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Face-hitting dominating sets in plane graphs: Alternative proof and linear-time algorithm

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In a recent paper, Francis, Illickan, Jose and Rajendraprasad showed that every plane graph G has (under some natural restrictions) a vertex-partition into two sets V_1 and V_2 such that each V_i is *dominating* (every vertex of G contains a vertex of V_i in its closed neighbourhood) and *face-hitting* (every face of G is incident to a vertex of V_i). Their proof works by considering a plane supergraph G' of G that has certain properties, and among all such graphs, taking one that has the fewest edges. As such, their proof is not obviously algorithmic. Their proof also relies on the 4-colour theorem, for which the fastest known algorithm is quadratic (and not simple).

In this paper, we give a new proof that every plane graph G has (under the same restrictions) a vertex-partition into two face-hitting dominating sets. Our proof is constructive, and requires nothing more complicated than splitting a graph into 2-connected components, finding an ear decomposition, and computing a perfect matching in a 3-regular 2-connected plane graph. For all these problems, linear-time algorithms are known and so we can find the vertex-partition in linear time.

A crucial ingredient in our proof is to find (for a 2-connected plane graph G) a plane supergraph G^+ that is a triangulation, possibly with parallel edges. Furthermore, for every vertex v (with one exception) the cyclic order of edges around v in G^+ contains two consecutive edges that are in G. We study this problem further, and show that except for the case where G is an odd cycle, we can actually find a triangulation where *every* vertex has such a pair of edges.

Keywords: plane graph, dominating set, face-hitting, ear decomposition, triangulating

1 Introduction

One of the oldest problems in graph theory is how to find a small *dominating set* in a graph G, i.e., a set S of vertices such that for every vertex v, either v is in S, or some neighbour of v is in S. The goal is to minimize $|S|$, and the corresponding decision problem is NP-complete and remains NP-complete even in planar graphs under restrictions such as being 4-regular or having maximum degree 3 (Garey and Johnson, 1979). However, in a planar graph one can efficiently approximate a minimum dominating set arbitrarily closely, using Baker's approximation scheme (Baker, 1994).

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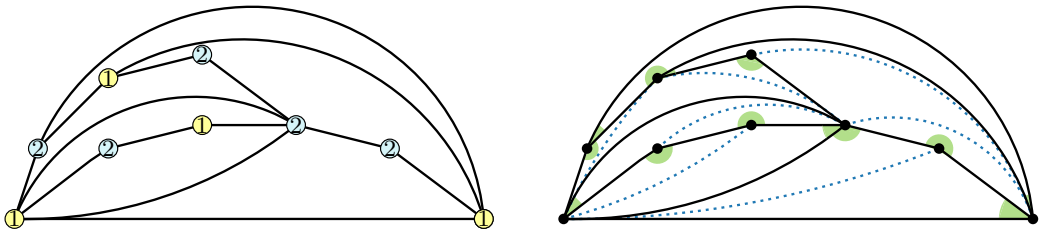
Related to finding a minimum dominating set is the question how big it may be required to be. Since a graph consisting of n isolated vertices clearly requires size n for any dominating set, one typically imposes some restriction on the class of graphs studied. One interesting result here was given by Matheson and Tarjan (1996), who showed that every n-vertex simple triangulated planar graph G has a dominating set of size at most $n/3$. (See Section 2 for definitions.) In fact, they show the stronger result that the vertices of G can be partitioned into three disjoint sets that each are dominating. By combining disjoint copies of K_4, one can easily construct planar triangulated graphs that require at least $n/4$ vertices in any dominating set, and the quest to bring the upper bound closer to $n/4$ has given rise to a number of interesting research insights. See the paper by Christiansen, Rotenberg, and Rutschmann (2024) for the currently best upper bound of $\frac{2}{7}n$ (if $n > 10$), and the references therein for more results for other related classes of planar graphs.

In a recent paper, Francis, Illickan, Jose, and Rajendraprasad (2024) studied an interesting variant of this problem, where they are looking for a vertex set that is not only a dominating set, but that also is *face-hitting*, i.e., they presume that the planar graph G comes with a fixed crossing-free drawing (it is *plane*), and they want a dominating set S where additionally every face of the drawing is incident to at least one vertex of S. It is known from an earlier construction by Bose, Shermer, Toussaint, and Zhu (1997) that there are simple triangulated planar graphs where every face-hitting vertex set has size at least $\lfloor n/2 \rfloor$. (In a triangulated planar graph, such a set is automatically also dominating.) Francis et al. match this bound, and in particular, show that any plane graph (under some natural restrictions, such as "no isolated vertices") has a vertex-partition into two face-hitting dominating sets; the smaller of these sets has size at most $\lfloor n/2 \rfloor$. (See also Figure 1(a).) This in turn can be used to improve the bound of Christiansen et al. (2024) in some triangulated planar graphs: If G has an independent set I of size exceeding $3n/7$, then $G \setminus I$ has fewer than $4n/7$ vertices, and by taking a small face-hitting dominating set of $G \setminus I$ one can obtain a dominating set of G of size less than $2n/7$.

From a mathematical point of view the result of Francis et al. is very nice since it matches the lower bound. But from an algorithmic point of view it is somewhat disappointing, since the authors do not discuss how to find the vertex-partition efficiently, and it is not obvious how to do so. In particular, a crucial ingredient of their proof is to use a plane supergraph G' of G that has the maximum number of so-called "happy vertices" and "happy faces", and among all such graphs, take the one with the fewest added edges. Clearly G' always exists, but it is not clear how to find it. From later properties that the authors show about G', it seems likely that one could simply start with $G' := G$, search for a violation of those properties, and then update G' according to the steps in their proofs. It is not clear how fast such a procedure would be, but it would likely take at least quadratic time.

A second bottleneck is that Francis et al. use the 4-colour theorem to colour the graph G' (and extract the vertex-partition by combining two colour-classes). While there is an algorithm to find the 4-colouring of a planar graph by Robertson, Sanders, Seymour, and Thomas (1997), the run-time is quadratic (with a very large constant factor).

In this paper, we therefore re-prove the result by Francis et al. in a different way that immediately leads to a linear-time algorithm to find the vertex-partition. Rather than relying on the 4-colour theorem, we find a bipartite subgraph H of G whose 2-colouring then gives the vertex-partition. We ensure that for every vertex and every face of G at least one incident edge is retained in H, so either colour-class is a face-hitting dominating set. We sketch here an outline of how to find H. This is easy if G is triangulated: remove the edges corresponding to a perfect matching in the dual of G (this always exists by Petersen's theorem). If G is not triangulated but 2-connected, then we first create a triangulated plane supergraph G^+



(a) (b)

Figure 1: (a) A plane graph G, not simple, with a partition into two face-hitting dominating sets. (b) A supergraph G^+ (added edges are blue dotted). Graph G^+ is a triangulation where every vertex is happy due to the green shaded angles.

of G where every vertex v (with one exception) is *happy*: in the cyclic order of edges around v in G^+ there are two consecutive edges that are in G. (See also Figure 1(b).) Graph G^+ is obtained by computing an ear decomposition of G, and triangulating each face suitably when its corresponding ear is added. Since vertices of G^+ are happy, removing the edges of a perfect matching in the dual of G^+ then again gives H. Finally, for a graph G that is not 2-connected, we simply combine the subgraphs of the 2-connected components. All steps can easily be implemented in linear time, leading us to our main result:

Theorem 1. *There exists a linear-time algorithm for the following task: Given a plane graph $G = (V, E)$ without isolated vertices or faces incident to at most two vertices, partition V into two vertex sets V_1, V_2 that both are dominating as well as face-hitting.*

The main new ingredient in our paper is an algorithm that finds, for a given 2-connected plane graph G, a triangulated supergraph G^+ with many happy vertices. This problem is related to other papers that dealt with triangulating a planar graph under some constraint, see e.g. Kant and Bodlaender (1997) for triangulating planar graphs while keeping the maximum degree small, and Biedl et al. (1997) for triangulating planar graphs while making them 4-connected. There are also many papers that consider triangulating geometric objects (such as a set of points or a polygon) under some constraints, see e.g. Bern et al. (1993) and the references therein. Finally, making vertices happy can be seen as the problem of assigning angles to vertices under some constraints, and as such our problem is also related to papers that studied angle-assignments (under different constraints), see e.g. Evans et al. (2021). But to our knowledge the question of triangulating a planar graph while making vertices happy has not been studied previously except by Francis et al. (2024). We feel that it deserves more attention, and therefore study it beyond what is needed as ingredient for the proof of Theorem 1. Specifically, we show the following.

