are done.

If the top left entry of T_r remains positive, let x be the number of non-positive entries in row 1 and let y be the number of non-positive entries in column 1, so that $x + y = M_1$. If $x \leq a_1$, then $y - b_1 \leq y \leq r - 1$, so $M_1 - a_1 - b_1 = x - a_1 + y - b_1 \leq r - 1$ and we are done. Similarly, we are done if $y \leq b_1$, so we assume neither is true.

For each $2 \leq i \leq r$, the $(1, i)$ entry of T_r is non-positive after the operations if and only if $a_1 + b_i \geq r + 1 - i$. Since x of these entries are non-positive, we have $\sum_{i=2}^{r} b_i \geq \sum_{i=r-x+1}^{r} \max\{r + 1 - i - a_1, 0\} = \sum_{i=1}^{x} \max\{i - a_1, 0\} = \sum_{k=1}^{x-a_1} k = \binom{x-a_1+1}{2}$. Similarly, we have $\sum_{i=2}^{r} a_i \geq \binom{y-b_1+1}{2}$. Like above, we may assume $M_1 - a_1 - b_1 = x - a_1 + y - b_1 \geq r$, as otherwise we are done. By convexity, $N \geq \sum_{i=2}^{r} a_i + \sum_{i=2}^{r} b_i \geq \binom{\lfloor \frac{r+2}{2} \rfloor}{2} + \binom{\lceil \frac{r+2}{2} \rceil}{2} = N_r$, so $M - N \leq \binom{r+1}{2} - N_r = \left\lfloor \frac{r^2}{4} \right\rfloor$, as required. □

Lemma 2.18. *If $m \geq 2N_r + 2$ and $p \geq 2^{r-1} + 2q + 1$, then*

$$\mathrm{forb}(m, 3, F(p-q, p, p, p-q)) = \mathrm{forb}(m, 3, p \cdot K_2) - qN_r.$$

Proof: Suppose N non-edges are used to fix M scarce pairs. Let $\mathcal{P}$ be the set of scarce pairs $(i, m+1-j)$ that remained unfixed. For every $(i, m+1-j) \in \mathcal{P}$, let the (i, j) entry of T_r after the N corresponding row or columns operations be $a_{i,j}$, so that by Lemma 2.14 there are $n_{i,j} := 2^{r-a_{i,j}}$ possible 1-mark columns with their unique marks at $(i, m+1-j)$. Since $(i, m+1-j)$ is not fixed, $a_{i,j} \geq 1$ and so $p-1-n_{i,j} \geq p-1-2^{r-1} \geq 2q$.

It follows that the maximum number of 1-mark columns plus columns with at least two marks is at most

$$\begin{aligned}
&(p-1)\left(\binom{m}{2} - \left(\binom{r+1}{2} - M\right)\right) - qN + \sum_{(i,m+1-j)\in\mathcal{P}} n_{i,j} + \frac{1}{2}\sum_{(i,m+1-j)\in\mathcal{P}} (p-1-n_{i,j}) \\
&= (p-1)\binom{m}{2} - qN - \frac{1}{2}\sum_{(i,m+1-j)\in\mathcal{P}} (p-1-n_{i,j}) \\
&\leq (p-1)\binom{m}{2} - qN - q\left(\binom{r+1}{2} - M\right) \\
&\leq (p-1)\binom{m}{2} - qN_r,
\end{aligned}$$

where the last inequality follows from Lemma 2.17.

There are at most $2^m + m2^{m-1}$ columns with no mark, as they together form a matrix in $\mathrm{Avoid}(m, 3, K_2)$. Hence, the total number of columns is at most $2^m + m2^{m-1} + (p-1)\binom{m}{2} - qN_r = \mathrm{forb}(m, 3, p \cdot K_2) - qN_r$. This combined with Corollary 2.16 completes the proof. □

Lemma 2.19. *Let $m \geq 2N_r + 2$, $qN_r \leq 2^{r-3}$, and $A \in \mathrm{Avoid}(m, 3, F(p-q, p, p, p-q))$. Suppose there is at least one scarce pair that is neither fixed nor weakly fixed, then A has at most $\mathrm{forb}(m, 3, p \cdot K_2) - qN_r$ columns.*