Theorem 2. *Let G be a 2-connected plane graph with at least three vertices. If G is not an odd-length cycle, then there exists a (possibly non-simple) triangulated plane supergraph G^+ without loops such that all vertices are happy. It can be found in linear time.*

We will also show that the restrictions "2-connected", "not an odd-length cycle" and "possibly non-simple" are required in the sense that for some graphs, not all vertices can be made happy otherwise.

2 Preliminaries

This section clarifies some notation for a graph $G = (V, E)$; for most standard notation, see for example Diestel's book (Diestel, 2012). Throughout, n is the number of vertices, and we assume that $n \geq 2$. A *loop* is an edge where the two endpoints coincide, while two *parallel edges* are two edges with the same set of endpoints. A graph is *simple* if it has no loops and no parallel edges. We do not assume that G is simple (except where explicitly stated below). A *k-walk* is a closed walk with k edges; it is called a *k-cycle* if no vertex repeats.

Recall that a vertex set S is *dominating* if for every $v \in V$ either $v \in S$ or some neighbour of v is in S. An *edge cover* is a set S of edges such that for every $v \in V$ at least one incident edge is in S. A *perfect matching* is a set M of edges where every vertex is incident to exactly one edge in M.

Throughout the paper, we will assume that G is connected, since otherwise we can obtain a partition into face-hitting dominating sets in each component and combine them to get one for G. Graph G is called *2-connected* if it does not contain a *cut-vertex*, i.e., a vertex v such that $G \setminus v$ has more connected components than G. A *2-connected component* C is a maximal subgraph of G that is 2-connected. For the borderline case of a vertex v with incident loop(s) in a graph with at least two vertices, we consider v to be a cut-vertex and each subgraph formed by taking v and one of the loops to be a *trivial* 2-connected component.

A *planar graph* is a graph G that can be drawn without crossing in the plane, or equivalently, on the sphere $\mathbb{S}^2$. Consider a crossing-free drawing Γ of G on the sphere. The maximal regions of $\mathbb{S}^2 \setminus \Gamma$ are called the *faces*. Since G is connected, drawing Γ can be described by specifying the *rotation scheme*, i.e., the clockwise cyclic order of edges at each vertex. In this paper, the input graph G will always be planar, and in fact it will be *plane*, i.e., it comes with a fixed rotation scheme. This rotation scheme determines the faces implicitly: Every face can be described via the walk that defines its *facial boundary*, and this walk can be read from the rotation scheme. An *angle $\angle uvw$ incident to vertex v* consists of two edges (u, v) and (v, w) that are consecutive in the cyclic order of edges at v. If $\deg(v) = 1$ then v has exactly one incident angle (using the same edge twice); if $\deg(v) \geq 2$ then at all incident angles the two edges are distinct. For every angle the two edges are consecutive on the facial boundary of some face F; we hence also call it an *angle incident to F*.

The *degree* of a face F is the number of angles incident to it, while the *size* of F is the number of vertices incident to F. For a 2-connected plane graph the two concepts coincide since the boundary of F is a cycle. A *bigon* is a face whose boundary consists of two parallel edges and so has degree 2. A *triangulated graph* is a plane graph where every face has degree 3, i.e., it is triangular. Note that this definition forbids bigons and faces bounded by loops, but in this paper we specifically *allow* a triangulated graph to have parallel edges or loops as long as they do not bound a face. (We use the term "simple triangulated graph" if no loops or parallel edges are allowed.) A *3^+-face* is a face with size 3 or more; this obviously excludes faces of degree 1 or 2, but it also excludes some faces of higher degrees, for example a triangular face that is bounded by a loop and two parallel edges.

Let S be either a vertex-set or an edge-set. A face F is *hit* by S if it is incident to at least one element of S. Set S is called *face-hitting* if it hits every face, and *3^+-face hitting* if it hits every 3^+-face.

In this paper, we are usually given a plane graph G, and we will work with some subgraph or planar supergraph of G obtained by deleting/adding edges. Without further mentioning we assume that these graphs use the rotation schemes obtained as explained below. For a subgraph H of G, we use the *inherited* rotation scheme, i.e., we keep the same cyclic order of edges at all vertices except for deleting edges that

are not in H. Any planar supergraph G^+ that we construct is a *plane supergraph*, i.e., it *respects the embedding of* G. This means that for any newly added edge $e = (v, w)$ there is a face F incident to both v and w, and we add the edge *inside face* F, which means that edge e gets inserted into the cyclic orders at v and w between the two edges of an angle that is incident to F. (We will do this only if G is 2-connected and the angles are hence unique.)

The *dual graph* of G is the graph $G^* = (\mathcal{F}, E^*)$ that has a vertex v_F for every face F of G, and a *dual edge* $e^* = (v_F, v_{F'})$ whenever faces F and F' share an edge e on their boundary; it includes a loop at vertex v_F whenever face F is incident twice to an edge e. The degree of vertex v_F in G^* equals the degree of face F in G. The dual graph G^* has a natural rotation scheme: For every vertex v_F of the dual graph, list the incident edges in the order in which their dual edges appeared around the facial boundary of face F.

It will sometimes be convenient to *choose an outer face of* G as follows. We have a spherical drawing Γ of G (this can be read from the rotation scheme). For any face F of our choice, we can pick a point p strictly inside F and project the drawing Γ into the plane using p as projection point. With this, every cycle of G then has an interior and exterior side. Also face F gets projected to an unbounded area of the plane, and we hence call it the *outer face* while all other faces are *interior*. An *interior vertex* is a vertex not incident to the outer face.

3 Bipartite Edge Covers

To prove Theorem 1, it suffices to prove the following result.

Theorem 3. *Let G be a connected plane graph with at least two vertices. Then G has a bipartite subgraph H whose edges form a 3^+-face-hitting edge cover of G. Furthermore, we can pre-specify a* reference-edge *$\hat{e}$ of G and require that H includes $\hat{e}$. Subgraph H can be found in linear time.*

To see why this implies Theorem 1, consider a 2-colouring of H, say into vertex-sets V_1 and V_2, and let $i \in \{1, 2\}$. Since $E(H)$ is an edge cover, for every vertex v there exists an edge (v, w) in H. Since H is 2-coloured, either $v \in V_i$ or $w \in V_i$, and so v is dominated by V_i. Since $E(H)$ is 3^+-face-hitting, for every 3^+-face F of G there exists an edge (x, y) incident to F in H; either $x \in V_i$ or $y \in V_i$, and so V_i is 3^+-face-hitting. Since by assumption of Theorem 1 all faces of G are 3^+-faces, the result holds. (We note here that Theorem 1, which we intentionally kept very similar to the statement in Francis et al. (2024), could be strengthened ever so slightly to permit faces of size 1 or 2, as long as they need not be hit.)

3.1 Triangulated graphs

We first prove Theorem 3 for a triangulated planar graph without loops, where this is very easy. For later use as a subroutine, we actually prove a stronger claim on the edges $E(H)$ of H: at every angle at least one edge is retained. This immediately implies that $E(H)$ is a face-hitting edge cover of G since by connectivity every vertex and every face is incident to an angle.

Lemma 4. *Let G be a triangulated planar graph with at least three vertices and no loops. Fix a reference-edge $\hat{e}$. Then G has a bipartite subgraph H that contains $\hat{e}$, and where for every angle $\angle uvw$ of G, at least one of the edges (u, v) and (v, w) is in H. Subgraph H can be found in linear time.*

Proof: The technique to find H is well known, see e.g. (Bose et al., 1996; Biedl et al., 2001), but we review it here to prove the additional claims and verify that it works even if G has parallel edges. Since

G is triangulated, its dual graph G^* is 3-regular. Graph G^* is also 2-connected, for otherwise it (as a 3-regular graph) would have an edge e^* whose removal disconnects G^*, and this edge is dual to a loop of G. By Petersen's theorem (Petersen, 1891), G^* therefore has a perfect matching M^*. In fact (see e.g. the proof of Petersen's theorem by Biedl et al. (2001)) we can require one edge of our choice to be in the perfect matching M^*; we choose it here to be an edge that shares exactly one endpoint with $\hat{e}^*$ so that $\hat{e}^*$ is *not* in M^*. This matching can be computed in linear time (Biedl et al., 2001).

Define M to be those edges of G whose dual edges are in M^*, and let $H := (V, E \setminus M)$, see also Figure 2. Since M^* is a perfect matching, all faces of H have degree 4, which makes H a bipartite subgraph of G. Since the dual of $\hat{e}$ is not in M^*, reference-edge $\hat{e}$ is in H. Consider any angle $\angle uvw$ of G, say incident to face F, and observe that its two edges (u, v) and (v, w) are distinct since G has no vertices of degree 1. Since M^* contains exactly one edge incident to v_F, at most one of (u, v) and (v, w) can be in M, and the other is retained in H. □

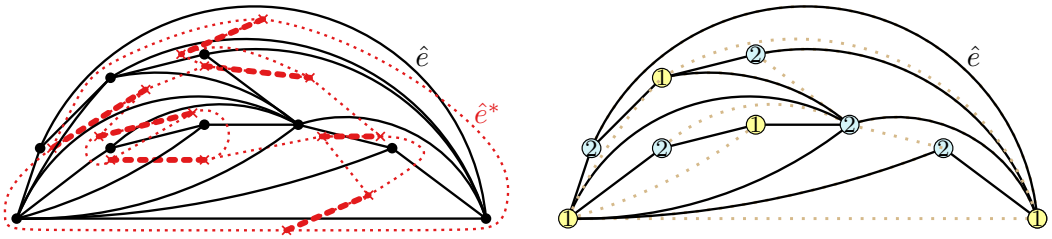


Figure 2: (a) The triangulation from Figure 1(b), with its dual graph (red dotted) and a perfect matching in it (thick dashed). (b) The resulting bipartite graph H; its 2-coloring gives the face-hitting dominating sets from Figure 1(a).

3.2 2-Connected Plane Graphs

Next we prove Theorem 3 for a 2-connected plane graph G in the special case where G has no bigons. The plan is to find a *plane triangulation* of G (i.e., a supergraph G^+ that is triangulated and respects the embedding of G), and apply Lemma 4 to G^+. For this to work out, we need that some angles of G are *preserved* in G^+, i.e., no edge of $G^+ \setminus G$ has been inserted into the angle, and so the angles are also angles of G^+. This concept can be applied to any plane supergraph G^+; when G^+ is a triangulation then "$\angle uvw$ has been preserved" is equivalent to what Francis et al. (2024) called being *happy*: G^+ contains a triangular face that includes the edges (u, v) and (v, w) of the angle. Call a vertex *happy* if it has at least one incident happy angle. The following result is crucial for proving Theorem 3, but we defer its proof to Section 4 where we will actually prove a stronger result.

Proposition 5. *Let G be a 2-connected plane graph with at least three vertices and no bigons. Fix a vertex s. Then G has a triangulated plane supergraph G^+ without loops in which all vertices except perhaps s are happy. Supergraph G^+ can be found in linear time.*

We also need an easy observation about angles incident to faces, which implies that there is no need to explicitly ensure that faces have incident happy angles (the approach taken by Francis et al. (2024)).

Observation 1. *Let G be a 2-connected plane graph with at least three vertices and no bigons, and let G^+ be any plane supergraph of G. Then every face F of G has at least two incident angles that are preserved in G^+.*

Proof: The boundary B of F is a cycle with at least three vertices since G is 2-connected without bigons. Consider the subgraph G_F of G^+ formed by B and all edges that were inserted inside F. This is a 2-connected graph with at least three vertices that can be drawn with all vertices on one face (it is *outerplanar*). It is well-known that such a graph has at least two vertices of degree exactly 2. Let w be one of these vertices. It lies on B, hence its two incident edges in G_F belong to B, and form an angle in G^+ since G_F includes all edges that we added inside F. So the angle incident to F at w is preserved. □

Corollary 6. *Theorem 3 holds for any 2-connected plane graph G with $n \geq 2$ vertices and no bigons.*

Proof: Clearly the theorem holds if $n = 2$ since G itself is then bipartite, so we may assume that $n \geq 3$. Let s be an endpoint of the reference-edge $\hat{e}$. Using Proposition 5, find a triangulated plane supergraph G^+ of G where all vertices except perhaps s are happy. Apply Lemma 4 to G^+ with respect to edge $\hat{e}$. This gives us, in linear time, a bipartite subgraph H^+ of G^+ that contains $\hat{e}$; in particular vertex s has $\hat{e}$ as an incident edge in $E(H^+) \cap E(G)$. For every vertex $v \neq s$ there exists an incident angle in G^+ where both edges belong to G. At least one of these edges is retained in H^+, so v also has at least one incident edge in $E(H^+) \cap E(G)$. Define H to be the bipartite subgraph of H^+ with edge set $E(H^+) \cap E(G)$; by the above $E(H)$ is an edge-cover. By Observation 1 edge set $E(H)$ is also face-hitting: For any face F of G, some incident angle was preserved in G^+, at least one edge of this angle is retained in H^+, and hence F has an incident edge in $E(H)$. □

As it turns out, bigons do not pose a problem either.

Corollary 7. *Theorem 3 holds for any 2-connected plane graph G with at least two vertices.*

Proof: Let G' be the subgraph of G obtained by deleting, for every bigon of G, one of the parallel edges that bound the bigon (choosing one that is not reference-edge $\hat{e}$, if $\hat{e}$ is in a bigon). Graph G' is 2-connected (adding parallel edges cannot increase vertex-connectivity) and has no bigons, and so by Corollary 6 we can find a bipartite subgraph H' such that $E(H')$ is a face-hitting edge cover of G'. Let H be the bipartite subgraph of G obtained by adding to H' all bigon-edges where one edge of the bigon was in H'. Clearly H is bipartite and all steps to find it can be implemented in linear time.