Proof: Let $\mathcal{P}_1$ be the set of scarce pairs that are weakly fixed and let $\mathcal{P}_2$ be the set of scarce pairs that are neither fixed nor weakly fixed. For every $(i, m+1-j) \in \mathcal{P} := \mathcal{P}_1 \cup \mathcal{P}_2$, let the (i, j) entry of T_r after the N operations be $a_{i,j}$, so $a_{i,j} = 1$ if $(i, m+1-j) \in \mathcal{P}_1$, and $a_{i,j} \geq 2$ if $(i, m+1-j) \in \mathcal{P}_2$. From Lemma 2.14, there are $n_{i,j} := 2^{r-a_{i,j}}$ possible 1-mark columns with their unique marks at $(i, m+1-j)$.

Therefore, the maximum number of columns is at most, like in the proof of Lemma 2.18,

$$\begin{aligned}
&2^m + m2^{m-1} + (p-1)\binom{m}{2} - qN - \frac{1}{2}\sum_{(i,j)\in\mathcal{P}} (p-1-n_{i,j}) \\
&< 2^m + m2^{m-1} + (p-1)\binom{m}{2} - qN - \frac{1}{2}\cdot 2^{r-2} \\
&\leq 2^m + m2^{m-1} + (p-1)\binom{m}{2} - qN_r \\
&= \mathrm{forb}(m, 3, p \cdot K_2) - qN_r,
\end{aligned}$$

as required, where the first inequality uses $\emptyset \neq \mathcal{P}_2 \subset \mathcal{P}$ and $p-1-n_{i,j} \geq p-1-2^{r-2} > 2^{r-2}$ for all $(i, m+1-j) \in \mathcal{P}_2$, while the second inequality holds because $\frac{1}{2}\cdot 2^{r-2} \geq qN_r$. □

Lemma 2.20. *If all scarce pairs are fixed or weakly fixed by N non-edges, then $N_r - N \leq \lceil \frac{r}{2} \rceil$.*

Proof: In the language of Lemma 2.17, all entries in the triangular array T_r are at most 1 after N row or column operations, and we want to show $N \geq N_r - \lceil \frac{r}{2} \rceil$. We may assume that all operations are performed on

the first $r-1$ rows and columns, as the entries in the last row and the last column are already at most 1. Now, apply this set of operations on the first $r-1$ rows and columns of T_r to the triangular array T_{r-1} instead, and note that they turn all $\binom{r}{2}$ entries in T_{r-1} non-positive. Thus, by Lemma 2.17 we have $\binom{r}{2} - N \leq \binom{r}{2} - N_{r-1}$, and so $N \geq N_{r-1} = N_r - \lceil \frac{r}{2} \rceil$. □

For $i \in [r]$ and $j \in [r+1-i]$, we say the (i,j) entry of T_r is fixed (or weakly fixed) if the corresponding scarce pair $(i, m+1-j)$ is fixed (or weakly fixed).

Lemma 2.21. *If all scarce pairs are fixed or weakly fixed by $N_r - t$ non-edges for some $0 \leq t \leq \lceil \frac{r}{2} \rceil$, then there are at least $t(\lfloor \frac{r}{2} \rfloor + 1)$ weakly fixed pairs.*

Proof: We use induction on r and work equivalently in the triangular array T_r. The $r = 1, 2$ cases are easy to verify, so we assume $r \geq 3$.