We claim that $E(H)$ is 3^+-face-hitting (and of course it is an edge cover since already $E(H')$ was one). Let F be any 3^+-face of G. If F was also a face of G', then it was hit by $E(H')$ and so also by $E(H)$. If F was not a face of G', then it was replaced in G' by a face F' that is, roughly speaking, the union of F and a number of bigons. Formally, the facial boundary of F' is the symmetric difference of the facial boundary of F with all those bigons where the deleted edge was incident to F. Some edge e' of F' was in $E(H')$, hence e' is either on F or it belongs to a bigon $\{e', e''\}$ with e'' incident to F. In the latter case we added e'' to H. Therefore either way some edge of F is in $E(H)$. □

3.3 Putting It All Together

We now put all ingredients together to prove Theorem 3 as follows. We already know the result for 2-connected graphs (Corollary 7). The way to generalize it to arbitrary connected graphs is very standard:

Compute the bipartite subgraph for each 2-connected component of G, and combine them to obtain a bipartite subgraph of G. However, if G is not simple then we have to be a bit more careful in choosing the reference-edge so that a 3^+-face is hit even if its boundary contains loops.

The details are as follows. For ease of description choose the outer face of G such that it contains the reference-edge $\hat{e}$. Compute the 2-connected components of G; this can be done in linear time (Hopcroft and Tarjan, 1973). Consider a non-trivial 2-connected component C (i.e., C has at least two vertices). If C does not contain $\hat{e}$, then choose as reference-edge $\hat{e}_C$ of C an arbitrary edge on the outer face of C in the induced planar embedding; otherwise set $\hat{e}_C = \hat{e}$. Apply Corollary 7 to C with $\hat{e}_C$ as reference-edge. This gives a bipartite subgraph H_C of C for which $E(H_C)$ is a 3^+-face hitting edge cover. Let H be the union of all these subgraphs; since 2-connected components intersect in at most one vertex and H has no loops, this is bipartite. Since (in a connected graph with at least two vertices) every vertex belongs to at least one non-trivial 2-connected component C, and is incident to an edge of H_C, the set $E(H)$ is an edge cover.

Now we must show that $E(H)$ hits any 3^+-face F of G. (Figure 3 illustrates some possibilities for F.) If F is a face of one 2-connected component C, then $E(H_C)$ (and hence $E(H)$) hits F. If F is not a face of one 2-connected component, then let B be the boundary walk of F and let $C_1, \ldots, C_\ell$ for $\ell \geq 2$ be the 2-connected components intersected by B. Note that $B \cap E(C_i)$ is a face of C_i (in the induced planar embedding of C_i) for all i. If this is a 3^+-face of C_i then $E(H_{C_i})$ will include an edge of it and so $E(H)$ hits F. If $B \cap E(C_i)$ is the outer face of C_i, and C_i is not a loop, then reference edge $\hat{e}_{C_i}$ was chosen on $B \cap E(C_i)$ and included in $E(H_{C_i})$, and so $E(H)$ hits F.

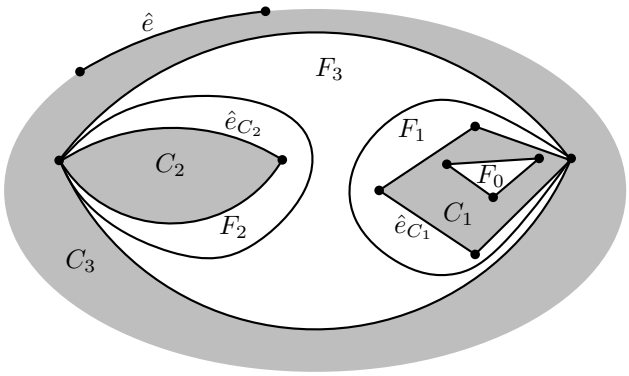


Figure 3: A graph that is not 2-connected; non-trivial 2-connected components are shaded. Face F_0 is a face of C_1, hence is hit by $E(H_{C_1})$. The boundary of F_1 meets C_1 in a 3^+-face, hence F_1 is hit by $E(H_{C_1})$. The boundary of face F_2 meets C_2 in the outer face of C_2, hence F_2 is hit by $\hat{e}_{C_2}$. Face F_3 is incident only to loops and a bigon of C_3 and not a 3^+-face.

So we are done unless for $i = 1, \ldots, \ell$, either $B \cap E(C_i)$ is a loop, or $B \cap E(C_i)$ bounds a bigon of C_i that is an inner face of C_i and hence contains face F in its interior. The latter can happen for at most one index i since face F cannot be interior to two bigons that both share edges with it. Therefore the boundary of F consists either only of loops, or of one bigon with some loops attached at its endpoints. Either way F has size at most 2 and is not a 3^+-face.

In summary, the edges of H are a 3^+-face-hitting edge cover, which proves Theorem 3 and hence

Theorem 1 (subject to proving Proposition 5, which will happen in Section 4.3).

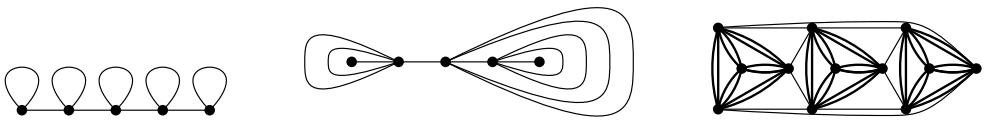
3.4 Hitting faces of smaller sizes

We briefly discuss here how the bounds change if we want to hit not only the 3^+-faces but in fact *all* faces with a dominating set S. (This makes a difference only if the graph is allowed to be not simple.)

If we permit faces of degree 1 (i.e., faces whose boundary is a loop), then we must include the unique vertex of each such face in S to be face-hitting, and so $|S| = n$ may be required since we may have a degree-1 face at every vertex. See Figure 4(a).

In fact, $|S| \approx n$ may be required even in a graph that does not have faces of degree 1 but may have loops. A face that is bounded by two parallel loops has degree 2, but still forces its unique vertex to be in any face-hitting vertex set S. We can build a graph that has such faces at $n-2$ vertices, without having a face of degree 1, see Figure 4(b). Hence $|S| \geq n-2$ may be required.

If we forbid loops, but permit bigons, then $|S| \geq \frac{3}{4}n$ may be required: Take $n/4$ copies of K_4, replace every edge by a bigon, and then connect the copies with arbitrary further edges to make the graph connected and planar (we can even achieve higher connectivity, see Figure 4(c)). To hit the bigons at each K_4, set S must include a *vertex cover* of K_4, i.e., a vertex set C such every edge has an endpoint in C. Any vertex cover of K_4 must include 3 out of 4 vertices, and so $|S| \geq \frac{3}{4}n$.



(a) (b) (c)

Figure 4: Graphs where any face-hitting dominating set S must be large. (a) With faces of degree 1, we may need $|S| = n$. (b) With loops, we may need $|S| \geq n-2$ even if all faces have degree 2 or more. (c) With bigons, we may need $|S| \geq \frac{3}{4}n$ even if there are no loops.

On the positive side, for any connected plane graph G, any vertex cover is a face-hitting dominating set since every vertex and face is incident to at least one edge. If G has no loops then it is 4-colourable, and the three smallest colour classes form a vertex cover of size at most $\frac{3}{4}n$. So the lower bound for plane graphs without loops can be matched. Note that due to the use of the 4-colour theorem, this proof only yields a quadratic-time (and difficult to implement) algorithm to find the vertex cover. Reducing this runtime to linear would be equivalent to finding an independent set of size at least $n/4$ in any planar graph in linear time, and no such results seem known (see Biedl and Wilkinson (2005) for related discussion).

4 Making Everyone Happy

To goal of this section is to prove Proposition 5, and more generally to study the problem of finding a triangulated plane supergraph of a given plane graph G such that many (ideally all) vertices are happy. As before, we assume that G is connected and has at least two vertices. We also assume that all faces of G have degree 3 or more, otherwise we cannot hope to make the graph triangulated by adding edges.

For ease of reference, we repeat and clarify here a few definitions. Let G be a plane graph and G^+ be a supergraph (not necessarily a triangulation). We say that an angle $\angle uvw$ of G^+ is *happy* if the incident

face in G^+ is a triangle, and both edges (u, v) and (v, w) are edges of G. We say that adding another edge (u, w) to G^+ *makes* $\angle uvw$ *happy* if it is happy afterwards due to this edge, i.e., (u, w) closed up a triangular face $\{u, v, w\}$ and both (u, v) and (v, w) are edges of G. Note that once an angle $\angle uvw$ has been made happy, it will remain happy even during later edge-additions since $\{u, v, w\}$ will remain a face. A vertex v of G^+ is *happy* there exists a happy angle $\angle uvw$, and v *is made happy* by adding an edge (u, w) if this made an angle $\angle uvw$ happy.

4.1 Some bad examples

We first give some examples of graphs that justify the restrictions/permission made on the input and output of Proposition 5 (this will also be a convenient warm-up). These graphs are illustrated in Figure 5. First, the proposition assumes that the input graph G is 2-connected. This is required: we can otherwise make only a fraction of the vertices happy in some graphs.

Lemma 8. *For n divisible by 3, there exists a connected plane n-vertex graph G where any plane triangulation has at most $\frac{2}{3}n$ happy vertices.*

Proof: Let C be a cycle with $n/3$ vertices, and attach two vertices x, y of degree 1 at every vertex v of the cycle. Fix a plane embedding and outerface such that x is inside C and y is outside. Now consider adding edges to create some plane triangulation of this graph. To make vertex x happy, we must make its unique incident angle happy, which can be done only by adding a loop at v that encloses x. (So in particular, we cannot make x happy at all if adding loops is forbidden.) To make vertex y happy, we must add another loop that encloses y, and the two loops together add edges inside all four angles incident to v. So if x, y are both happy then v cannot be happy, meaning that out of each triple of vertices at least one is never made happy. □

Next, the proposition permits parallel edges in the triangulation. Again this is required: we can otherwise make only a constant number of vertices happy in some graphs.

Lemma 9. *For any $n \geq 4$, there exists a 2-connected plane n-vertex graph G where any simple triangulation has at most four happy vertices.*

Proof: Consider the graph $K_{2,n-2}$, i.e., two vertices a, a' are both adjacent to all of $n-2$ vertices $b_0, \ldots, b_{n-3}$. This has a unique rotation scheme (up to renaming), and its faces are bounded by the 4-cycles $\{a, b_i, a', b_{i+1}\}$ for $i = 0, \ldots, n-3$, addition modulo $n-2$. To triangulate such a face, we can either insert (a, a') or (b_i, b_{i+1}), and a vertex b_i is made happy only if it is incident to a face where we inserted (a, a'). But in a simple triangulation we can only have one copy of (a, a'), so at most a, a' and two of the vertices $b_0, \ldots, b_{n-3}$ can be happy. □

Finally, the proposition permits that one vertex s is not happy. Again this is required.

Lemma 10. *For any $n \geq 5$ odd, any triangulation of the n-cycle has at most $n - 1$ happy vertices.*

Proof: If all vertices were happy, then in one of the two faces we would have at least $\lceil n/2 \rceil$ happy angles, and by n odd and the pigeonhole principle hence have two consecutive ones. This is impossible, since the edge that makes an angle happy necessarily gets inserted into the two adjacent angles of the face. □

Surprisingly, as we will prove below, C_n (for $n \geq 5$ odd) is the *only* 2-connected graph where we have to have an unhappy vertex if the triangulation is allowed to have parallel edges.

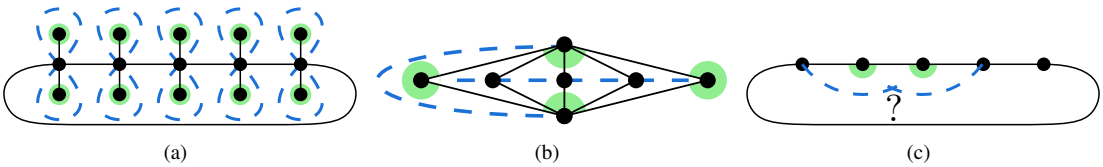


Figure 5: Graphs where triangulations have few happy angles (shaded green). (a) A graph with cutvertices. (b) Demanding a simple triangulation. (c) We cannot make two consecutive angles happy.

4.2 Triangulating a single face

Towards the proof of Proposition 5, we first study how to triangulate a single face (modelled here as triangulating the interior of a cycle). We show a stronger statement: we can prescribe which angles become happy (presuming none of them are consecutive, which is obviously necessary). In fact, we can even prescribe which angles should *not* be happy (as long as at least two are allowed to be happy); this is not a property that will be used later, but we find it interesting in its own right.

Lemma 11. *Let C_n be an n-cycle for $n \geq 3$, and let A be any subset of the vertices such that no two of them are consecutive on the cycle. There exists a simple triangulation of the interior of C_n such that all interior angles at vertices of A are happy. It can be found in linear time.*

If $|A| \geq 2$ then we can furthermore ensure that an interior angle of C_n is happy if and only if it belongs to a vertex of A.

Proof: Nothing is to show if $n = 3$ since then C_n is already a triangulated graph and all angles are happy. So assume that $n \geq 4$ and pad A, if needed, with extra vertices to achieve $|A| \geq 2$. If $n = 4$ then by $|A| \geq 2$ and assumption we alternatingly have vertices in A and not in A along C_n; adding an edge between the two vertices not in A then satisfies all conditions.

So assume that $n \geq 5$ and consult Figure 6 for an illustration of the following steps. We first add, for every vertex $v \in A$, an edge inside C_n between its two neighbours v', v'' on C_n; the naming is such that v' is the ccw neighbour of v. This makes the interior angle incident to v happy and does not add a bigon by $n \geq 5$. It also maintains planarity since vertices of A are not consecutive along C_n, and so none of the added edges cross each other.

Let F be the face of the resulting graph that is incident to all added edges. If we did not care about accidentally creating further happy vertices, then we could simply triangulate F arbitrarily. To make *only* the vertices of A happy, we have to be a bit more careful. Recall that $|A| \geq 2$, and let v, w be two vertices in A. Their neighbours v', v'', w', w'' are incident to F (where possibly $v'' = w'$ or $w'' = v'$). If $w' \neq v''$ then add an edge (v', w') (if $w' = v''$ then this edge exists already). Then add edges from v' to everyone that is strictly between v'' and w' on F in clockwise order, and add edges from w' to everyone that is strictly between w'' and v' on F in clockwise order. This does not make any other interior angle of C_n happy since each of the resulting triangular faces has at most one edge on C_n. □

4.3 Triangulating all faces

We know from Lemma 10 that odd-length cycles will be a special case for Proposition 5, and so we prove the result for them separately.