For every $1 \leq i \leq \lceil \frac{r+1}{2} \rceil$, let a_i be the number of operations performed on row i of T_r and let b_i be the number of operations performed on column i of T_r. For every $1 \leq i \leq \lceil \frac{r+1}{2} \rceil$, since the pair $(1, m+1-i)$ is fixed or weakly fixed, the $(1,i)$ entry of T_r, which was $r-i+1$ at the beginning, is now at most 1. Thus, $r-i+1-a_1-b_i \leq 1$, and so $a_1 + b_i \geq r-i$. In particular, $a_1 + b_1 \geq r-1$. Moreover, if the $(1,i)$ entry is non-positive, then $r-i+1-a_1-b_i \leq 0$, and so $a_1 + b_i \geq r-i+1$. Similarly, by looking at the $(i,1)$ entry of T_r for every $1 \leq i \leq \lceil \frac{r+1}{2} \rceil$, we see that $a_i + b_1 \geq r-i$, and $a_i + b_1 \geq r-i+1$ if this entry is non-positive.

Consider the subtriangle of size $r-2$ obtained by removing the first row and column from T_r. Let $N_{r-2} - S$ be the number of operations performed in this subtriangle. Then $r-1 \leq a_1 + b_1 = N_r - t - (N_{r-2} - S)$, so $S \geq r - 1 - N_r + N_{r-2} + t = t-1$. Also, since all entries in this subtriangle are fixed or weakly fixed, we have $N_{r-2} - S \geq N_{r-2} - \lceil \frac{r-2}{2} \rceil = N_{r-2} - \lceil \frac{r}{2} \rceil + 1$ by Lemma 2.20. It follows that $S \leq \lceil \frac{r}{2} \rceil - 1$, and we can apply the induction hypothesis to see that there are at least $S(\lfloor \frac{r-2}{2} \rfloor + 1) = S \lfloor \frac{r}{2} \rfloor$ weakly fixed entries in this subtriangle.

If $S \geq t+1$, then as $S \leq \lceil \frac{r}{2} \rceil - 1$ from above, we have $t \leq \lceil \frac{r}{2} \rceil - 2$. From the induction hypothesis, the subtriangle contains at least $(t+1)\lfloor \frac{r}{2} \rfloor \geq t(\lfloor \frac{r}{2} \rfloor + 1)$ weakly fixed entries, and hence so does the original triangle, as required. Otherwise, $S = t + x$ for $x = -1$ or 0. Then, we have $a_1 + b_1 = N_r - t - (N_{r-2} - S) = r + x$. Therefore,

$$\sum_{i=2}^{\lceil \frac{r+1}{2} \rceil} (a_i + b_1) + \sum_{i=2}^{\lceil \frac{r+1}{2} \rceil} (a_1 + b_i) \leq \left\lceil \frac{r-1}{2} \right\rceil (r+x) + (N_{r-2} - S)$$
$$= \begin{cases} \frac{3}{4}r^2 + \frac{1}{2}(x-1)r - x - t, & \text{if } r \text{ is even,} \\ \frac{3}{4}r^2 + \frac{1}{2}(x-2)r + \frac{1}{4}(1 - 6x - 4t), & \text{if } r \text{ is odd.} \end{cases}$$

On the other hand,

$$\sum_{i=2}^{\lceil \frac{r+1}{2} \rceil} (r-i+1) + \sum_{i=2}^{\lceil \frac{r+1}{2} \rceil} (r-i+1) = \begin{cases} \frac{3}{4}r^2 - \frac{1}{2}r, & \text{if } r \text{ is even,} \\ \frac{3}{4}r^2 - r + \frac{1}{4}, & \text{if } r \text{ is odd.} \end{cases}$$

Therefore, if r is even, then at least

$$\frac{3}{4}r^2 - \frac{1}{2}r - \left(\frac{3}{4}r^2 + \frac{1}{2}(x-1)r - x - t \right) = x + t - \frac{1}{2}xr$$

of the inequalities $a_i + b_1 \geq r-i+1$ for $2 \leq i \leq \frac{r}{2}+1$ and $a_1 + b_i \geq r-i+1$ for $2 \leq i \leq \frac{r}{2}+1$ must fail to hold, which leads to $x + t - \frac{1}{2}xr$ weakly fixed entries in the first row or column. If $x = -1$, this is $\frac{1}{2}r + t - 1$, which along with the weakly fixed $(1,1)$ entry and the at least $(t-1)\frac{r}{2}$ weakly fixed entries in the subtriangle guaranteed by the induction step above gives the required $t(\frac{r}{2}+1)$ weakly fixed entries overall. If $x = 0$, this is t, and there are at least $t\frac{r}{2}$ weakly fixed entries in the subtriangle by induction, so we have the required $t(\frac{r}{2}+1)$ overall as well.