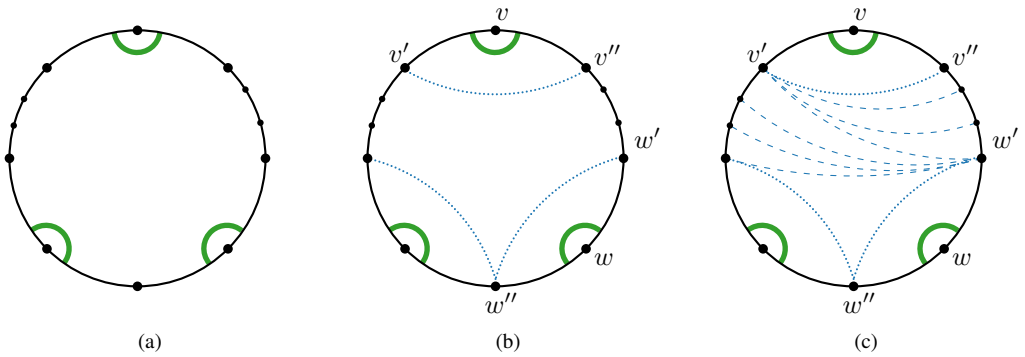


Figure 6: Triangulating the interior such that a prescribed set of angles (indicated by green circles) is happy.

Observation 2. *Proposition 5 holds for any odd-length cycle C_n.*

Proof: Recall that we are given one vertex s that need not be made happy. Enumerate the vertices of C_n as $s, v_1, \ldots, v_{n-1}$. Apply Lemma 11 to triangulate the interior of C_n to make $v_2, v_4, \ldots, v_{n-1}$ happy, and apply it again to triangulate the exterior of C_n to make $v_1, v_3, \ldots, v_{n-2}$ happy. This can clearly be implemented in linear time. □

As promised, for all other 2-connected plane graphs we can make all vertices happy. Put differently, we now prove Theorem 2, which together with Observation 2 implies Proposition 5. We restate the theorem here for ease of reference, and then devote the rest of the subsection to proving it.

Theorem 2. *Let G be a 2-connected plane graph with at least three vertices. If G is not an odd-length cycle, then there exists a (possibly non-simple) triangulated plane supergraph G^+ without loops such that all vertices are happy. It can be found in linear time.*

Crucial to the proof of this theorem is the idea of building up the graph G "piece by piece", adding one face at a time, and triangulating this face while maintaining a suitable invariant. This "building up the graph" uses a well-known tool that we define first.

Definition 12. *An* ear decomposition *of a graph G is a collection $\langle P_1, \ldots, P_f \rangle$ of paths ("*ears*") in G with the following properties:*

- *Every edge of G belongs to exactly one path.*
- *P_1 is a cycle, i.e., its endpoints coincide.*
- *For $i > 1$, P_i is a path with two distinct endpoints. The endpoints of P_i are in $V_{i-1} := V(P_1) \cup \ldots \cup V(P_{i-1})$, while the interior vertices of P_i (if any) are not in V_{i-1}.*

It is well-known that a graph G with at least three vertices is 2-connected if and only if it has an ear decomposition (Whitney, 1931). Given an ear decomposition, we define (for $i \geq 1$) the *partial graph* G_i to be the graph with vertices V_i and all edges of $P_1, \ldots, P_i$; note that G_i is 2-connected since it has an ear decomposition.

It is known that if G is a plane graph, then we can put extra conditions on ear decompositions. We have not been able to find an explicit reference for this result, and hence provide a short proof, mostly based on the insights in Rosenstiehl and Tarjan (1986) and illustrated in Figure 7.

Lemma 13. *Let G be a 2-connected plane graph with at least three vertices. Then for any face F of G, there exists an ear decomposition $\langle P_1, \ldots, P_f \rangle$ of G and a choice of outer face of G such that P_1 bounds face F, and for $i > 1$ path P_i lies on the outer face of the partial graph G_i in the induced embedding of G_i. Such an ear decomposition can be found in linear time.*

Proof: Let (s, t) be an arbitrary edge on F. Since G is 2-connected, it has a vertex ordering $v_1, \ldots, v_n$ such that $v_1 = s$, $v_n = t$, and every other vertex has at least one earlier and at least one later neighbour (Lempel et al., 1967). This can be computed in linear time (Even and Tarjan, 1976). Direct every edge from the lower-indexed to the higher-indexed endpoint to obtain a *bipolar orientation* (Rosenstiehl and Tarjan, 1986), i.e., an acyclic edge-orientation where s is the only source and t is the only sink. Consider the dual graph G^*, and direct it such that every dual edge e^* crosses the corresponding edge e of G left-to-right. If we delete the edge dual to (s, t) from G^*, then the resulting orientation is also a bipolar orientation, with the source and sink the two faces incident to (s, t) (Rosenstiehl and Tarjan, 1986). Up to reversal of edges we may therefore assume that v_F is the source of G^* while the vertex of the outer face is the sink. Compute a topological order of G^* to enumerate the faces of G as $F_1, \ldots, F_{f+1}$, with $F_1 = F$ and F_{f+1} the outer face.

Set P_1 to be the boundary of $F{=}F_1$. To define P_i for $2 \leq i \leq f$, consider the interior face F_i. Since we have a bipolar orientation of G^*, vertex v_{F_i} is neither a source nor a sink. It is known that its outgoing edges are consecutive in the clockwise order of edges, as are its incoming edges (Tamassia and Tollis, 1986). Translated from G^* to G, therefore the boundary of F_i is split into two non-empty parts, one part with edges shared with smaller-indexed faces (hence in earlier paths), and one part with edges shared with larger-indexed faces. Define P_i to be to be this latter part. One easily verifies all required properties. □

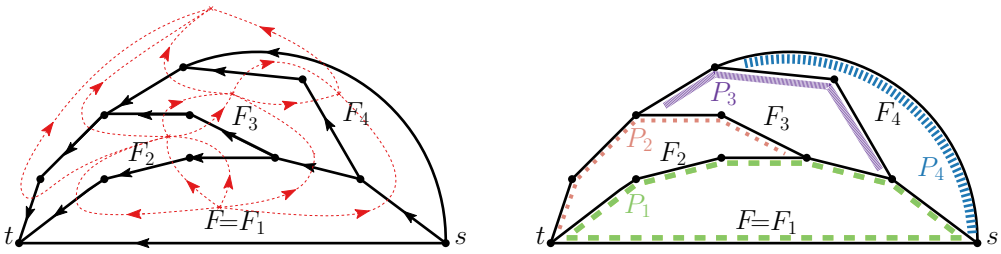


Figure 7: Finding an ear decomposition of a plane 2-connected graph via a topological order in the bipolar orientation of the dual graph.

Continuing the proof of Theorem 2, we compute an ear decomposition $\langle P_1, \ldots, P_f \rangle$ of the given graph G by using Lemma 13, choosing as face F one of even degree if there are such faces, and an arbitrary one otherwise. This also enumerates the interior faces as $F_1, \ldots, F_f$, with P_i on the boundary of F_i for $i = 1, \ldots, f$. As outlined earlier, the plan is to triangulate F_i, for $i = 1, \ldots, f$, such that the resulting supergraph G_i^+ of G_i satisfies the following **invariant**:

(a) Every interior vertex of G_i^+ is happy.

(b) For any edge e on the outer face of G_i^+, at least one endpoint of e is *interior-happy*, i.e., it is incident to a happy angle at an interior face.

Since we build G_i^+ by triangulating $F_1, \ldots, F_i$, and always do so by appealing to Lemma 11, it also follows automatically that the outer face of G_i^+ is the same as the outer face of G_i, every interior face of G_i^+ has degree 3, and no loops are added.

Base case if G has a face of even degree: If G has a face of even degree, then with our choice of ear decomposition face F_1 has even degree. See also Figure 8(a). We can then appeal to Lemma 11 to triangulate F_1 while making every second interior angle happy. Since G_1 has no interior vertices, the invariant then clearly holds for the resulting G_1^+.

Base case if G has only faces of odd degree: In this case we will triangulate faces F_1 and F_2 at the same time such that the invariant holds for G_2^+. Note first that F_2 must exist since otherwise G would be a cycle of odd length. It also must share some edges with F_1 since the endpoints of P_2 are distinct. So G_2 consists of three disjoint paths, call them $\pi_a = \langle a_1, \ldots, a_p\rangle$, $\pi_b = \langle b_1, \ldots, b_q\rangle$, and $\pi_c = \langle c_1, \ldots, c_r\rangle$, with $a_1 = b_1 = c_1$ and $a_p = b_q = c_r$. We assume that the naming is such that F_1 borders π_a and π_b while F_2 borders π_b and π_c, hence the interior vertices of π_b are not on the outer face of G_2. See also Figure 8(b) and (c). We know that $p + q$ is odd, as is $q + r$, since all faces have odd degree. In consequence the outer face of G_2 (which is bounded by the cycle formed by $\pi_a \cup \pi_c$) has even degree. Set A_0 to be every other vertex on the outer face, choosing among the two possibilities such that a_1 is *not* in A_0. For $i \in \{1, 2\}$, write A_0^i for all those vertices in A_0 that are incident to face F_i.

We now have two subcases. If q is even (hence p and r are odd) then vertex b_q is *not* in A_0. Set A_1 to be $\{b_3, b_5, \ldots, b_{q-1}\}$ and A_2 to be $\{b_2, b_4, \ldots, b_{q-2}\}$. Observe that for $i \in \{1, 2\}$ the set $A_0^i \cup A_i$ consists of vertices on F_i, no two of which are consecutive. We can hence appeal to Lemma 11 to make the angles incident to F_i at $(A_0 \cap V(F_i)) \cup A_i$ happy by triangulating F_i. Thus in the resulting graph G_2^+ all interior vertices of π_b are happy, and every other vertex on the outer face is interior-happy, and the invariant holds.

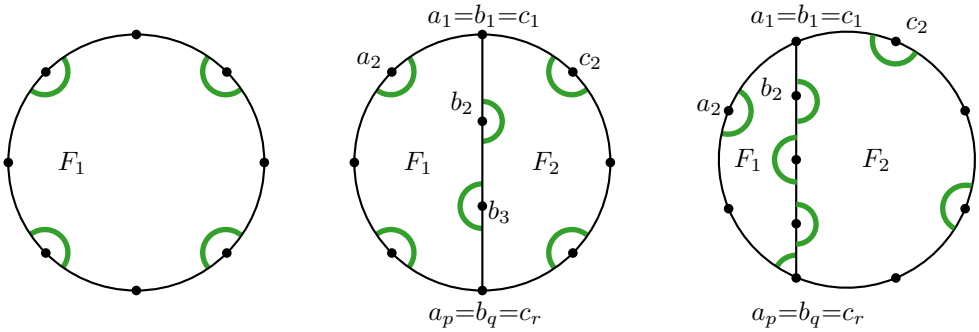
If q is odd, then define A_1 and A_2 similarly, but this time $A_1 := \{b_3, b_5, \ldots, b_q\}$ includes b_q while $A_2 = \{b_2, b_4, \ldots, b_{q-1}\}$. Observe that we had $b_q \in A_0$, and we now, for $i = 1, 2$, appeal to Lemma 11 to triangulate face F_i using set $(A_0^i \setminus b_q) \cup A_i$ after verifying that no two of these vertices are consecutive on F_i. This makes all interior vertices of π_b happy in G_3^+. It also makes all vertices of A_0 happy since $b_q \in A_1$, and the invariant holds.

Step: Assume now that G_{i-1}^+ (for some $i \geq 2$) satisfies the invariant. Let $P_i = \langle y_0, y_1, \ldots, y_k, y_{k+1}\rangle$ for some $k \geq 0$ be the next ear, so $y_0 \neq y_{k+1}$ are vertices of G_{i-1} while all other vertices of P_i (if any) are new. Enumerate the interior face F_i incident to P_i as

$$F_i = \langle y_0, y_1, \ldots, y_k, y_{k+1}{=}z_0, z_1, \ldots, z_\ell, z_{\ell+1} = y_0\rangle,$$

so $\langle z_0, z_1, \ldots, z_\ell, z_{\ell+1}\rangle$ for some $\ell \geq 0$ was a path along the outer face of G_{i-1} between the ends of P_i. The plan is again to appeal to Lemma 11, and to this end, we define a set A as follows (see also Figure 9).

First we want to ensure Invariant (a), i.e., that all interior vertices of G_i^+ are happy. This holds for all such vertices except $z_1, \ldots, z_\ell$ by induction. So set A_1 to be the set of all vertices of $z_1, \ldots, z_\ell$ that were not interior-happy in G_{i-1}^+. Since the invariant held for G_{i-1}^+, no two of them were consecutive on the outer face of G_{i-1}, hence no two of them are consecutive on F_i, not even if $k = 0$ since $y_0, y_{k+1} \notin A_1$.



(a) (b) (c)

Figure 8: Three base cases: (a) F_1 has even degree. (b) All faces have odd degree, and q is even. (c) All faces have odd degree, and q is odd.

Next we want to ensure Invariant (b), i.e., that for any edge on the outer face at least one endpoint is interior-happy. This holds for all edges except the ones of P_i by induction. To ensure it for P_i, we have have two cases.

1. In the first case, one of y_0 and y_{k+1} (say y_{k+1}) is interior-happy already. We set $A_2 := \{y_1, y_3, \ldots, y_{2\lceil k/2 \rceil - 1}\}$, i.e., we take every odd-indexed vertex of P_i until y_{k-1} or y_k, depending on the parity of k. See Figure 9(a-b). Note that every edge in P_i has one endpoint in $A_2 \cup \{y_{k+1}\}$, so creating happy angles at the vertices of A_2 suffices to satisfy (b) since y_{k+1} is already interior-happy. Also none of the vertices in A_2 is consecutive on F_i with a vertex of A_1 since $y_0, y_{k+1} \notin A_1$.

2. In the second case, neither y_0 nor y_{k+1} was already interior-happy. This implies that $\ell \geq 1$, since by Invariant (b) they then must not be consecutive on the outer face of G^+_{i-1}. Again using Invariant (b), this implies that z_1 and z_ℓ *were* interior-happy, and in particular, do not belong to A_1. Set $A_2 := \{y_0, y_2, \ldots, y_{2\lceil k/2 \rceil}\}$, i.e., we take every even-indexed vertex of P_i until y_k or y_{k+1}, depending on the parity of k. See Figure 9(c). Note that none of these vertices (not even y_0 or y_{k+1}) are consecutive in F_i with a vertex of A_1 since $z_1, z_\ell \notin A_1$.

So we have chosen A_1 and A_2 such that no two vertices of $A := A_1 \cup A_2$ are consecutive along F_i, and such that creating happy angles at the vertices in A ensures the invariant. We hence use Lemma 11 with A to triangulate F_i and obtain the desired graph G^+_i

Triangulating the outer face. Using the algorithm inherent in the above inductive proof, we can build G^+_n in linear time since triangulating face F_i takes time proportional to its degree. With this all interior vertices are happy. Let A be the set of vertices on the outer face of G^+_n that are not interior-happy. By Invariant (b) no two of these are consecutive on the outer face, so by appealing to Lemma 11 once more with the outer face and A, we have hence proved Theorem 2 and Proposition 5.