If r is odd, then at least

$$\frac{3}{4}r^2 - r + \frac{1}{4} - \left(\frac{3}{4}r^2 + \frac{1}{2}(x-2)r + \frac{1}{4}(1-6x-4t)\right) = \frac{3}{2}x + t - \frac{1}{2}xr$$

of the inequalities $a_i + b_1 \geq r - i + 1$ for $2 \leq i \leq \left\lceil \frac{r+1}{2} \right\rceil$ and $a_1 + b_i \geq r - i + 1$ for $2 \leq i \leq \left\lceil \frac{r+1}{2} \right\rceil$ must fail to hold, leading to $\frac{3}{2}x + t - \frac{1}{2}xr$ weakly fixed entries in the first row or column. If $x = -1$, this is $\frac{1}{2}r + t - \frac{3}{2} = \left\lfloor \frac{r}{2} \right\rfloor + t - 1$, which along with the weakly fixed $(1,1)$ entry and the at least $(t-1)\left\lfloor \frac{r}{2} \right\rfloor$ weakly fixed entries in the subtriangle guaranteed by induction gives the required $t(\left\lfloor \frac{r}{2} \right\rfloor + 1)$ weakly fixed entries overall. If $x = 0$, this is t, and there are at least $t\left\lfloor \frac{r}{2} \right\rfloor$ weakly fixed entries in the subtriangle by induction, so we have the required $t(\left\lfloor \frac{r}{2} \right\rfloor + 1)$ overall as well, finishing the proof. □

Proof of Theorem 1.6: We have already seen in Lemma 2.18 that equality holds if $p \geq 2^{r-1} + 2q + 1$. If $p \leq 2^{r-1} + 2q$ and $qN_r \leq 2^{r-3}$, then by Lemma 2.19, the number of columns is at most $\mathrm{forb}(m, 3, p \cdot K_2) - qN_r$ if at least one scarce pair is neither fixed nor weakly fixed. Now assume all scarce pairs fixed or weakly fixed and $\frac{1}{2}(\left\lfloor \frac{r}{2} \right\rfloor + 1)(p - 1 - 2^{r-1}) \geq q$.

Suppose there are $N = N_r - t$ non-edges, and they fix M pairs, then by Lemma 2.20 we have $t \leq \left\lceil \frac{r}{2} \right\rceil$. Using the notation in the proof of Lemma 2.19, the number of columns with at least 1 mark is at most

$$\begin{aligned}
&(p-1)\binom{m}{2} - qN - \frac{1}{2}\sum_{(i,m+1-j)\in\mathcal{P}_1}(p - 1 - n_{i,j}) \\
&= (p-1)\binom{m}{2} - qN - \frac{1}{2}|\mathcal{P}_1|(p - 1 - 2^{r-1}) \\
&= (p-1)\binom{m}{2} - qN_r + qt - \frac{1}{2}|\mathcal{P}_1|(p - 1 - 2^{r-1}).
\end{aligned}$$

If $t \leq 0$, then this is at most $(p-1)\binom{m}{2} - qN_r$. If $1 \leq t \leq \left\lceil \frac{r}{2} \right\rceil$, then Lemma 2.21 implies that $|\mathcal{P}_1| \geq t\left(\left\lfloor \frac{r}{2} \right\rfloor + 1\right)$, so this is at most

$$(p-1)\binom{m}{2} - qN_r + qt - \frac{1}{2}t\left(\left\lfloor \frac{r}{2} \right\rfloor + 1\right)(p - 1 - 2^{r-1}) \leq (p-1)\binom{m}{2} - qN_r.$$

Thus, in both cases, the maximum number of columns is at most

$$2^m + m2^{m-1} + (p-1)\binom{m}{2} - qN_r = \mathrm{forb}(m, 3, p \cdot K_2) - qN_r.$$