5 Further Thoughts

In this paper, we gave a new proof that every simple planar connected graph with at least two vertices has a vertex partition into two face-hitting dominating sets, with the goal of developing a linear-time algorithm

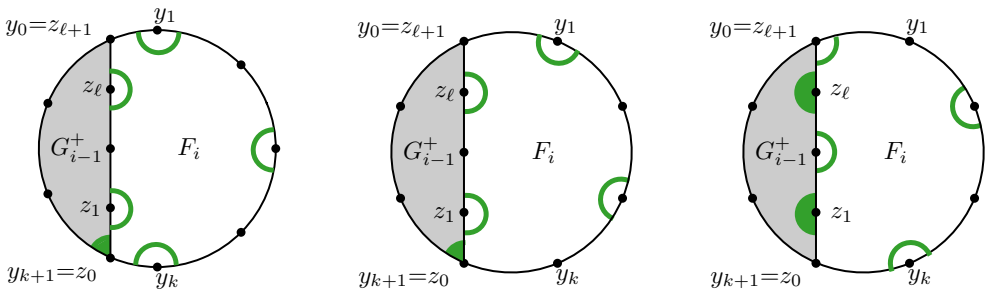


Figure 9: Three examples of the step: (a) y_{k+1} was interior-happy, k odd. (b) y_{k+1} was interior-happy, k even. (c) Neither end of P_i was interior-happy. Vertices that were already happy in G^+_{i-1} are indicated with filled circular arcs.

to find this vertex partition. The proof extends to graphs that are not simple, as long as we are content with hitting only those faces that are incident to at least three distinct vertices. We also studied one crucial ingredient, the problem of triangulating a given plane graph such that (almost) all vertices are happy. We characterized exactly when this can be done for a 2-connected graph if parallel edges are allowed, and gave various examples where it cannot be done if the graph is not 2-connected or the triangulation must be simple.

One natural open problem concerns bounds for face-hitting dominating sets in 4-connected planar graphs. The lower-bound example of Bose et al. (1997) has 3-cycles that are not faces, i.e., it is not 4-connected. Can we improve the upper bound on the size of face-hitting dominating sets for 4-connected planar graphs, at least for triangulations? Another open problem concerns making (almost) all vertices happy in plane graphs that are not necessarily 2-connected. Are vertices of degree 1 the only obstacle, so can we make (almost) all vertices happy in a connected plane graph with minimum degree 2, or at least in any 2-edge-connected plane graph?

References

Brenda S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *J. ACM*, 41(1): 153–180, 1994. doi: 10.1145/174644.174650.

Marshall W. Bern, Herbert Edelsbrunner, David Eppstein, Scott A. Mitchell, and Tiow Seng Tan. Edge insertion for optimal triangulations. *Discret. Comput. Geom.*, 10:47–65, 1993. doi: 10.1007/BF02573962.

Therese Biedl and Dana F. Wilkinson. Bounded-degree independent sets in planar graphs. *Theory Comput. Syst.*, 38(3):253–278, 2005. doi: 10.1007/S00224-005-1139-0.

Therese Biedl, Goos Kant, and Michael Kaufmann. On triangulating planar graphs under the four-connectivity constraint. *Algorithmica*, 19(4):427–446, 1997. doi: 10.1007/PL00009182.

Therese Biedl, Prosenjit Bose, Erik D. Demaine, and Anna Lubiw. Efficient algorithms for Petersen's matching theorem. *J. Algorithms*, 38(1):110–134, 2001. doi: 10.1006/JAGM.2000.1132.

Prosenjit Bose, David G. Kirkpatrick, and Zaiqing Li. Efficient algorithms for guarding or illuminating the surface of a polyhedral terrain. In Frank Fiala, Evangelos Kranakis, and Jörg-Rüdiger Sack, editors, *Canadian Conference on Computational Geometry*, pages 217–222. Carleton University Press, 1996. URL http://www.cccg.ca/proceedings/1996/cccg1996_0037.pdf.

Prosenjit Bose, Thomas C. Shermer, Godfried T. Toussaint, and Binhai Zhu. Guarding polyhedral terrains. *Comput. Geom.*, 7:173–185, 1997. doi: 10.1016/0925-7721(95)00034-8.

Aleksander B. G. Christiansen, Eva Rotenberg, and Daniel Rutschmann. Triangulations admit dominating sets of size 2*n*/7. In David P. Woodruff, editor, *ACM-SIAM Symposium on Discrete Algorithms, SODA 2024*, pages 1194–1240. SIAM, 2024. doi: 10.1137/1.9781611977912.47.

Reinhard Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012. ISBN 978-3-642-14278-9.

William S. Evans, Ellen Gethner, Jack Spalding-Jamieson, and Alexander Wolff. Angle covers: Algorithms and complexity. *J. Graph Algorithms Appl.*, 25(2):643–661, 2021. doi: 10.7155/JGAA.00576.

Shimon Even and Robert Endre Tarjan. Computing an *st*-numbering. *Theor. Comput. Sci.*, 2(3):339–344, 1976. doi: 10.1016/0304-3975(76)90086-4.

P. Francis, Abraham M. Illickan, Lijo M. Jose, and Deepak Rajendraprasad. Face-hitting dominating sets in planar graphs. In Daniel Král and Martin Milanic, editors, *International Workshop on Graph-Theoretic Concepts in Computer Science, WG 2024*, Lecture Notes in Computer Science, pages 211–219. Springer, 2024. doi: 10.1007/978-3-031-75409-8_15.

Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979. ISBN 0-7167-1044-7.

John E. Hopcroft and Robert Endre Tarjan. Efficient algorithms for graph manipulation [H] (algorithm 447). *Commun. ACM*, 16(6):372–378, 1973. doi: 10.1145/362248.362272.

Goos Kant and Hans L. Bodlaender. Triangulating planar graphs while minimizing the maximum degree. *Inf. Comput.*, 135(1):1–14, 1997. doi: 10.1006/INCO.1997.2635.

Abraham Lempel, Shimon Even, and Israel Cederbaum. An algorithm for planarity testing of graphs. In Pierre Rosenstiehl, editor, *Theory of Graphs, International Symposium Rome 1966*, pages 215–232. Gordon and Breach, 1967.

Lesley R. Matheson and Robert Endre Tarjan. Dominating sets in planar graphs. *Eur. J. Comb.*, 17(6): 565–568, 1996. doi: 10.1006/EUJC.1996.0048.

Julius Petersen. Die Theorie der regulären graphs (The theory of regular graphs). *Acta Mathematica*, 15: 193–220, 1891.

Neil Robertson, Daniel P. Sanders, Paul D. Seymour, and Robin Thomas. The four-colour theorem. *J. Comb. Theory B*, 70(1):2–44, 1997. doi: 10.1006/JCTB.1997.1750.

Pierre Rosenstiehl and Robert Endre Tarjan. Rectilinear planar layouts and bipolar orientations of planar graphs. *Discret. Comput. Geom.*, 1:343–353, 1986. doi: 10.1007/BF02187706.

Roberto Tamassia and Ioannis G. Tollis. A unified approach a visibility representation of planar graphs. *Discret. Comput. Geom.*, 1:321–341, 1986. doi: 10.1007/BF02187705.

Hassler Whitney. Non-separable and planar graphs. *Proceedings of the National Academy of Sciences*, 17(2):125–127, 1931. doi: 10.1073/pnas.17.2.125.