Together with Corollary 2.16, this completes the proof. □

Proof of Corollary 1.7: If $p = 2$, then $N_r = N_0 = 0$. From Dillon and Sali (2021), $g_2 = 0$, so we have $\mathrm{forb}(m, 3, 2 \cdot K_2) = \mathrm{forb}(m, 3, 2 \cdot I_2)$ for $m \geq 2$ by Proposition 1.4. Since $\mathrm{forb}(m, 3, 2 \cdot I_2) \leq \mathrm{forb}(m, 3, F(1,2,2,1)) \leq \mathrm{forb}(m, 3, 2 \cdot K_2)$, we have $\mathrm{forb}(m, 3, F(1,2,2,1)) = \mathrm{forb}(m, 3, 2 \cdot K_2) = \mathrm{forb}(m, 3, 2 \cdot K_2) - N_0$. If $p = 3$, then $N_r = N_1 = 1$. By Corollary 2.9, noting that we do not need to apply the Lemma 1.3, for $m \geq 4$ we have $\mathrm{forb}(m, 3, F(2,3,3,2)) = \mathrm{forb}(m, 3, 3 \cdot K_2) - 1 = \mathrm{forb}(m, 3, 3 \cdot K_2) - N_1$. If $p = 4$, then $N_r = N_2 = 2$. By Proposition 2.10, again without the need for Lemma 1.3, for $m \geq 6$ we have $\mathrm{forb}(m, 3, F(3,4,4,3)) = \mathrm{forb}(m, 3, 4 \cdot K_2) - 2 = \mathrm{forb}(m, 3, 4 \cdot K_2) - N_2$. Therefore, the statement holds for $p = 2, 3, 4$.

For $p \geq 5$, we have $r \geq 2$. It follows from Corollary 2.16 that $\mathrm{forb}(m, 3, F(p-1, p, p, p-1)) \geq \mathrm{forb}(m, 3, p \cdot K_2) - N_r$, so we just need to prove the reverse inequality. If all scarce pairs are fixed or weakly fixed, then as $\frac{1}{2}(\left\lfloor \frac{r}{2} \right\rfloor + 1)(p - 1 - 2^{r-1}) \geq 1$, the proof of Theorem 1.6 in the case when all scarce pairs are fixed or weakly fixed shows the maximum number of columns is $\mathrm{forb}(m, 3, p \cdot K_2) - N_r$.

Therefore, we just need to consider the case when at least one scarce pair is neither fixed nor weakly fixed. Assume that N non-edges are used and $\mathcal{P}$ is the set of scarce pairs $(i, m+1-j)$ that are not fixed. Then, as in the proof of Lemma 2.19, the maximum possible number of columns is

$$2^m + m2^{m-1} + (p-1)\binom{m}{2} - N - \frac{1}{2}\sum_{(i,m+1-j)\in\mathcal{P}}(p-1-n_{i,j}).$$

This is at most $\mathrm{forb}(m, 3, p \cdot K_2) - N_r$ if and only if

$$N + \frac{1}{2}\sum_{(i,m+1-j)\in\mathcal{P}}(p-1-n_{i,j}) \geq N_r. \tag{1}$$

Let w_f denote the number of weakly fixed pairs and n_f denote the number of neither fixed nor weakly fixed ones, so that $w_f + n_f = |\mathcal{P}|$. If $(i, m+1-j)$ is neither fixed nor weakly fixed, then $p-1-n_{i,j} \geq p-1-2^{r-2} \geq 2^{r-2}+1$, so

$$N + \frac{1}{2}\sum_{(i,j)\in\mathcal{P}}(p-1-n_{i,j}) \geq N + \frac{1}{2}w_f + (2^{r-3}+\frac{1}{2})n_f. \tag{2}$$

We will prove that for all r,

$$N + \frac{1}{2}w_f + (2^{r-3}+\frac{1}{2})n_f \geq N_r, \tag{3}$$

which then implies (1) and finishes the proof.

If $r \geq 7$, then $2^{r-3} \geq N_r$, and as $n_f \geq 1$, (3) holds. The values of N_r for $2 \leq r \leq 6$ are as follows.

r	2	3	4	5	6
N_r	2	4	6	9	12

If $r = 2$, then we are done if $N \geq 1$ as $n_f \geq 1$. If $N = 0$, then $w_f = 2$ and (3) holds as well.

For $r = 3$, we need $N + \frac{1}{2}w_f + \frac{3}{2}n_f \geq 4$. If $N \geq 3$, then the inequality holds as $n_f \geq 1$. If $N = 2$, (3) holds as at least one of the three diagonal entries is weakly fixed, while $N \leq 1$ implies $n_f \geq 2$ and $w_f \geq 2$, so (3) still holds.

For $r = 4$, we need $N + \frac{1}{2}w_f + \frac{5}{2}n_f \geq 6$, which holds if $N \geq 4$ as $n_f \geq 1$. If $N = 3$, it holds as $w_f \geq 1$. If $N = 1, 2$, then at least two of the three entries in the upper left corner of T_4 $\begin{matrix}4 & 3\\ 3\end{matrix}$ are not at least weakly fixed, so $n_f \geq 2$ and (3) holds. If $N = 0$, $n_f = 6$ and (3) holds.

For $r = 5$, we need $N + \frac{1}{2}w_f + \frac{9}{2}n_f \geq 9$, which holds if $N \geq 5$. If $N = 4$, then (3) holds as $w_f \geq 1$. If $N \leq 3$, then $n_f \geq 2$ and (3) holds, as at least two of the three upper left corner entries of T_5 $\begin{matrix}5 & 4\\ 4\end{matrix}$ are not at least weakly fixed.

Finally for $r = 6$, we need $N + \frac{1}{2}w_f + \frac{17}{2}n_f \geq 12$, which holds if $N \geq 4$. If $N \leq 3$, then $n_f \geq 3$ and (3) holds as all three entries of the upper left corner of T_6 $\begin{matrix}6 & 5\\ 5\end{matrix}$ are not at least weakly fixed. This proves (3) for all $r \geq 2$ and finishes the proof. □

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References

N. Alon. On the density of sets of vectors. *Discrete Mathematics*, 46(2):199–202, 1983.

R. Anstee and L. Lu. Unavoidable multicoloured families of configurations, 2014. arXiv:1409.4123.

R. Anstee and A. Sali. A survey of forbidden configuration results. *The Electronic Journal of Combinatorics*, page DS20, 2025.

R. P. Anstee, J. Dawson, L. Lu, and A. Sali. Multivalued matrices and forbidden configurations. *Discrete Mathematics*, 346(9):113492, 2023.

T. Dillon and A. Sali. Exponential multivalued forbidden configurations. *Discrete Mathematics & Theoretical Computer Science*, 23(Combinatorics), 2021.

K. Ellis, B. Liu, and A. Sali. Multi-symbol forbidden configurations. *Discrete Applied Mathematics*, 276:24 – 36, 2020. doi: 10.1016/j.dam.2019.11.010.

Z. Füredi and A. Sali. Optimal multivalued shattering. *SIAM Journal on Discrete Mathematics*, 26:737–744, 2012.

I. Heller and C. B. Tompkins. *14. An extension of a theorem of Dantzig's*, pages 247–254. Princeton: Princeton University Press, 1957. ISBN 9781400881987. doi: 10.1515/9781400881987-015.

N. Sauer. On the density of families of sets. *Journal of Combinatorial Theory, Series A*, 13:145–147, 1972.

S. Shelah. A combinatorial problem: Stability and order for models and theories in infinitary language. *Pacific Journal of Mathematics*, 41:247–261, 1972.

J. M. Steele. Existence of submatrices with all possible columns. *Journal of Combinatorial Theory, Series A*, 24(1):84–88, 1978.

V. Vapnik and A. Chervonenkis. On the uniform convergence of relative frequencies of events to their probabilities. *Theory of Probability and Its Applications*, 16:264–280, 1971.